SIOG 231: GEOMAGNETISM AND ELECTROMAGNETISM

Chapter 10: Main Field Modeling and Regularization

1. Introduction

In our discussions of the field so far we have not dealt with the issue of how to determine the magnetic field from a practical set of observations. In an earlier lecture it was stated that the spherical harmonic (SH) representation provides a unique description of any magnetic field represented by a harmonic potential, and we saw how in principle this allows unique separation of the internal and external field contributions to spherical harmonic expansions. So far we have seen two different representations of a scalar potential for the geomagnetic field. We've used both $\mathbf{B} = -\nabla\Psi$ and $\Omega = rB_r$ (Neumann Boundary Value Problem) and supposed that we had perfect knowledge of the field to write a general solution to Laplace's equation in the form of an SH expansion.

Continuing with the specialization to the internal field coefficients, we note that the field coefficients for $\Omega(r, \theta, \phi) = rB_r(r, \theta, \phi)$ on $S(a)$, the surface of the sphere of radius a, can in principle be derived from the equation below:

$$\beta_l^m = \int_{S(a)} B_r(a, \theta, \phi) Y_l^m(\theta, \phi)^* \frac{d^2\mathbf{r}}{a^2} \tag{56}$$

but this evidently requires knowledge of B_r all over $S(a)$, which is not possible in practice. We will need a strategy for dealing with incomplete and noisy observations. But before we tackle the question of finding approximate models, (and here we will follow custom and say that a geomagnetic field model is a finite collection of SH coefficients), we discuss the mathematical question of what are sufficient measurements to define the field uniquely.

2. Geomagnetic Elements

Equation (56) relies on knowledge of the radial magnetic field, but these are not the only kind of observations that are made; typically when surface survey, observatory, and satellite data are involved we may need to consider all of the following kinds of observations:

B_r, B_θ, B_ϕ – orthogonal components of the magnetic field in geocentric reference frame.

X, Y, Z – orthogonal components of the geomagnetic field in local coordinate system, directed north, east, and downwards respectively. This is the geodetic (or geographic) coordinate system.

$H = (X^2 + Y^2)^{\frac{1}{2}}$ – horizontal magnetic field intensity

$B = (B_\theta^2 + B_\phi^2 + B_r^2)^{\frac{1}{2}}$ or $F = (X^2 + Y^2 + Z^2)^{\frac{1}{2}}$ – total field intensity

$D = \tan^{-1}(Y/X)$ – declination

$I = \tan^{-1}(Z/H)$ - inclination.

If we are prepared to accept the approximation that the Earth is a sphere, then we can write

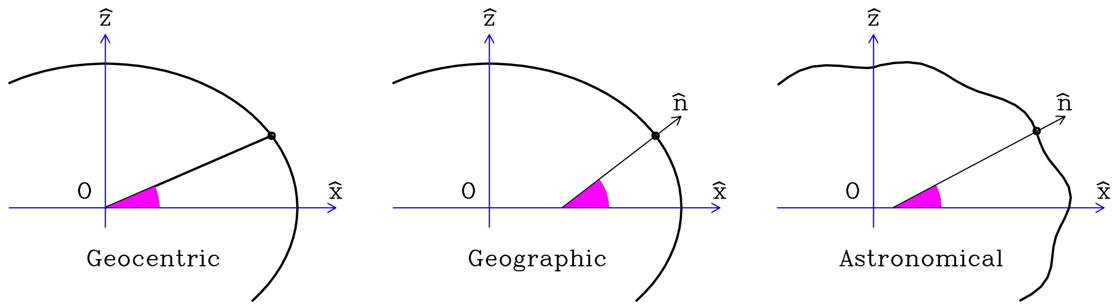
$$X \approx -B_\theta; \quad Y \approx B_\phi; \quad Z \approx -B_r. \tag{57}$$

But as you will have seen in gravity lectures the shape of the Earth is much better approximated by a spheroid or ellipsoid of revolution, with equatorial radius a, polar radius b, eccentricity, e

$$e^2 = \frac{(a^2 - b^2)}{a^2} \tag{58}$$

and flattening

$$f = \frac{(a - b)}{a} = 1/298.257. \tag{59}$$



In paleomagnetic field modeling it's really not necessary to take account of the difference between geodetic and geocentric latitude but in modern field modeling (usually based on both surface and satellite observations) the data are accurate enough for it to really matter. Since we make our SH models in geocentric coordinates and surface observations are measured in geographic coordinates they must first be transformed into geocentric.

> *Exercise:*
> What is the size of the error you make if you neglect to correct for the ellipsoidal shape of the earth and use geocentric latitude and longitude in calculating the field from a spherical harmonic model?

