SIOG 231: GEOMAGNETISM AND ELECTROMAGNETISM

Chapter 11: The Lithospheric Magnetic Field

1. Introduction

Magnetic anomalies are the signals of the internal geomagnetic field left behind after the part generated by Earth's core has been removed from the observations. Most of the mantle and some of the lower portions of the crust are above the Curie point of magnetic minerals; thus the source of the magnetic anomalies must lie in the crust or uppermost mantle. We see this from the Lowes spectrum; if there were significant mantle sources, the spectrum would not die away exponentially, but more as a power law, which is the case for the gravitational SH spectrum. Magnetic anomalies have of course played a very important role in earth sciences: the correlation of lineated magnetic anomalies roughly parallel to mid-oceanic ridge crests with the geomagnetic reversal time scale provided the final evidence for seafloor spreading. With the advent of the satellite era they are now studied on a global scale through Comprehensive Magnetic Field Modeling (e.g. Sabaka et al., 2020, https://doi.org/10.1186/s40623-020-01210-5) and compilations of aeromagnetic and marine survey data merged to provide the World Digital Magnetic Anomaly Map.

2. Regional Surveys of the Scalar Field

Regional magnetic surveys, both on land and under the oceans used to be performed by taking spot readings of the magnetic field. Now, however, aircraft or shipboard surveys are commonly carried out with a towed proton precession magnetometer, which often measures only the field intensity not its direction. In order to study magnetic anomalies a standard core field model plus large scale external fields, $\mathbf{B}_0$, is subtracted from the observations; this is often called the *regional magnetic field*. Suppose the sum of all contributions to the field is $\mathbf{B}$. The *total field anomaly* is the difference between the magnitudes of the observed and core magnetic fields:

$$\Delta T = |\mathbf{B}| - |\mathbf{B}_0|. \tag{150}$$

Now we will let $\Delta\mathbf{B}$ be the contribution to $\mathbf{B}$ of some anomalous magnetic source. Then

$$\mathbf{B} = \mathbf{B}_0 + \Delta\mathbf{B}. \tag{151}$$

We would like to know $\Delta\mathbf{B}$ to study the crustal source, so

$$\begin{aligned}\Delta T &= |\mathbf{B}_0 + \Delta\mathbf{B}| - |\mathbf{B}_0| \\ &= \sqrt{|\mathbf{B}_0|^2 + |\Delta\mathbf{B}|^2 + 2\Delta\mathbf{B}\cdot\mathbf{B}_0} - |\mathbf{B}_0|.\end{aligned} \tag{152}$$

Neglecting quantities $O|\Delta\mathbf{B}|^2/|\mathbf{B}_0|^2$ we find

$$\begin{aligned}\Delta T &= |\mathbf{B}_0|[1 + 2\Delta\mathbf{B}\cdot\mathbf{B}_0/|\mathbf{B}_0|^2]^{\frac{1}{2}} - |\mathbf{B}_0| \\ &= |\mathbf{B}_0|[1 + \Delta\mathbf{B}\cdot\mathbf{B}_0/|\mathbf{B}_0|^2] - |\mathbf{B}_0| \\ &= \Delta\mathbf{B}\cdot\mathbf{B}_0/|\mathbf{B}_0| = \Delta\mathbf{B}\cdot\hat{\mathbf{B}}_0.\end{aligned} \tag{153}$$

Thus $\Delta T \approx \Delta\mathbf{B}\cdot\hat{\mathbf{B}}_0$, that is, it is the component of the anomaly field in the direction of the regional field, provided the anomaly field is small in magnitude relative to the total field. The validity of this approximation depends on the size of $\Delta\mathbf{B}$ relative to $\mathbf{B}$. Typical crustal magnetic anomalies range in magnitude from a few nT to several thousand nT, but are usually less than 5,000 nT, so this provides an adequate representation for total field anomalies.

Is the total field harmonic? Approximately, to the extent that the regional field direction remains constant in the survey region. We can see this easily, writing $\Delta\mathbf{B} = -\nabla V$ with harmonic V,

$$\nabla^2 \Delta T = -\nabla^2(\nabla V \cdot \hat{\mathbf{B}}_0) = -\hat{\mathbf{B}}_0 \cdot \nabla\nabla^2 V = 0. \tag{154}$$

If $\mathbf{B}_0$ is not effectively constant, the algebra is messy and the subject not worth much effort.

As with satellite observations, the reason one prefers to measure the field magnitude $|\mathbf{B}|$ at sea is that there is no need to keep an accurately oriented platform, and total field data can be made to ± 1 nT in 60,000 nT with very robust instruments, and ± 0.1 nT with only a little more effort. As we asked in 3.4.1 we can inquire whether knowledge of ΔT is actually sufficient to describe the harmonic field of crustal sources. If $\mathbf{B}_0$ is effectively constant, and not horizontal, then knowledge of ΔT is fully equivalent to knowledge of $\Delta\mathbf{B}$ itself, since one can then construct $\Delta\mathbf{B}$ from ΔT. But as usual, things aren't quite so simple. One collects survey data on long lines in the oceans, and it really is not possible to know $\mathbf{B}$ everywhere on a surface. If instead of ΔT on a long line one actually has the vector data $\Delta\mathbf{B}$, then valuable information can be obtained about the accuracy of the measurements and other useful things, like across track lineation, by looking at the correlations between the components. See Parker and O'Brien, *J. Geophys. Res.* 102, pp 24815-24, 1997. Over the past two decades there has been a move towards measuring the vector field in some marine surveys, particularly on Japanese ships and on near-seafloor instruments.

