# SIOG 231: GEOMAGNETISM AND ELECTROMAGNETISM

## Chapter 15: The One Dimensional Earth

### Introduction

Our next step is to move from the half-space solutions to models in which conductivity varies with depth $\sigma(z)$. The one dimensional Earth could be a flat Earth, in the case of an MT sounding, or a radially symmetric Earth, in the case of GDS. While radial symmetry is a good approximation to deep Earth sounding, few people nowadays would rely on 1D interpretation for MT studies. However, understanding the behavior of the 1D problem is a good first step in understanding how the MT method works. Fitting 1D problems provides useful quality control on MT data. Finally, the 1D solution can serve as a background for higher dimensional problems that use a scattered field approach.

Derived from first principles, the flat Earth and radial solutions are different, but we will see that there is a mapping from one to the other, so we will concentrate on the Cartesian problem.

### Analytical solutions

Again, we will consider a horizontal $x$-directed magnetic field at the surface that varies harmonically in time with frequency $\omega$:

$$B_x = B_o e^{i\omega t} \quad .$$

We have already established that the resulting electric field will be only in the $y$-direction, which can be written as an electric field that varies with depth only:

$$E_y = E(z) e^{i\omega t} \quad .$$

Note that $E(z)$ is complex, which can be thought of as an amplitude and phase. We can again show that $\nabla \cdot \mathbf{E} = 0$ by taking the divergence of $\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E}$ (which is just Ampere's and Ohm's Laws)

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_o \nabla \cdot \sigma(z) \mathbf{E} \quad .$$

The left hand side of this equation is zero from vector identity (I1: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$), and we can apply vector identity (I4 :$\nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s \nabla \cdot \mathbf{A}$) to the right hand side to get

$$0 = \sigma(z) \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \sigma(z) \quad .$$

Since in the case we are considering here, $\mathbf{E}$ is horizontal (in the $y$-direction) and $\sigma$ only varies in $z$, $\mathbf{E} \cdot \nabla \sigma(z) = 0$. Thus, we are left with $\nabla \cdot \mathbf{E} = 0$, and so the diffusion equation we derived before still holds

$$\nabla^2 \mathbf{E} = i\omega \mu_o \sigma \mathbf{E}$$

(remember, to get to this equation before we had to assume that $\sigma$ was constant everywhere). Substituting our expression above for $E_y$ we have

$$\frac{d^2 E}{dz^2} = i\omega \mu_o \sigma(z) E(z) \quad .$$

We are going to need some boundary conditions. We know that $\mathbf{E} \to 0$ as $z \to \infty$, but we could also set $E(h) = 0$ at some large depth $h$ by considering there to be an infinite (or at least large) conductor at $h$. (If we set $h$ to the core-mantle boundary, this is actually physically realistic.)

Now we need to use the applied magnetic field to set the boundary condition at the surface. Because

$$\frac{\partial \mathbf{B}}{\partial t} = i\omega \mathbf{B}$$

Faraday's Law ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$) becomes

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B}$$

and we can get an expression for $\mathbf{B}$:

$$\mathbf{B} = -\frac{1}{i\omega} \nabla \times \mathbf{E} \quad .$$

If we look at the curl operator

$$\nabla \times \mathbf{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

there is no $E_x$ term, so those derivatives are zero, and in 1D there is no variation in the $x$ and $y$ directions, so the only non-zero term in the curl operator is the $-\partial A_y/\partial z$ in the $x$ position (which is good because $\mathbf{B}$ is in the $x$-direction)

$$\mathbf{B} = \frac{1}{i\omega} \frac{dE_y}{dz}$$

Now we can use the source field at the surface, $B_o$, to get a boundary condition on $E$:

$$E'(0) = i\omega B_o$$

where the prime means differentiation with respect to $z$ ($dE_y/dz$ here). Along with $E(h) = 0$ we have the two boundary conditions we need to solve for $E(z)$ if $\sigma(z)$ is known.

We also have a new expression for admittance $c$ at the surface:

$$c(\omega) = -\frac{E_y}{i\omega B_x} = -\frac{E(0)}{E'(0)} \quad .$$

Note that $B_0$ has disappeared, because both $E$ and $E'$ scale with $B_0$. This is good – we don't want our MT response function to depend on source field strength. Since we can choose any value we like for $B_0$, one useful trick is to choose it such that

$$E'(0) = -1$$

so that $E(0) = c(\omega)$.