As we already noted, it is impossible to acquire complete knowledge of the radial magnetic field (or indeed any other component) on any spherical surface. Our data are always incomplete and noisy, and consequently there will always be ambiguity in the models derived from them. Because of this it might be argued that the issue of uniqueness in the case of complete and perfect data is of purely academic interest. However, experience in making magnetic field models based exclusively on one kind of observation has shown that this is not the case.

3. Fundamental Limitations on Uniqueness

Non-Uniqueness and $|\mathbf{B}|$

The fact that B_r on a surface $S(a)$ uniquely determines the field (because it is the solution of a Neumann boundary value problem for Laplace's equation) might lead one to hope that complete measurements of $|\mathbf{B}|$ (the magnitude of the field) would uniquely specify the internal magnetic field to within a sign. Until relatively recently almost all marine observations have been of $|\mathbf{B}|$, and so are many satellite measurements. It is also the case that scalar observations are generally more accurate than their vector counterparts so there might be advantages to just using those data in modeling. For the case of intensity data alone, George Backus (now Professor Emeritus at IGPP) showed in 1968 (*Q. J. Mech and Appl. Math* 21, pp 195-221, 1968; and *J. Geophys. Res.* 75, pp 6339-41, 1970) by constructing a counter example that knowledge of $|\mathbf{B}|$ was not sufficient. Backus' counter example is constructed in the following way: first find a magnetic field with potential Ψ_M that is everywhere orthogonal to a dipole field; then $\Psi_D + \Psi_M$ and $\Psi_D - \Psi_M$ have the same B everywhere.

One might reasonably ask whether this counter-example matters or is merely an artificial mathematical construct. If it does not one would surely be better off using scalar measurements, with typical accuracy of $<$ 1nT, than vector components which are usually only accurate to several nT because of the need to orient a low earth orbiting measurement platform within an Earth centered reference frame. However the effect is real. Independent field models using exclusively vector or scalar Magsat data predict similar intensity at Earth's surface, but show differences of up to 2000 nT in the individual vector components. Backus' example provided an explanation of the poor quality of directional information predicted by models based on B alone, especially in equatorial regions. It reflects the fact that optimization in the $|\mathbf{B}|$ model is insensitive to errors perpendicular to the field, while the full vector is sensitive to misfit in all directions. This *perpendicular error effect* is also referred to as the *Backus effect*.

Early satellites had only measured total field intensity, and Backus' result led to the launching in 1979 of the first vector field satellite, known as Magsat. Later satellites (Oersted, CHAMP, Swarm) launched to monitor Earth's magnetic field from low Earth orbit have routinely taken vector as well as scalar measurements and use one or more star cameras to provide a means of transforming vector observations from the satellite coordinates in which measurements are made to the geocentric coordinate system used for Earth observations.

Subsequently, Khokhlov, Hulot and Le Mouël (*Geophys. J. Internat.* 130, pp 701-3, 1997) showed that if knowledge of the location of the dip equator is added to knowledge of $|\mathbf{B}|$ everywhere then uniqueness of the solution is guaranteed. The practical usefulness of this result is debatable, and full field **B** measurements are now routine.

Non-Uniqueness and $\hat{\mathbf{B}}$

Another potentially interesting case is exact knowledge of the direction of the field all over Earth's surface, $S(a)$, which one might imagine could determine the field to within a multiplicative scaling constant. Paleomagnetic observations can determine ancient field directions much more reliably than field intensities, so one might hope to discover the shape of the geomagnetic field in those epochs when paleomagnetic data are plentiful. Similarly to the intensity case, (despite an earlier inaccurate argument to the contrary) it was shown in 1990 by Proctor and Gubbins, (*Geophys. J. Internat.* 100, pp 69-79, 1990) that $\hat{\mathbf{B}}$ on a spherical surface does not determine the field to within a scalar multiple.

The counter examples might again be argued to be questionable representatives of Earth-like fields, being axisymmetric, antisymmetric with respect to the Equator, and octupolar in type. This result too has been investigated further and it has now been shown that if the magnetic field only has two poles on the surface $S(a)$ then knowledge of its direction everywhere allows the geomagnetic field to be recovered, except for a constant multiplier (see Hulot, Khokhlov and Le Mouël, *Geophys. J. Internat.* 129, pp 347-54, 1997 for details).