3. Magnetic Permeability and Susceptibility

In chapter 2 we outlined the relationship between magnetic displacement $\mathbf{H}$ and magnetic induction $\mathbf{B}$

$$\mathbf{H} = \mathbf{B}/\mu_0 \; - \; \mathbf{M} \tag{155}$$

with $\mathbf{M}$ representing the magnetic polarization or magnetization of the material. In the region where we make measurements it is unnecessary to distinguish between $\mathbf{B}$ and $\mathbf{H}$ because there are no currents flowing and no magnetization. Inside the crust, however, $\mathbf{M}$ depends on the atomic and macroscopic properties of the material. Materials can acquire a component of magnetization in the presence of an external magnetic field (such as that generated in Earth's core). The so-called *induced magnetization* is often considered to be proportional in magnitude to and along the direction of the external field and

$$\mathbf{M} = \chi\mathbf{H} \tag{156}$$

and χ is called the *magnetic susceptibility*. Then we can write

$$\mathbf{B} = \mu_0(1 \; + \; \chi)\mathbf{H} = \mu\mathbf{H} \tag{157}$$

where μ is the *magnetic permeability*. In practice χ may be dependent on field intensity, negative or need to be represented by a tensor (magnetically anisotropic materials).

Although diamagnetism due to perturbations of electron orbits in an applied field and paramagnetism (perturbations of atomic magnetic moments) are important physical processes, these are insignificant contributors to the geomagnetic field. The important contributions come from materials with atomic moments that interact strongly with each other as a result of quantum mechanical exchange interactions. These are called *ferrimagnetic* materials and they can carry either an *induced* or *remanent* magnetization, and are the most important contributors to the lithospheric geomagnetic field. The total magnetization of a rock will result from the sum of these two contributions

$$\mathbf{M} = \mathbf{M}_i + \mathbf{M}_r = \chi\mathbf{H} + \mathbf{M}_r. \tag{158}$$

The stability and acquisition of magnetization depends on temperature and history of exposure to magnetic fields: above the Curie temperature thermal perturbations destroy the spontaneous magnetization so that the only remaining magnetization is from diamagnetic or paramagnetic effects.

Iron-nickels are important in extra-terrestrial magnetic studies. Iron-oxides, iron-oxyhydroxides, and iron sulfides are most important for paleomagnetic studies. Magnetite (Fe_3O_4, Curie temperature 580°C) and its solid solutions with ulvospinel Fe_2TiO_4) are the most important magnetic minerals in crustal rocks (see Figure 5.1.1) and both Curie temperature and magnetization vary substantially with composition of the solid solution. Hematite, greigite and pyrhotite play an important role in paleomagnetic studies, and for much more on this topic see Lisa Tauxe's online book *Essentials of Paleomagnetism* at https://earthref.org/MagIC/books/Tauxe/Essentials/ .

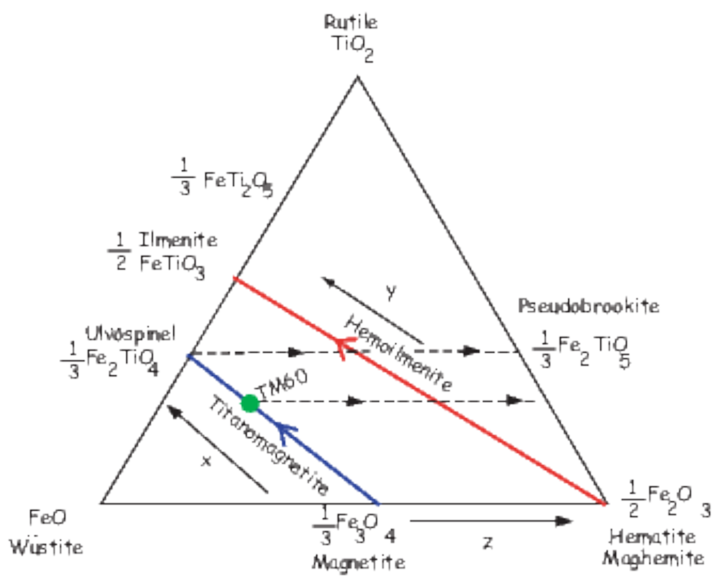


Figure 5.1.1: Ternary diagram showing solid solutions of iron and titanium oxides

3:1 A Digression: Back to Some Physics of Magnetism

The existence of a magnetic field in a volume of space means that there is a change of energy within the volume, and furthermore there is an energy gradient so that a force is produced. The force can be detected by the acceleration of a charged particle moving in the field, by the force on a current carrying conductor, by the torque on a magnetic dipole (*e.g.*a bar magnet, or compass needle), or even by reorientation of spins of electrons within certain types of atoms.

Magnetic fields are caused by electrical charge in motion. In 1819 Oersted discovered that electric currents produce a magnetic force when he observed that a magnetic needle is deflected at right angles to a conductor carrying a current. Magnetic fields are also produced by permanent magnets: although there is no conventional electric current in them, there are orbital motions and spins of electrons (sometimes called “Amperian currents") which lead to a magnetization within the material and a magnetic field outside. The cooperative behavior that leads to permanent magnetization is a quantum mechanical effect: however for our purposes we can describe the macroscopic effects of the observed magnetization and associated magnetic fields using classical electromagnetic theory. Magnetic fields exert a force on both current-carrying conductors and permanent magnets.