As we shall see, we can use these equations to predict $c(\omega)$ given $\sigma(z)$. Weidelt (1972) showed that for a one-dimensional Earth, if you know $c(\omega)$ exactly (that is, error-less measurements for all frequencies), then you can recover $\sigma(z)$ exactly. This is nice to know, but meaningless in practice. Since apparent resistivity $\rho_a(\omega)$ and phase $\phi$ can be computed from $c$, any solution for $c$ applies to the standard magnetotelluric data as well.

Solutions to partial differential equations such as the one we derived for $d^2E/dz^2$ above have been the subject of research for hundreds of years. Bob recommends Zwillinger (1997) for a good summary of approaches. One classical approach is to assume $\sigma(z)$ takes on the form of an equation for which the solutions are known analytically, such as a power law in depth or an exponential in depth. This is the approach that Lahiri and Price (1939) used for the first attempt at estimating conductivity structure from the observed data. One can also use initial value problems, spectral methods, eigenfunction expansions. Parker and Whaler (1981) reduced the problem to a Sturm-Liouville system to come up with an analytical inversion algorithm for 1D c-response data that guarantees a least-squares solution. This is an extremely valuable algorithm, as well as a useful result for conceptualizing inverse methods, but the math is challenging to say the least (let me know if you are interested - I have one of Bob's lectures on the subject). However, the code is freely available and can be used to rigorously establish if an MT or GDS data set is compatible with a 1D solution. Parker and Booker (1996) extended this approach to apparent resistivity and phase, which can be used to check the compatibility of resistivity and phase curves, and even set bounds on missing data.

There is only one approach in use routinely for the forward 1D MT problem − the stack of layers − which we will derive below. However, while the finite difference algorithm is no longer used routinely, it serves as a useful precursor to higher dimensional modeling. To my knowledge the finite element solution was never applied to 1D MT, but again it is much easier to introduce it in the context of 1D modeling, and we shall do so here.

**Finite Difference Schemes**

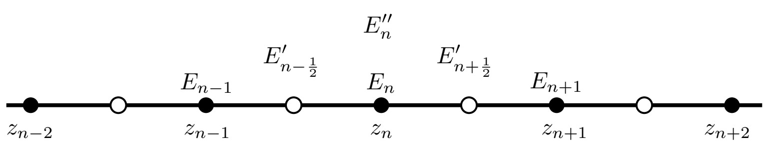
If we assume $\sigma(z)$ is a smooth function, we can sample it at a set of $N$ discrete points on on $z = (0, h)$. For simplicity, let's assume the points are evenly spaced with a constant $\Delta z$ between successive points. We call these points nodes, and call them $z_n$ for $n = 1, 2, ..., N$. We will call $E(z_n)$ simply $E_n$. Now the first derivative $E'(z_n)$ can be approximated by a forward difference

$$E'(z_n) = \frac{E_{n+1} - E_n}{\Delta z} + O(\Delta z)$$

and the second derivative

$$E''(z_n) = \frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} + O(\Delta z^2) \quad .$$

There is an implied staggered grid involved here, since the forward and backward differences are centered between the nodes. Here we are only interested in the second difference, so we don't need to keep track of the staggered grid, but in higher dimensions one might want to.




Now we can approximate our differential equation with a set of linear approximations

$$\frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} - i\omega\mu_0\sigma_n E_n = 0$$

for

$$n = 2, 3, ..., N - 1 \quad .$$

We need something for $n = 1$ and $n = N$, but these are our boundary conditions

$$\frac{E_2 - E_1}{\Delta z} = -1$$

and

$$E_N = 0 \quad .$$

All the above can be set up as a system of $N$ linear equations

$$\mathbf{A}x = b$$

and solved using a matrix inversion. This will be the subject of Homework 3. In principle 2D and 3D finite difference modeling is pretty much the same as this, but the trick with numerical approaches is to make things (a) more accurate and (b) faster. Both are possible even in our toy 1D problem. Note that we have made $\Delta z$ constant here, but this need not be the case.

**Finite Element Schemes**

Again, nobody solves the 1D MT problem using finite elements, but many people solve the 2D and 3D problems that way, so it is useful to introduce them in a simpler dimension.