For much more discussion of uniqueness questions, including some extensions to the case where both external and internal field are considered see Sabaka *et al.* (2010, *Handbook of Geomathematics*, doi: 10.1007/978-3-642-01546-5_17). The essential result of importance for geomagnetic field modeling using observations from satellites and geomagnetic observatories is summarized in their Figure 4 and reproduced below. This figure moves somewhat beyond our assumption that all observations are taken in a source free region, as LEO satellites are typically fly at 400-1000 km elevation, outside of the neutral atmosphere, and in the F-region above the E-region of Earth's ionosphere. Despite intrinsic non-uniqueness one can recover sufficient information for field modeling to be a useful activity. Our discussion of the basics of how to construct geomagnetic field models in the next section is much simpler; we will ignore any external fields and initially focus on reconstruction of the main part of the geomagnetic field generated in Earth's core.

4. Construction of Field Models

We will suppose here that we have a finite collection of inaccurate observations of orthogonal components of the geomagnetic field $B_j = \mathbf{B} \cdot \hat{\mathbf{r}}_j$ in geocentric coordinates. Our goal is to derive from these observations the spherical harmonic coefficients that best represent the real geomagnetic field. Clearly, we cannot use (56) because our knowledge of $B_r(a, \theta, \phi)$ is incomplete. Also it would not make use of our measurements of the tangential part of the field. Making a field model is a way of mapping the field everywhere outside the source region and is analogous to (and in some respects identical with) that of interpolation to find a curve passing through a finite number of data. Many curves will do the job, and we cannot choose which one is

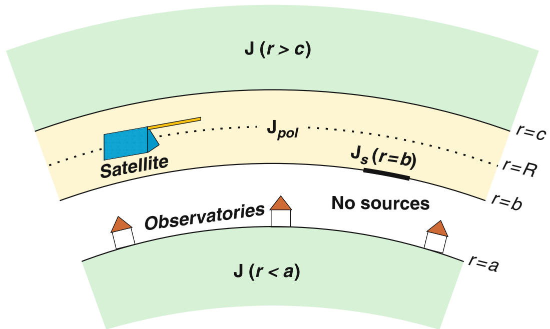


Fig. 4
Uniqueness of a magnetic field recovered from partial information within a current-carrying shell. In this special case relevant to geomagnetism, it is assumed that any source can lie below $r = a$ (*internal* J($r < a$) sources), and above $r = c$ (*external* J($r > c$) sources), no sources can lie within the lower subshell ($a < r < b$, the neutral atmosphere), a spherical sheet current can lie at $r = b$ (the *E*-region $J_s(r = b)$ sources), and only poloidal sources can lie within the upper subshell ($b < r < c$, the *F*-region ionosphere). The knowledge of B on a sphere $r = R$ in the upper subshell (as provided by, e.g., a satellite) and of enough components of B on the sphere $r = a$ (as provided by, e.g., observatories at the Earth's surface), is then enough to recover the field produced by most sources in many places (see text for details)

more desirable without supplying additional information of some kind. One extra thing we know is that the magnetic field obeys a differential equation, but that turns out not to be enough information by itself. When the observations are not exact, but uncertain as in all real situations the issue is even more complicated. We obviously shouldn't expect the interpolant to pass through all (perhaps even any) of the observations. In the case where we would like to downward continue a geomagnetic field model to the core-mantle boundary, we need to be especially careful about how we deal with noise in the data; if we fit models with small scale structure derived from this noise, then it may dominate the real signal after downward continuation.

4:0.0 Least Squares Estimation

We will start with the time-honored technique for the construction of geomagnetic field models (invented by Gauss for this very purpose!), which was that of least squares estimation of the spherical harmonic coefficients in a truncated spherical harmonic expansion for the measured field components. Instead of the exact expansion with infinitely many terms, we decide ahead of time to model the data with an expansion truncated to degree L: so the scalar potential becomes

$$\Psi(r, \theta, \phi) = a \sum_{l=1}^{L} \sum_{m=-l}^{l} b_l^m \left(\frac{a}{r}\right)^{l+1} Y_l^m(\theta, \phi) \tag{60}$$

and we know

$$\mathbf{B} = -\nabla\Psi. \tag{61}$$

Once again we are using fully-normalized spherical harmonic functions here which simplifies things from the theoretical perspective when we want to discuss regularized field models in the next section. If we measure all three orthogonal components of the field B_r, B_θ, B_ϕ in a geocentric coordinate system at P locations we have a set of $M = 3P$ observations of magnetic elements at sites designated $\mathbf{r}_p = (\hat{r}_p, \hat{\theta}_p, \hat{\phi}_p)$, the vector field is written $\mathbf{B}(\mathbf{r}_p), p = 1, \ldots, P$. Now we let $\hat{\mathbf{s}}_{p_i}$, for $i = 1, \ldots, 3$ be the unit vector along one of the orthogonal (r, θ, ϕ) directions at location $\mathbf{r}_j$, where the locations $\mathbf{r}_j$ will not be distinct because of the vector observations. Then

$$
\begin{aligned}
d_j &= \hat{\mathbf{s}}_{p_i} \cdot \mathbf{B}(\mathbf{r}_j), \quad j = 1, ..., M \\
&= a^{l+2} \sum_{l=1}^{L} \sum_{m=-l}^{l} b_l^m \hat{\mathbf{s}}_{p_i} \cdot \nabla \left[\frac{Y_l^m(\hat{\mathbf{r}}_j)}{r_j^{l+1}} \right] + \epsilon_j .
\end{aligned} \tag{62}
$$

Note that we have allowed for uncertainty in each observation through ϵ_j, and that although we said each location has 3 vector components measured it doesn't matter if some of them turn out to be missing.