So far we have not been very explicit about distinguishing magnetic field from magnetic induction and magnetization. The distinction is often not clearly made and units are used interchangeably, especially between **H** and **B**, so it is important to be careful. To simplify things we will only consider SI units. There are three kinds of magnetic vectors:

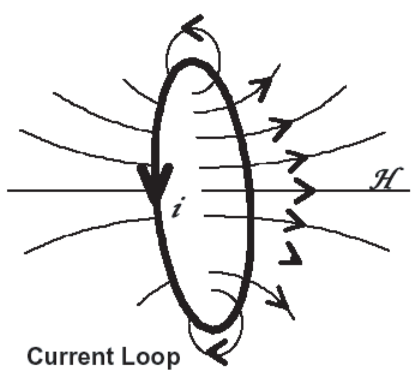
(1) Magnetic field **H**, measured in A/m;

(2) Magnetization **M**; also known as the intensity of magnetization or magnetic moment per unit volume ($\mathbf{M} = \frac{\mathbf{m}}{V}$ with **m** the magnetic moment). It is also measured in A/m

(3) Magnetic induction **B**, measured in *Tesla*.

These quantities are related through the equation $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$, where μ_0 is the permeability of free space ($\mu_0 = 4\pi \times 10^{-7} NA^{-2}$).

The primary role of electric currents in generating magnetic fields is illustrated in Figure 5.2.1, where current, i, flows in a loop of radius r. The loop has a magnetic moment **m** associated with it.



A magnetic field H will be produced at the center of the loop given by

$$H = \frac{i}{2r} \qquad \text{[Amperes/meter, A /m]}$$

The current loop has a magnetic moment, m, associated with it

m = i x Area [Am^2]

Figure 5.2.1: A current loop has a magnetic moment **m** and generates a magnetic field

Gravitational, electrostatic and magnetic forces all have fields associated with them. The field is a property of the space in which the force acts, and the pattern of a field is portrayed by field lines. At any point in a field the direction of the force is tangential to the field line, and the intensity of the force is proportional to the density of field lines. Unlike gravitational and electrostatic forces, both of which act centrally to the source of the force and follow an inverse square law, magnetic fields vary with azimuth and even in the simplest case of a magnetic dipole the field strength falls off as the cube of distance. Although magnetic monopoles do not really exist, they provide a useful mechanism for describing the effects of magnetic fields and the concept of a magnetic potential analogous to gravitational potential.

3:2 Magnetic Poles and Dipoles and their Potentials

In 1785 Coulomb showed that the force between the ends (or magnetic poles) of long thin magnetized steel needles obeyed an inverse square law. Thus supposing that monopoles existed and we have two with strengths p_1 and p_2 we can write an inverse square law for the force between them, a Coulomb's Law for magnetic poles

$$F(r) = K\frac{p_1 p_2}{r^2}. \tag{5.3.1}$$

By analogy with the gravitational case we can define the magnetic field associated with a monopole p as

$$B(r) = K\frac{p}{r^2}$$

In SI units $K = \frac{\mu_0}{4\pi}$ where $\mu_0 = 4\pi \times 10^{-7} NA^{-2}$ (or equivalently henry/meter) is the permeability constant.

The magnetic potential W at a distance r from a pole of strength p is defined in terms of the work required to move a pole of unit strength from position r to infinity

$$W = -\int_r^{-\infty} B dr = \frac{\mu_0 p}{4\pi r} \tag{5.3.2}$$

In contrast to electrostatic charges magnetic poles cannot exist in isolation: each positive pole must be paired with a corresponding negative pole to form a magnetic dipole. (One consequence of this is that $\nabla \cdot \mathbf{B} = 0$.) We can construct the potential for a dipole by summing the potential for two equal but opposite poles p and $-p$ located a distance d apart, and then letting their separation become infinitesimally small compared with the distance to the point of observation.

Referring to figure 5.3.1 we can calculate the potential W at a distance r from the mid-point of a pair of poles in a direction that makes an angle θ to the axis passing through both poles p and $-p$. if the distances to the respective poles are r_+ and r_- respectively the net potential at (r, θ) will be

$$W = \frac{\mu_0 p}{4\pi}(\frac{1}{r_+} - \frac{1}{r_-}) = \frac{\mu_0 p}{4\pi}(\frac{r_- - r_+}{r_+ r_-}) \tag{5.3.3}$$

Now we suppose that $d << r$ and use the approximations

$$r_+ \approx r - \frac{d}{2}\cos\theta$$
$$r_- \approx r + \frac{d}{2}\cos\theta'$$

Since $d << r$ we can write $\theta \approx \theta'$ and on negelcting terms of order $(d/r)^2$ we find