If we have a partial differential equation

$$u''(x) = f(x)$$

on $x = [0, 1]$, the "weak formulation" is given by

$$\int_0^1 f(x)v(x)dx = \int_0^1 u''(x)v(x)dx$$

for any smooth test function $v(x)$ with boundary conditions $v(0) = v(1) = 0$. Recall how to integrate by parts:

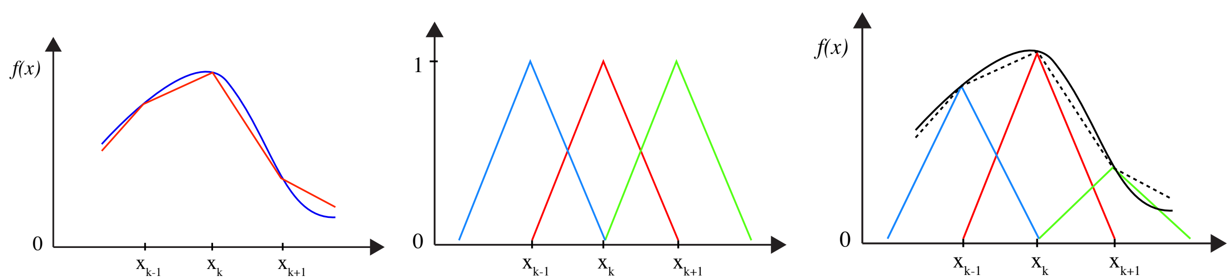
$$\int_a^b v(x)u'(x)dx = v(x)u(x)\big|_a^b - \int_a^b v'(x)u(x)dx$$

So the right hand side of our weak formulation becomes

$$\int_0^1 u''(x)v(x)dx = u(x)'v(x)\big|_0^1 - \int_0^1 u'(x)v'(x)dx \equiv -\phi(u, v)$$

where our boundary conditions on $v(x)$ gets rid of $u(x)'v(x)\big|_0^1$. So now our weak formulation is

$$\int_0^1 f(x)v(x)dx = -\int_0^1 u'(x)v'(x)dx \equiv -\phi(u, v)$$




If we divide our real axis up into $n$ elements, defined by $n+1$ nodes, we can approximate our function $f(x)$ by piecewise linear functions. The basis for these functions are tent functions defined by

$$v_k(x) = \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}} & x \in [x_{k-1}, x_k] \\ \frac{x_{k+1} - x}{x_{k+1} - x_k} & x \in [x_k, x_{k+1}] \\ 0 & \text{otherwise} \end{cases}$$

and our approximate function can be written as

$$f(x) = \sum_{k=1}^{n} f_k v_k(x)$$

where the $f_k$ are the values of the function at the nodes, and similarly our unknown function $u$ is

$$u(x) = \sum_{k=1}^{n} u_k v_k(x)$$

with derivative

$$u'(x) = \sum_{k=1}^{n} u_k v_k'(x)$$

and our weak formulation can be written as

$$-\sum_{k=1}^{n} u_k \phi(v_k, v) = \sum_{k=1}^{n} f_k \int_0^1 v_k v(x) dx$$

Now we introduce our test functions, $v(x) = v_j(x)$ for $j = 1, ..., n$ and our weak formulation is

$$-\sum_{k=1}^{n} u_k \phi(v_k, v_j) = \sum_{k=1}^{n} f_k \int_0^1 v_k v_j dx \quad \text{for } j = 1, ..., n$$

This can be cast as a linear system

$$-\mathbf{Lu} = \mathbf{b}$$

where

$$L_{ij} = \phi(v_i, v_j) = \int_0^1 v_i'(x) v_j'(x) dx$$

$$b_j = \sum_{k=1}^{n} f_k \int_0^1 v_k v_j dx$$

and $\mathbf{u}$ is our vector of unknown $u_k$. Note that because the test/tent functions are zero everywhere when $|j - k| > 1$ or $|j - i| > 1$, the integration needs only be carried out for three cases. Further, because of the simplicity of the test functions, the integrals are analytic. In particular,

$$v_k'(x) = \begin{cases} \frac{1}{x_k - x_{k-1}} & x \in [x_{k-1}, x_k] \\ \frac{-1}{x_{k+1} - x_k} & x \in [x_k, x_{k+1}] \\ 0 & \text{otherwise} \end{cases}$$