The data d_j are linear functionals of the b_l^m, the SH coefficients that specify the field. With an appropriate indexing scheme for the b_l^m we can write a prediction for our observations d_j as a matrix equation.

$$
\mathbf{d} = \mathbf{G}\mathbf{b} + \mathbf{e} \tag{63}
$$

with vectors $\mathbf{d}$ representing the M data, $\mathbf{e}$ noise in measurements both $\in R^M$, $\mathbf{b} \in R^K$ is a vector containing an ordered list of the spherical harmonic coefficients b_l^m. $\mathbf{G}$ is an $M \times K$ matrix that tell us how to predict the observations based on their positions $\mathbf{r}_j$ and the various $Y_l^m(\theta, \phi)$. For each d_j we can compute g_{jk} (with $j = 1, \ldots, M$ and $k = 1, \ldots, K$) the contribution of the relevant spherical harmonic at that point; the vector $\mathbf{e}$ contains the misfits between the model predictions and the actual measurements. The total number of parameters, the length of the vector $\mathbf{b}$, is determined by the truncation level: $K = L(L + 2)$. Thus

$$
\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_M \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1^{-1} \\ b_1^0 \\ b_1^1 \\ \vdots \\ b_L^L \end{bmatrix} \qquad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_M \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots & g_{1K} \\ g_{21} & g_{22} & g_{23} & \cdots & g_{2K} \\ g_{31} & g_{32} & g_{33} & \cdots & g_{3K} \\ \vdots & & & & \\ g_{M1} & g_{M2} & g_{M3} & \cdots & g_{MK} \end{bmatrix} \tag{64}
$$

The value of L is chosen so that K is (much) less than M, the number of data, so there are fewer free parameters than data to be fit. This means that it is impossible to choose $\mathbf{b}$ to get an exact match to the data, and so $\mathbf{e}$ is not a vector of zeros. Least squares estimation involves finding the values for $\mathbf{b}$ that minimize $||\mathbf{e}||^2 = ||\mathbf{d} - \mathbf{G}\mathbf{b}||^2$, where the notation $|| \cdot ||$ is called a norm – in this case it is the ordinary length of the vector. The idea here is to do the best job possible with the available free variables and make the model predictions as close to the data as they can be, as measured by the length of the misfit vector.

Straightforward calculus can be used to show that the LS solution vector $\tilde{\mathbf{b}}$ can be written in terms of the solution to the normal equations:

$$
\tilde{\mathbf{b}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{d}. \tag{65}
$$

Note that (65) is for several reasons not a good way to find $\tilde{\mathbf{b}}$ in a computer – first, linear systems of equations ought never to be solved by calculating a matrix inverse (it wastes time and is inaccurate); second, there is a clever way of writing the LS equations that avoids a serious numerical precision problem arising in (65). (All this is essentially invisible in Matlab and Python!) A result from statistics, known as the Gauss-Markov Theorem, shows that provided the misfits are due to random, uncorrelated perturbations with zero mean, and have a common variance, then the least squares solution is the best linear unbiased estimate (BLUE) available, in the sense that the resulting coefficients have the smallest variance (and thus the smallest uncertainty in the result) amongst such estimates. Also, the expected value of $||\mathbf{e}||^2$

$$
\mathcal{E}[||\mathbf{e}||^2] = (M - K)\sigma^2 \tag{66}
$$

where σ^2 is the variance of the noise process. If the noise has a known covariance structure, then the theory can readily be adjusted to take this into account.

A fundamental problem with this approach is that although we might have an idea of the size of the uncertainty in the various observations, and we could choose the truncation level of the spherical harmonic expansion so (66) is approximately satisfied, we do not know that the K spherical harmonics we have chosen adequately describe the geomagnetic field. In other words, the misfit has two sources, not one: measurement error and an insufficiently detailed model. To guarantee a complete description of the real field $L = \infty$; in that case the Gauss-Markov theorem does not apply, nor does (66), and we have no uncertainty estimates for our model. Truncation at finite L corresponds to an assumption about simplicity in the model that has no physical basis – the resulting field model may be biased by the truncation procedure.