$$r_- - r_+ \approx d\cos\theta$$

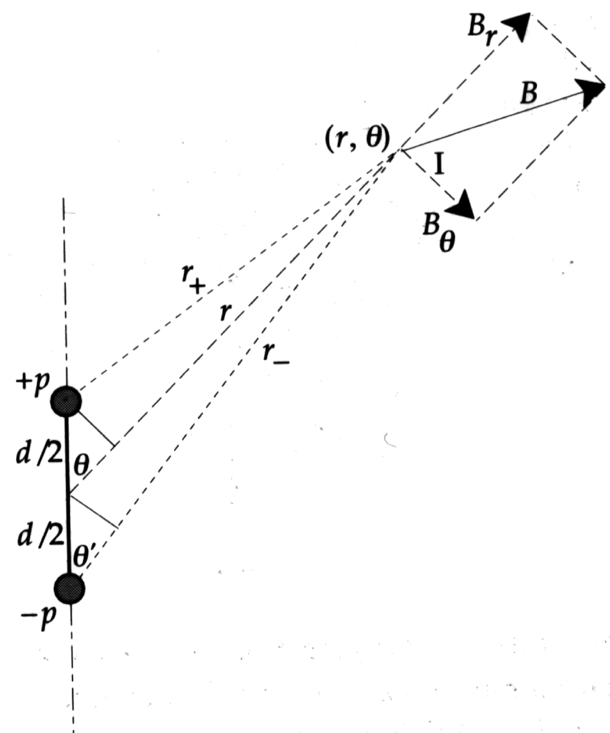


Figure 5.3.1: Building a magnetic dipole from two fictitious monopoles

$$r_+r_- \approx r^2 - \frac{d^2}{4}\cos^2\theta \approx r^2$$

and that the dipole potential at the point (r, θ) is

$$W(r, \theta) = \frac{\mu_0}{4\pi}\frac{(dp)\cos\theta}{r^2} = \frac{\mu_0 m\cos\theta}{4\pi r^2}$$

We call the quantity $m = (dp)$ the magnetic moment of the dipole. Note that it is a vector quantity: our coordinate system is defined relative to the axis of the dipole.

3:3 A Further Digression on Dipoles

From the perspective of potential theory, the dipole represents the point source obtained by allowing two point charges of opposite sign to approach each other, in such a way that the product of the charges and the separation $p = qd$ is fixed, as the distance is decreased. This picture is artificial in every realistic system, since there are no point electric dipoles in electricity, there are no point charges in magnetism, and there are no negative sign masses in gravity. None-the-less the dipole is fundamental for static magnetism because it appears as the first term in the SH expansion of an internal magnetic field. When $\mathbf{B} = -\nabla\Psi$, we have in a fully normalized form for fields with interior sources:

$$\Psi(\mathbf{r}) = \sum_{l=1}^{\infty}(\frac{1}{r})^{l+1}\sum_{m=-l}^{l} b_l^m Y_l^m(\hat{\mathbf{r}}) \tag{5.4.1}$$

where $r > 0$ here and in all our manipulations. The scalar magnetic potential of an axial dipole (the one symmetric about the z axis) is just the $l = 1$, $m = 0$ term

$$\Psi_D(\mathbf{r}) = \frac{b_1^0 Y_1^0(\hat{\mathbf{r}})}{r^2} = \frac{N_1^0 b_1^0 \cos\theta}{r^2}. \tag{5.4.2}$$

This agrees of course with the traditional multipole way of writing it: for an axial point dipole at the coordinate origin

$$\Psi_D(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\cos\theta}{r^2}, \qquad r > 0 \tag{5.4.3}$$

where m is the *dipole moment*. The constant $\frac{\mu_0}{4\pi}$ makes its appearance because the field variable $\mathbf{H}$ is conventionally attached to the scalar potential (recall here $\nabla \times \mathbf{H} = 0$), and the 4π is a feature of the SI system. In that system the units of magnetic dipole moment are A m^2, also sometimes written as J T^{-1}.

In geomagnetism it is more traditional to use the Gauss coefficients in the semi-normalized expansion; recall that

$$\Psi(\mathbf{r}) = a \sum_{l=1}^{\infty} (\frac{a}{r})^{l+1} \sum_{m=0}^{l} \hat{N}_l^m [g_l^m \cos m\phi + h_l^m \sin m\phi] P_l^m(\cos\theta). \tag{5.4.4}$$

Then for the axial dipole term we extract the single term

$$\Psi_D(\mathbf{r}) = \frac{a^3 g_1^0 \cos\theta}{r^2}. \tag{5.4.5}$$

Notice a simplification here because the normalizing constant $\hat{N}_1^0 = 1$. Identifying coefficients in (5.4.3) and (5.4.5), we see that

$$m = \frac{4\pi a^3 g_1^0}{\mu_0}. \tag{5.4.6}$$

The multipole interpretation of (5.4.3) leads to another way of expressing Ψ_D. Since the separation of the fictitious charges is a vector quantity, so is the dipole moment. In (5.4.2) - (5.4.5) we have looked only at an axial dipole, that is, the one with circular symmetry about $\hat{\mathbf{z}}$. Clearly (5.4.3) can be written in a way that works for a dipole pointing in any direction:

$$\Psi_D(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{r^3} \tag{5.4.7}$$

where $\mathbf{m}$ is the vector dipole moment. You will ready confirm that the Gauss coefficients $g_1^1,\ h_1^1$ or the fully normalized $b_1^{\pm1}$ express the potentials due to x and y components of the vector $\mathbf{m}$. For the general case (5.4.6) must be revised:

$$m = \frac{4\pi a^3((g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2))^{\frac{1}{2}}}{\mu_0}. \tag{5.4.8}$$

Another way that (5.4.7) has already been expressed, in the Equivalent Source Theorem (see Gravity section 12), is

$$\Psi_D(\mathbf{r}) == -\frac{\mu_0}{4\pi}\mathbf{m}\cdot\nabla\frac{1}{r}. \tag{5.4.9}$$

In electromagnetic theory the physical model for a magnetic dipole is not the pair of opposite sign poles, but the tiny current loop. To the accuracy set by classical physics, electrons and other elementary particles are point dipoles; and the circulation of electrons within atoms comprises elementary current loops. We will merely state a number of results about dipoles related to this approach. Proofs will be found in most books on classical electromagnetism or in *Foundations of Geomagnetism*, Chapter 2. The natural vehicle for this theory is of course the vector potential $\mathbf{A}$ with $\mathbf{B} = \nabla \times \mathbf{A}$. Then the analog of (5.4.9) is

$$\mathbf{A}_D(\mathbf{r}) = -\frac{\mu_0}{4\pi}\mathbf{m} \times \nabla\frac{1}{r} \tag{5.4.10}$$

where $\mathbf{m}$, the dipole moment can be computed for a set of divergence-free currents by

$$\mathbf{m} = \frac{1}{2}\int_V (\mathbf{s} \times \mathbf{j})\, d^3\mathbf{s}. \tag{5.4.11}$$