How do we apply this to the 1D MT problem? Our diffusion equation

$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

can be written in 1D with $z$ positive down as

$$-E''(z) + i\omega\mu_o\sigma(z)E(z) = 0$$

where $E''$ is the second derivative with respect to $z$. The weak form of this becomes

$$-\int_0^Z E''(z)v(z)dz + \int_0^Z i\omega\mu_o\sigma(z)E(z)v(z)dz = 0$$

where $Z$ is some large depth. We do the integration by parts trick

$$-\big[E'(Z)v(Z) - E'(0)v(0) - \int_0^Z E'(z)v'(z)dz\big] + \int_0^Z i\omega\mu_o\sigma(z)E(z)v(z)dz = 0$$

If we anticipate that the basis for out test function $v$ will go to zero at $z = 0$ and $z = Z$ the we have

$$\int_0^Z \big[E'(z)v'(z) + i\omega\mu_o\sigma(z)E(z)v(z)\big]dz = 0$$

Now we expand $E$ in terms of our basis functions

$$E(z) = \sum_{k=1}^{n} E_k v_k(z)$$

we have

$$\int_0^Z \Big[ \sum_{k=1}^{n} E_k v_k'(z) v'(z) + i\omega\mu_o\sigma(z) \sum_{k=1}^{n} E_k v_k(z) v(z) \Big] dz = 0$$

or

$$\sum_{k=1}^{n} \int_0^Z \big[ v_k'(z) v'(z) + i\omega\mu_o\sigma(z) v_k(z) v(z) \big] dz E_k = 0$$

Now we introduce our test functions $v(z) = v_j(z),\ j = 1, ..., n$

$$\sum_{k=1}^{n} \int_{\Omega_j} \big[ v_k'(z) v_j'(z) + i\omega\mu_o\sigma(z) v_k(z) v_j(z) \big] dz E_k = 0 \quad \text{for } j = 1, ..., n$$

where now we only need to integrate over the support basis of each element $\Omega_j$. Our conductivity, the input to the problem, can be taken as piecewise constant between the nodes. This is a linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$A_{jk} = \int_{z_{k-1}}^{z_{k+1}} \big[ v_k'(z) v_j'(z) + i\omega\mu_o\sigma(z) v_k(z) v_j(z) \big] dz$$

$$x_k = E_k$$

and $\mathbf{b} = 0$. Although mighty tedious, these integrals can be readily calculated. If we fix the element size at $\Delta z$, and make conductivity between $z = [z_{k-1}, z_k]$ constant and given by $\sigma_k$, we have

$$v_k(z) = \begin{cases} \frac{z - z_{k-1}}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{z_{k+1} - z}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad v_k'(z) = \begin{cases} \frac{1}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{-1}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & \text{otherwise} \end{cases}$$

so for $j = k$

$$A_{jk} = \int_{z_{k-1}}^{z_{k+1}} \frac{1}{\Delta z^2} dz + \int_{z_{k-1}}^{z_k} i\omega\mu_o\sigma_k \frac{(z - z_{k-1})^2}{\Delta z^2} dz + \int_{z_k}^{z_{k+1}} i\omega\mu_o\sigma_{k+1} \frac{(z_{k+1} - z)^2}{\Delta z^2} dz \quad j = k$$

which if my integration is correct is equal to

$$A_{jk} = \frac{1}{\Delta z^2} \Big[ z_{k+1} - z_{k-1} + i\omega\mu_o\sigma_k \big[ \frac{1}{3}(z_k^3 - z_{k-1}^3) - z_{k-1} z_k^2 + z_{k-1}^2 z_k \big] + i\omega\mu_o\sigma_{k+1} \big[ \frac{1}{3}(z_{k+1}^3 - z_k^3) + z_{k+1} z_k^2 - z_{k+1}^2 z_k \big] \Big]$$

for $j = k - 1$

$$A_{jk} = \frac{1}{\Delta z^2} \Big[ z_{k-1} - z_k + i\omega\mu_o\sigma_k \big[ \frac{1}{3}(z_{k-1}^3 - z_k^3) + z_{k-1} z_k^2 - z_{k-1}^2 z_k \big] \Big]$$