4:0.1 Regularization – an Alternative to Least Squares

An alternative to LS fitting that has been very widely used since the mid 1980s and is the basis of almost all modern geomagnetic field modeling is to choose "simple" or minimum complexity models for the field of an explicit kind. To illustrate the concept we return to our one-dimensional interpolation problem, where we suppose we have pairs of observations (x_k, y_k) as shown in Figure 3.4.4.1 and we want to represent them throughout the domain (x_1, x_N) with a smooth function $f(x)$.

One widely used solution to the problem of interpolation is to use what is known as cubic spline interpolation. Data are connected by piecewise cubic polynomials, with continuous derivatives up to second order. The cubic spline interpolant has the property that it is the smoothest curve connecting the points, in the precise sense of minimizing the *integrated squared value (ISV) of the second derivative*:

$$\int_{x_1}^{x_M} \left[\frac{d^2 f}{dx^2} \right]^2 dx. \tag{67}$$

The spline solution is the solid line in the Figure 3.4.4.1; any other curves, like the dashed ones, have a larger value for the integral (67). In physical terms we can think of (67) as the quadratic approximation to the stored elastic energy in a thin beam deformed to pas through the points, with the shape of the beam given by the spline curve.

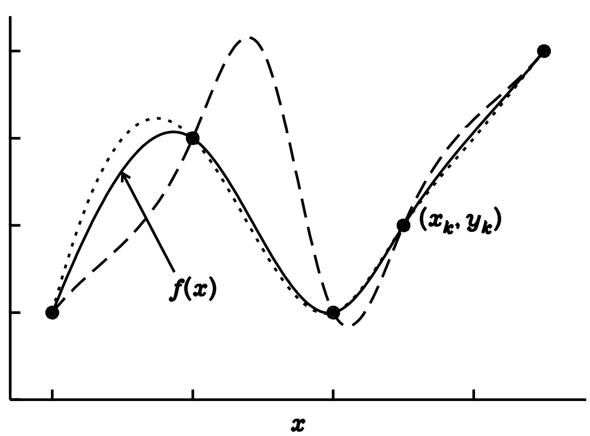


Figure 3.4.4.1

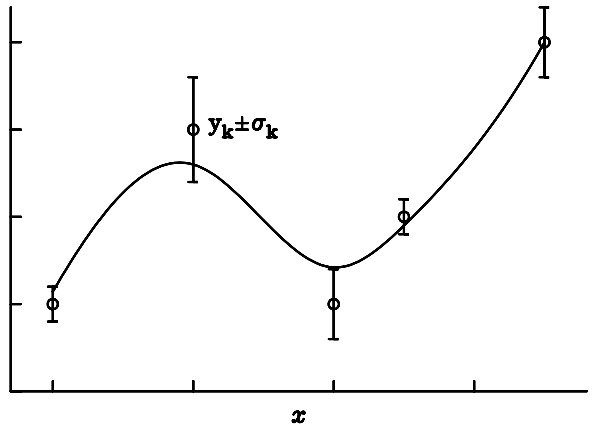


Figure 3.4.4.2

If the data are noisy we can still model them by the same kind of curve, but we no longer require the curve to pass exactly through the model points. Instead, we ask for a satisfactory fit, usually defined in terms of norm of the misfit. So we seek the curve with the smallest ISV second derivative, that has a target misfit. This is an example of a constrained minimization, and we can solve it with a Lagrange multiplier. The solution to the constrained problem is determined by finding the stationary points of an *objective functional*, U, of the following kind:

$$U = \sum_{j=1}^{M} \frac{(f(x_j) \ - \ y_j)^2}{\sigma_j^2} \ + \ \lambda \int_{x_1}^{x_M} \left[\frac{d^2 f}{dx^2}\right]^2 dx \tag{68}$$

subject to

$$\sum_{j=1}^{M} \frac{(f(x_j) - y_j)^2}{\sigma_j^2} = T. \tag{69}$$

The size of the Lagrange multiplier λ is dictated by the constraint that the data be fit to a reasonable tolerance level. If the uncertainty in the observations, σ_j, is known, one reasonable choice for T is the expected value, written here as $\mathcal{E}$, of a chi-squared random variable with M degrees of freedom and

$$\mathcal{E}[\chi_M^2] = M$$

The same ideas are used in geomagnetic field modeling. However, the vector nature of the field makes the choice of penalty functional more complicated – we want to find some property of the magnetic field that can be used like (67) to minimize wiggliness or complexity in our models of the geomagnetic field, either at Earth's surface ($r = a$) or at the CMB (core-mantle boundary, $r = c$). The general idea of constructing models that minimize a penalty other than just data misfit is called *regularization*.