We will demonstrate the equivalence of (5.4.9) and (5.4.10) later. For an elementary circular current loop with radius r and normal $\hat{\mathbf{n}}$, (5.4.11) integrates to

$$\mathbf{m} = \hat{\mathbf{n}}\pi r^2 j \tag{5.4.12}$$

so that the magnitude of the dipole moment is just the area times the current; it is this picture that motivates the usual units of amp meter-squared for dipole moment.

What happens if we carry through the calculation to produce the actual field, rather than merely displaying the potentials? The grand result, proved in the next Section is this: the vector field $\mathbf{B}$ at $\mathbf{r}$ due to a dipole $\mathbf{m}$ at O is

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\nabla(\mathbf{m} \cdot \nabla\frac{1}{r}) = -\frac{\mu_0}{4\pi}\mathbf{m} \cdot \nabla\nabla\frac{1}{r} \tag{5.4.13}$$

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\Big(\frac{3\mathbf{m} \cdot \mathbf{r}\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\Big). \tag{5.4.14}$$

Finally, we can use (5.4.13) to show the equivalence of (5.4.9) and (5.4.10). Note that since $\mathbf{m}$ is constant, vector identity #5 for the curl shows that

$$\mathbf{A}_D(\mathbf{r}) = -\frac{\mu_0}{4\pi}\nabla \times \Big(\frac{\mathbf{m}}{r}\Big). \tag{5.4.15}$$

Then, taking the curl of (5.4.15) together with identity #9 yields

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\nabla \times \nabla \times (\frac{\mathbf{m}}{r}) \tag{5.4.16}$$

and

$$= -\frac{\mu_0}{4\pi}[\nabla\nabla \cdot (\frac{\mathbf{m}}{r}) \ - \ \nabla^2(\frac{\mathbf{m}}{r})]. \tag{5.4.17}$$

Again, since $\mathbf{m}$ is constant, the second term is just $\mathbf{m}\nabla^2(1/r)$ which vanishes, and we can use identity #4 to expand the remaining term:

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\nabla(\mathbf{m} \cdot \nabla\frac{1}{r} \ + \ \frac{1}{r}\nabla \cdot \mathbf{m}) \tag{5.4.18}$$

$$= -\frac{\mu_0}{4\pi}\nabla \mathbf{m}\cdot\nabla\frac{1}{r} = -\frac{\mu_0}{4\pi}\mathbf{m}\cdot\nabla\nabla\frac{1}{r} \tag{5.4.19}$$

which is identical to (5.4.13).

4. Crustal Magnetic Models

The ultimate goal of a regional magnetic survey is to make some inferences about the spatial distribution and nature of the magnetic sources generating the anomalies and hence to draw conclusions about the geological processes active in the earth. The first step in understanding this problem is to calculate the fields from a model: that is for a given distribution of magnetization predict the expected observations; this is called solving the *forward problem*.

How can we do this? We start by considering Equation 5.4.9 in the form of the magnetic scalar potential Ψ at position $\mathbf{r}$ due to a point dipole $\mathbf{m}$ located at $\mathbf{s}$:

$$\Psi(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{m}(\mathbf{s})\cdot\nabla_s\frac{1}{|\mathbf{r} - \mathbf{s}|}. \tag{159}$$

To find the potential for a distribution of magnetization, we consider it to be a sum of contributions from elemental dipoles (recall the magnetization vector $\mathbf{M}$ is just a density of dipole moment per unit volume):

$$\Psi(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \mathbf{M}(\mathbf{s})\cdot\nabla_{\mathbf{s}}\frac{1}{|\mathbf{r} - \mathbf{s}|}d^3\mathbf{s} \tag{160}$$

where V is the magnetized source region. Then to get the magnetic field

$$\begin{aligned}\mathbf{B} = -\nabla_{\mathbf{r}}\Psi(\mathbf{r}) &= -\frac{\mu_0}{4\pi}\nabla_{\mathbf{r}}\int_V \mathbf{M}(\mathbf{s})\cdot\nabla_{\mathbf{s}}\frac{1}{|\mathbf{r} - \mathbf{s}|}d^3\mathbf{s}\\ &= -\frac{\mu_0}{4\pi}\int_V \mathbf{M}(\mathbf{s})\cdot\nabla_r\nabla_s\frac{1}{|\mathbf{r} - \mathbf{s}|}\ d^3\mathbf{s}.\end{aligned} \tag{161}$$

You can easily verify that

$$\nabla_r\nabla_s\frac{1}{|\mathbf{r} - \mathbf{s}|} = -\nabla_s\nabla_s\frac{1}{|\mathbf{r} - \mathbf{s}|}. \tag{162}$$

Now let's evaluate (162). By using the summation convention (recall Gravity notes by Bob Parker, Section 1, Page 1) and keeping calm, it's quite straightforward. First some abbreviations: let $\mathbf{R} = \mathbf{r} - \mathbf{s}$; $\partial_j = \partial/\partial s_j$ and also recall from Part I that $\partial_j s_k = \delta_{jk} = -\partial_j R_k$.

Then we can write $\frac{1}{|\mathbf{r}-\mathbf{s}|} = (R_kR_k)^{-\frac{1}{2}}$ and in summation notation (162) becomes

$$\begin{aligned}G_{ij} &= \partial_i\partial_j(R_kR_k)^{-\frac{1}{2}} = \partial_i[-\frac{1}{2}(R_kR_k)^{-3/2}\partial_j(R_kR_k)]\\ &= \partial_i[-\frac{1}{2}(R_kR_k)^{-3/2}(\partial_jR_kR_k + R_k\partial_jR_k)]\\ &= \partial_i[-\frac{1}{2}(R_kR_k)^{-3/2}(-\delta_{jk}R_k - R_k\delta_{jk})] = \partial_i[R_j(R_kR_k)^{-3/2}]\\ &= (\partial_iR_j)(R_kR_k)^{-3/2}\ +\ R_j\partial_i(R_kR_k)^{-3/2}\\ &= -\delta_{ij}(R_kR_k)^{-3/2}\ +\ R_j[-3/2(R_kR_k)^{-5/2}](-2R_k\delta_{ik})\\ &= \frac{3R_iR_j}{(R_kR_k)^{5/2}} - \frac{\delta_{ij}}{(R_kR_k)^{3/2}}.\end{aligned} \tag{163}$$

Substituting (163) into (161) and restoring vector notation gives us

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \left[\frac{3\mathbf{M}(\mathbf{s})\cdot\mathbf{R}\mathbf{R}}{|\mathbf{R}|^5} - \frac{\mathbf{M}(\mathbf{s})}{|\mathbf{R}|^3}\right] d^3\mathbf{s} \\ &= \frac{\mu_0}{4\pi}\int_V [\mathbf{G}(\mathbf{r}-\mathbf{s})\cdot\mathbf{M}(\mathbf{s})]\ d^3\mathbf{s}\end{aligned} \tag{164}$$

where you will recall $\mathbf{R} = \mathbf{r} - \mathbf{s}$. The expression in brackets is nothing more than the vector magnetic field at $\mathbf{r}$ from a point dipole, moment M at $\mathbf{s}$ as we asserted in equation (5.4.14) Most geophysicists seem afraid of this formula and it doesn't appear very often in texts.

Finally if we have measured the total field anomaly and the regional core field has constant direction $\hat{\mathbf{B}}_0$, then

$$\begin{aligned}\Delta T = \Delta\mathbf{B}\cdot\hat{\mathbf{B}}_0 &= \frac{\mu_0}{4\pi}\int_V \left[3\mathbf{M}\cdot\mathbf{R}\hat{\mathbf{B}}_0\cdot\frac{\mathbf{R}}{|\mathbf{R}|^5} - \frac{\mathbf{M}\cdot\hat{\mathbf{B}}_0}{|\mathbf{R}|^3}\right] d^3\mathbf{s} \\ &= \frac{\mu_0}{4\pi}\int_V \mathbf{G}'(\mathbf{r}-\mathbf{s})\cdot\mathbf{M}(\mathbf{s})\, d^3\mathbf{s}.\end{aligned} \tag{165}$$

Note that here

$$\mathbf{G}'(\mathbf{r}-\mathbf{s}) = \hat{\mathbf{B}}_0.\nabla_r\nabla_r\frac{1}{|\mathbf{r}-\mathbf{s}|} = \left[\frac{3(\mathbf{r}-\mathbf{s})\hat{\mathbf{B}}_0\cdot(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^5} - \frac{\hat{\mathbf{B}}_0}{|\mathbf{r}-\mathbf{s}|^3}\right]$$

Solving the forward problem thus comes down to evaluating the above integral for an appropriately shaped body. In practice this is usually done by making approximations to the shape of the body.

For compact regions such as seamounts (which are isolated submarine volcanos) a favorite plan is to divide the region V into a set of smaller elementary shapes, like cuboids, over which the integral can be done exactly, then sum. See Blakely's book for some references to this approach. Sometimes it is permissible to take $\mathbf{M}$ = constant in space. Then one can apply identities and get (165) into a surface integral with the divergence theorem. The surface can be conveniently approximated by a set of triangular faces, called a *tessellation*; see Parker, Shure, and Hildebrand, *Rev. of Geophys.* 25, pp 17-40, 1987.

In the early days of marine magnetic survey work it was discovered, as you will know, that in many places the magnetic anomaly pattern takes the form of a series of stripes, caused by reversals of the ancient geomagnetic field and linear emplacement at the ocean ridges. This geometry allows a simplification by assuming that the magnetization $\mathbf{M}$ is constant in one direction, usually identified with the y axis. Then (165) can be reduced to an integral of the x-z plane, and is much easier to do; the formula for a polygon shape is quite simple.

Often it can be assumed that the magnetic layer is very thin, so that one can ignore variations of $\mathbf{M}$ in the vertical z direction. You will appreciate that when the thin-layer assumption is made, (165) has the form of a convolution and then Fourier methods can be invoked. This is very popular in marine magnetic work, both for the older single-profile data sets and the more modern surveys of an area. Again Blakely devotes a lot of space to this issue.

Exercise:

(a) As suggested, with $\mathbf{M}$ constant rewrite (165) in a form that you can apply the divergence theorem and reduce the integral to a surface form.

(b) Find out about Poisson's relationship. Explain what it is and when its use might be appropriate.