One candidate penalty function is

$$E = \int_{r>a} |\mathbf{B}|^2 d^3\mathbf{r}. \tag{70}$$

Since $|\mathbf{B}(\mathbf{r})|^2/2\mu_0$ is the energy density of the magnetic field at $\mathbf{r}$, $E/2\mu_0$ is the total energy stored in $\mathbf{B}$ outside the sphere of radius a or c. We can reduce this integral to a manageable form in terms of spherical harmonics. We write $\mathbf{B} = -\nabla\Psi$, then

$$E = \int_{r>a} \nabla\Psi \cdot \nabla\Psi \; d^3\mathbf{r}. \tag{71}$$

Next we make use of a familiar vector identity (number 4, in our list: $\nabla \cdot (s\mathbf{A}) = \nabla s \cdot \mathbf{A} + s\nabla \cdot \mathbf{A}$, and letting $\mathbf{A} = \nabla\Psi$ and $s = \Psi$) followed by Laplace's equation to write

$$\begin{aligned} E &= \int [\nabla \cdot (\Psi\nabla\Psi) \; - \; \nabla^2\Psi] \; d^3\mathbf{r} \\ &= \int \nabla \cdot (\Psi\nabla\Psi) \; d^3\mathbf{r}. \end{aligned} \tag{72}$$

Using Gauss' Divergence Theorem we can rewrite the volume integral in terms of a surface integral over $S(a)$

$$E = \int_{S(a)} -\Psi \frac{\partial\Psi}{\partial r} \; d^2\mathbf{r}. \tag{73}$$

Now we simply substitute the spherical harmonic expansion (60) for the potential Ψ.

$$\begin{aligned} E &= \int_{S(a)} [a \sum_{l,m} b_l^m Y_l^m(\hat{\mathbf{r}})][\sum_{l',m'} (l' \; + \; 1) b_{l'}^{m'} Y_{l'}^{m'}(\hat{\mathbf{r}})]^* a^2 d^2\hat{\mathbf{r}} \\ &= a^3 \sum_{l,m} \sum_{l',m'} (l' \; + \; 1) \; b_l^m (b_{l'}^{m'})^* \int_{S(a)} Y_l^m(\hat{\mathbf{r}}) Y_{l'}^{m'}(\hat{\mathbf{r}})^* d^2\hat{\mathbf{r}} \\ &= a^3 \sum_{l=1}^{\infty} \sum_{m=-l}^{l} (l \; + \; 1) |b_l^m|^2. \end{aligned} \tag{74}$$

Hence E can be written as a positive weighted sum of the squared absolute values of the spherical harmonic coefficients. This sum is now in a form we can use in minimizing an objective function like (71) for the magnetic field. It's usually the case that the factor of a^3 is absorbed into the Lagrange multiplier in (71), and then the weighting of the squared coefficients just becomes a polynomial in l.

The weighting by increasing l means that higher degree (shorter wavelength) contributions to the model will be strongly penalized in minimizing the objective functional. (74) is one example of a set of norms (measures of size) of the kind

$$||\mathbf{B}||_w^2 = \sum_{l=1}^{\infty} w_l \sum_{m=-l}^{l} |b_l^m|^2, \quad w_l > 0. \tag{75}$$

Many interesting properties corresponding to smoothness or small size of the field can be written in this form, for example,

$$\int_{S(a)} \mathbf{B} \cdot \mathbf{B} \; d^2\hat{\mathbf{r}} \qquad w_l = (2l \; + \; 1)(l \; + \; 1) \tag{76}$$

which you might recognize as the spatial geomagnetic power spectrum as a function of wavelength for a fully normalized SHE of the internal magnetic field.

Here are a few more examples of functions used to penalize short wavelength structure in the field model by applying successively heavier weight for large l:

$$\int_{S(a)} [\nabla_1(\hat{\mathbf{r}} \cdot \mathbf{B})]^2 d^2\hat{\mathbf{r}} \quad w_l = l(l \; + \; 1)^2(l \; + \; \frac{1}{2}) \quad \text{surface gradient of } B_r \tag{77}$$

$$\int_{S(a)} [\nabla_1^2(\hat{\mathbf{r}} \cdot \mathbf{B})]^2 d^2\hat{\mathbf{r}} \quad w_l = l^2(l+1)^4 \quad \text{surface Laplacian of } B_r \tag{78}$$

$$\int_{r<a} \mathbf{J}_T \cdot \mathbf{J}_T d^3\mathbf{r} \qquad w_l \approx (l+1)(2l+1)^2(2l+3). \tag{79}$$

In (79) $\mathbf{J}_T$ is the toroidal part of the current flow in Earth's core, whose significance will be discussed later. These ideas were first set out in a paper by Shure, Parker, and Backus, *Phys. Earth Planet. Inter.* 28, pp 215-29, 1982.

4:0.2 Results - Gauss Coefficients

We have already discussed that in honor of Karl Friedrich Gauss, the expansion coefficients are invariably called *Gauss coefficients*. In gravity the SHE is called Stokes' expansion. There are some differences in geomagnetism. First, the fundamental one, that makes geomagnetism more interesting than gravity – the field is changing detectably on times scales of a human lifetime (not to mention the largest changes involve geomagnetic reversals). So the coefficients must always be dated and as we have seen splines (linear, cubic, or higher order) are often used to represent the time variations. Secondly, and almost trivially, geomagnetists never use fully normalized spherical harmonics – instead they employ a theoretically awkward real (as opposed to complex) representation in which the basis functions (spherical harmonics) are normalized so that

$$\int_{S(1)} (Y_l^m)^2 d^2\hat{\mathbf{r}} = \frac{4\pi}{2l+1}. \tag{80}$$

Then the scalar potential is written

$$\Psi(r, \theta, \phi) = a \sum_{l=1}^{\infty} \left(\frac{a}{r}\right)^{l+1} \sum_{m=0}^{l} N_{lm}(g_l^m \cos m\phi + h_l^m \sin m\phi) P_l^m(\cos\theta) \tag{81}$$

where $a = 6,371.2$ km, the mean Earth radius, the P_l^m are the Associated Legendre functions from Part I, and

$$\begin{aligned} N_{lm} &= 1, \qquad\qquad m = 0 \\ &= \sqrt{\frac{2(l-m)!}{(l+m)!}}, \quad m > 0. \end{aligned} \tag{82}$$

In practice, the N_{lm} are not usually included in (81) by the geomagnetic community. Instead they are implicitly integrated into the P_l^m which are identified as Schmidt semi-normalized harmonics and you will find it simply written as

$$\Psi(r, \theta, \phi) = a \sum_{l=1}^{\infty} \left(\frac{a}{r}\right)^{l+1} \sum_{m=0}^{l} (g_l^m \cos m\phi + h_l^m \sin m\phi) P_l^m(\cos\theta) \tag{81a}$$

As you might expect this can cause a lot of confusion, especially in sharing computer codes. You will find the numbers g_l^m, h_l^m tabulated in many places. There are official models designated IGRF for the International Geomagnetic Reference Field, agreed upon every five years by the International Association for Geomagnetism and Aeronomy as a good approximation to the field. The values are in nanotesla. The linearized rate of change of each Gauss coefficient is also given for interpolation or extrapolation across the five year period. We are currently at Version 13, IGRF (2015-2020), and the model coefficients since 1900 can be found at http://www.ngdc.noaa.gov/IAGA/vmod/igrf.html. The website also features Fortran and Python code for evaluating the magnetic field either at a single location or on a grid. In the next figure we see various views of the radial component of the magnetic field from IGRF 2020: top is B_r at Earth's surface; middle the non-dipole contribution to B_r at Earth's surface and the bottom panel gives B_r downward continued to the core-mantle boundary.

Below, the (now 21 year old) IGRF-9, for the year 2000 is tabulated up to degree and order 10. IGRF models come with a Health Warning about their limitations, and there can be significant differences between their predictions and what you measure at a given time and place. Recent versions extend to degree and order 13, but they do not include crustal anomalies or detailed time variations both of which require a more comprehensive approach to modeling than what we have described here.

Gauss Coefficients for IGRF-2000

l	m	g_l^m	h_l^m	l	m	g_l^m	h_l^m	l	m	g_l^m	h_l^m
1	0	-29615	0	6	2	74	64	9	0	5	0
1	1	-1728	5186	6	3	-161	65	9	1	9	-20
2	0	-2267	0	6	4	-5	-61	9	2	3	13
2	1	3072	-2478	6	5	17	1	9	3	-8	12
2	2	1672	-458	6	6	-91	44	9	4	6	-6
3	0	1341	0	7	0	79	0	9	5	-9	-8
3	1	-2290	-227	7	1	-74	-65	9	6	-2	9
3	2	1253	296	7	2	0	-24	9	7	9	4
3	3	715	-492	7	3	33	6	9	8	-4	-8
4	0	935	0	7	4	9	24	9	9	-8	5
4	1	787	272	7	5	7	15	10	0	-2	0
4	2	251	-232	7	6	8	-25	10	1	-6	1
4	3	-405	119	7	7	-2	-6	10	2	2	0
4	4	110	-304	8	0	25	0	10	3	-3	4
5	0	-217	0	8	1	6	12	10	4	0	5
5	1	351	44	8	2	-9	-22	10	5	4	-6
5	2	222	172	8	3	-8	8	10	6	1	-1
5	3	-131	-134	8	4	-17	-21	10	7	2	-3
5	4	-169	-40	8	5	9	15	10	8	4	0
5	5	-12	107	8	6	7	9	10	9	0	-2
6	0	72	0	8	7	-8	-16	10	10	-1	-8
6	1	68	-17	8	8	-7	-3				