5. The Magnetic Annihilator and Runcorn's Theorem

The object of the process of magnetic modeling is to learn about the state of the crust, first its magnetic state, and then, if we're lucky, other things too. In the last section we assumed that $\mathbf{M}$ was known and we calculated $\Delta\mathbf{B}$ from it, but in reality we don't know the magnetization and we do know the anomaly. We need to reverse the process, which is called solving the *inverse problem*. A major question in any inverse problem is that of uniqueness: does a (complete and exact) set of data determine the unknown magnetization, or is there ambiguity, even with ideal data? It turns out the inverse problem for magnetization is *ill-posed*, which means that there are infinitely many possible solutions to choose from, unless further restrictions or simplifying assumptions are brought in. You might perhaps have come to this conclusion from the equivalent source theorem for gravity, (see Section 12 of Parker's notes on trying to infer density inside a body), which informs us that the potential due to any internal density distribution can be reproduced by equivalent sources distributed on the surface of the body. This theorem applies equally well here: if $\mathbf{B} = -\nabla\Psi$ and all magnetic sources are confined to compact region V

$$\Psi(\mathbf{r}) = \frac{1}{4\pi}\int_{\partial V} d^2\mathbf{s}\,\left[\Psi\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}-\mathbf{s}|} \;-\; \frac{1}{|\mathbf{r}-\mathbf{s}|}\frac{\partial\Psi}{\partial n}\right]. \tag{166}$$

This equation tell us that we can mimic the potential of a magnetized body by poles and dipoles on the surface, whatever the true interior distribution of $\mathbf{M}$ may be. The need for (mono)poles is a bit disturbing, however.

Here is another, perhaps more startling example. In (165) let the magnetization be $\mathbf{N} = \nabla q$ where $q(\mathbf{s})$ is *any* smooth function that vanishes on the boundary ∂V. It will be seen from (162) that $\mathbf{G} = \nabla p$ and $\nabla^2 p = 0$ provided $\mathbf{r}$ is outside V, which it always is. Then

$$\begin{aligned}\frac{4\pi}{\mu_0}\Delta T &= \int_V \nabla p\cdot\nabla q\; d^3\mathbf{s} = \int_V [\nabla\cdot(q\nabla p) \;-\; q\nabla^2 p]\; d^3\mathbf{s}\\ &= \int_{\partial V} q\nabla p\cdot\hat{\mathbf{s}}d^2\mathbf{s} = 0.\end{aligned} \tag{167}$$

Thus the anomaly caused by this whole family of functions is zero. Hence if we have a magnetization $\mathbf{M}$ that matches observation, then so does $\mathbf{M}+\mathbf{N}$. The function $\mathbf{N}$ is an example of a magnetic *annihilator*, a function with no observable magnetic field, yet nonzero internal magnetization. Of course we have already encountered toroidal magnetic fields, whose currents in a sphere are also annihilator sources.

The existence of annihilators means that we cannot ever know what a magnetic source is based solely on its magnetic anomaly. The problem is ill-posed.

It was assumed in the marine world that if the magnetic layer was very thin, this problem would be eliminated. But Parker and Huestis (*J. Geophys. Res.,* 79, pp 1587-93, 1974) showed there was ambiguity even then, in the form of a single function which can be added in arbitrary amounts to any solution, without disturbing the agreement.

Finally, an important example of an annihilator was discovered by Keith Runcorn in studies of the moon's magnetic field (*Nature* 253, 1042, pp 701-3, 1975). Runcorn was convinced the moon had once had an internal dynamo which has ceased to operate because the lunar core became solid. Samples from the moon have proved to be magnetic, yet there appears to be almost no lunar magnetic field. Critics of Runcorn asked, If there is a general lunar magnetization from the ancient dynamo, why can't we see the field from these fossil sources? Here is Runcorn's surprising answer: If the magnetization of the moon is induced magnetization in a shell, no matter what form the internal dynamo field was like, the resulting magnetization would be an annihilator – no observable magnetic field even though the rocks could be strongly magnetic.

Here is a proof. Consider the induced lunar magnetization in the shell $c \leq r \leq b$: we suppose $\mathbf{M} = \kappa \nabla \Phi$ where $\kappa = -\chi/\mu_0$ assumed constant, and so $\nabla^2 \Phi = 0$, since Φ is the scalar potential of the dynamo source in the moon's core. Now consider the scalar potential Ψ from the induced sources, outside the shell. Let $R = |\mathbf{r} - \mathbf{s}|$, then by (160):

$$\Psi(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \mathbf{M} \cdot \nabla \frac{1}{R}\, d^3\mathbf{s} = \frac{\mu_0}{4\pi} \int_V \kappa \nabla \Phi \cdot \nabla \frac{1}{R}\, d^3\mathbf{s}. \tag{169}$$

We can now apply our favorite vector identity and the divergence theorem:

$$\begin{aligned} \Psi &= \frac{\mu_0 \kappa}{4\pi} \int_V d^3\mathbf{s} \left[\nabla \cdot (\frac{1}{R} \nabla \Phi) \; - \; \frac{1}{R} \nabla^2 \Phi \right] \\ &= \frac{\mu_0 \kappa}{4\pi} \left[\int_{S(b)} \; - \; \int_{S(c)} \right] d^2\mathbf{s}\, \frac{1}{R}\, \partial_r \Phi = \Psi_b \; - \; \Psi_c \end{aligned} \tag{170}$$

where we have used the fact Φ is harmonic; notice in (170) that the normals on $S(c)$ and $S(b)$ point in opposite directions and so the two surface integrals must be subtracted. Let us look at the contribution Ψ_b from the outer surface $S(b)$; we introduce the SH expansion of $1/R$ from Part I (equation 138), and an expansion in c_l^m for the potential Φ:

$$\begin{aligned} \Psi_b &= \frac{\mu_0 \kappa}{4\pi} \int_{S(b)} d^2\mathbf{s} \left[\sum_{l,m} \frac{4\pi}{2l+1} \frac{b^l}{r^{l+1}} Y_l^m(\hat{\mathbf{r}}) Y_l^m(\hat{\mathbf{s}})^* \right] \left[\sum_{l',m'} -\frac{l'+1}{b^{l'+2}} c_{l'}^{m'} Y_{l'}^{m'}(\hat{\mathbf{s}}) \right] \\ &= -\frac{\mu_0 \kappa}{b^2} \sum_{l,m} \sum_{l',m'} c_{l'}^{m'} \frac{l'+1}{2l+1} \frac{1}{r^{l+1}} Y_l^m(\hat{\mathbf{r}}) \int_{S(1)} Y_{l'}^{m'}(\hat{\mathbf{s}}) Y_l^m(\hat{\mathbf{s}})^*\, b^2 d^2\hat{\mathbf{s}} \\ &= -\mu_0 \kappa \sum_{l,m} \frac{l+1}{2l+1} \frac{c_l^m}{r^{l+1}} Y_l^m(\hat{\mathbf{r}}). \end{aligned} \tag{171}$$

The remarkable thing about (171) is that it is independent of the radius of the surface b; so the same answer will be obtained for Ψ_c, the integral over $S(c)$. But $\Psi = \Psi_b - \Psi_c$, and so $\Psi(\mathbf{r})$ is identically zero for all $\mathbf{r}$.

This argument is applicable to the Earth to some degree. The induced crustal magnetization from a uniform medium in a shell will produce no observable anomaly. You will easily see this generalizes to any series of shells, so χ can vary with depth, and the annihilator property persists. Thus the crustal fields we see are due to lateral variability, changes in layer thickness, and other departures from uniformity. Of course the Earth's crust is so heterogeneous we are not surprised to see very large anomalies almost everywhere. Why would one expect the upper regions of the moon to be less heterogeneous than the Earth's crust?

5:1 Results – Magnetic Anomalies Everywhere

The major triumph for magnetic anomalies has been the discovery of the Vine-Mathews magnetic stripes, the discovery of sea-floor spreading, and the mapping of the ages of the ocean basins. Reversal of direction results in a very strong, short wavelength contrast in magnetization which gives rise to intense magnetic anomalies (200-2,000 nT) in the scale range 10-100 km at the sea surface. Traditionally these anomalies have been modeled by thin layers (500 - 1,000 m thick) with blocks of magnetization where the direction is constant, except for changes in sign, and the intensity is constant too. This kind of model never fits the observations exactly, but does a reasonably good job if geological ages are needed. If one wishes to get more detail, and allow more flexibility in the models the annihilator remains a problem. The next step in sophistication is to allow the models to vary in intensity with x, and to make the layer follow the topographic variations. The magnetic annihilator then has a large scale component and another that varies like the bathymetry. Adding or subtracting this function can make the magnetization change sign and appear more-or-less correlated with topographic relief. When one is seeking geomagnetic intensity histories, the ambiguity is troublesome.

One of the happy exercises carried out on profile data is to apply spatial linear filters to change the apparent dip of the magnetization vector. It can be shown that there is a filter which after application to an anomaly results in the profile that would have been observed at the north pole, with vertical magnetization. This activity is called *reduction to the pole* (see Blakely for more). The idea here is that magnetizations measured or acquired in different latitudes yield anomalies that differ widely in appearance even if the underlying block pattern is similar; reduction to the pole makes profiles much easier to compare, and also offers some information about the latitude of formation of that piece of crust.

Another (but less successful) application of geomagnetism to marine anomalies is the analysis of the anomalies from seamounts. The major piece of information required here is the average direction of magnetization, which gives the usual paleomagnetic clues about where the volcano was formed. To overcome a serious ambiguity problem the first studies simply assumed the magnetization vector within the seamount was quite constant, both in direction and in intensity. Now with only three parameters to fit nonuniqueness disappears. Of course the constant-$\mathbf{M}$ models don't fit the observations very well at all, but marine geologists have generally ignored this difficulty. Various methodologies have been invented for improving the fits and estimating the error in the resulting direction. One approach that worked reasonably well recognized the observation from drilling the oceanic crust that direction of $\mathbf{M}$ is much more nearly constant than $|\mathbf{M}|$, which varies by more than one order of magnitude in a single body. So we ask, What uni-directional magnetizations fit the given anomaly pattern, if any? It turns out that in some cases, quite a close clustering of directions will fit the anomaly, thus providing a direction and uncertainty. For details see Parker, R. L., A theory of ideal bodies for seamount magnetism, *J. Geophys. Res.* B10, pp 16101-12, 1991.

Magnetic anomalies observed near the surface over land are generally smaller in amplitude than the marine ones. This is assumed to be because continental rocks are on the whole comprised of much less magnetic types: granitic, metamorphic and sedimentary rocks are orders of magnitude less magnetizable, and have no thermoremanent component. But aeromagnetic surveys are relatively cheap and are routinely used to locate and delineate ore bodies. At the longest wavelength ($> 1{,}000$ km) however, as mapped by satellite, the continents have larger magnetic anomalies than the oceans. It is assumed that this is because on the longest scales induced magnetization is at work, and the extra thickness of the continental crust (or greater depth to Curie isotherm) gives the continents an advantage.