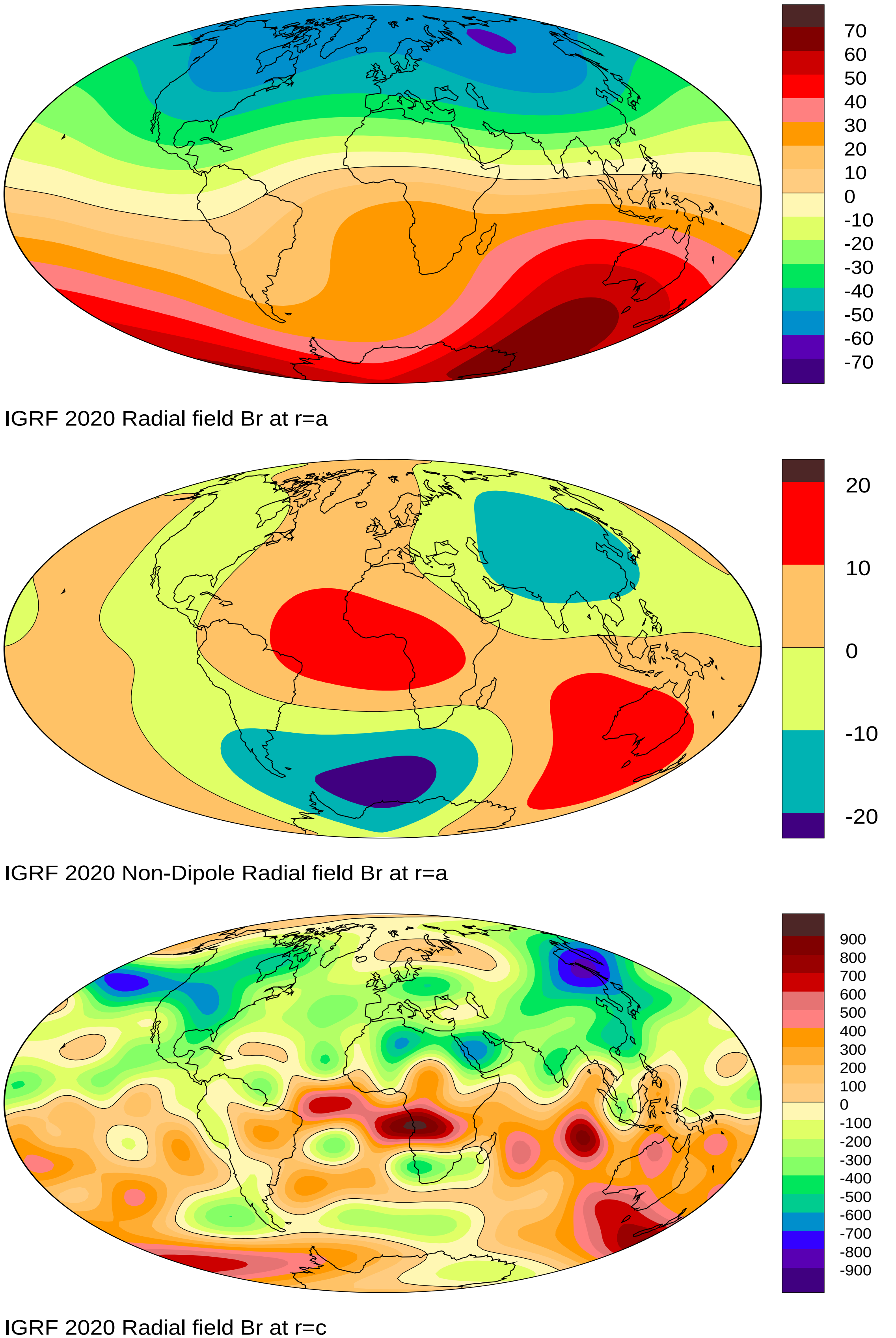


Figure 3.4.5. Radial magnetic field for 13th Generation IGRF 2020: Top panel is at Earth's surface; Middle has only non-dipole contributions to the field; Bottom has been downward continued to the CMB. Field values are in μT.