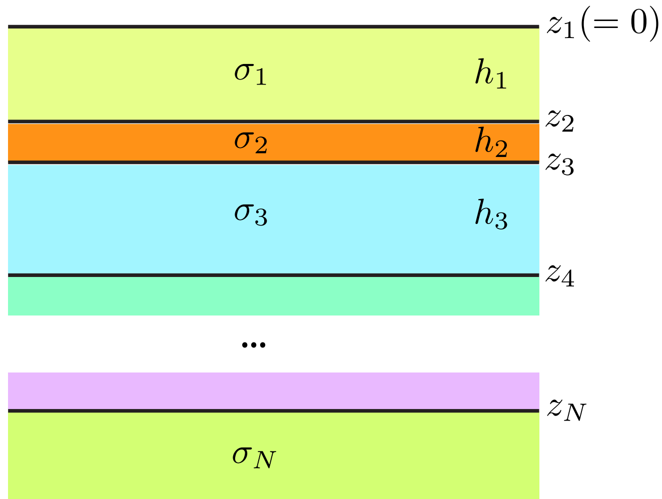
for $j = k + 1$

$$A_{jk} = \frac{1}{\Delta z^2} \Big[ z_k - z_{k+1} + i\omega\mu_o\sigma_{k+1} \big[ \frac{1}{3}(z_k^3 - z_{k+1}^3) - z_{k+1} z_k^2 + z_{k+1}^2 z_k \big] \Big]$$

Of course, we still need some boundary conditions. We could use the same as we did for the finite difference problem.

**Stack of Layers**

All of the analytical solutions and the finite difference solution mentioned above assume that $\sigma$ is smooth in $z$. But, this need not be so. We can consider $\sigma$ to be piece-wise constant, that is a stack of horizontal layers. It turns out, as we shall show below, that there is an exact solution to this geometry. For some geological situations, layering may be more realistic than smooth functions: consider sediments overlying igneous basement, horizontal sills of dolerite, seawater overlying seabed, etc. And, where the geology might produce a smooth variation in conductivity, we can simply increase the number of layers to approximate the model as closely as we please.




In the finite difference and finite element problems above, we could have allowed the node spacing to be variable. Here we will jump right in and consider $N$ layers with varying thicknesses $h_n$ and associated conductivities $\sigma_n$. We count down from the top, so the top of the $n$-th layer is at depth $z_n$ and $h_n = z_{n+1} - z_n$. Note that the thickness of layer $N$ is infinite because our layering stops there. Because $\sigma_n$ is constant in layer $n$, we can solve our PDE exactly. In fact, we have already done so: the solutions are of the form

$$E(z) = C_1 e^{k_n z} + C_2 e^{-k_n z}, \quad z_n \le z \le z_{n+1}$$

where $k_n = (i\omega\mu_0\sigma_n)^{1/2}$ is the complex wavenumber in layer $n$. Now we pull a trick and write this equation as

$$E(z) = A\cosh\big(k_n(z - z_{z+1})\big) + B\sinh\big(k_n(z - z_{n+1})\big), \quad z_n \le z \le z_{n+1}$$

which we can do because $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. The equations are the same – if you work through the algebra you will find that

$$C_1 = \frac{A+B}{2e^{k_n z_{n+1}}}$$

and

$$C_2 = \frac{A-B}{2e^{k_n z_{n+1}}}$$

Differentiating with respect to $z$:

$$E'(z) = k_n[A\sinh\big(k_n(z - z_{z+1})\big) + B\cosh\big(k_n(z - z_{n+1})\big)], \quad z_n \le z \le z_{n+1} \quad .$$

Now we can compute the admittance $c$ at the bottom of the layer by setting $z = z_{n+1}$ (making $z - z_{n+1} = 0$)

$$c_{n+1} = -\frac{E(z_{n+1})}{E'(z_{n+1})} = -\frac{A}{k_j B}$$

($\sinh(0) = 0$ and $\cosh(0) = 1$) and the top of the layer by setting $z = z_n$ (making $z - z_{n+1} = -h_n$)

$$c_n = -\frac{E(z_n)}{E'(z_n)} = -\frac{A\cosh(k_n h_n) - B\sinh(k_n h_n)}{k_j[-A\sinh(k_n h_n) + B\cosh(k_n h_n)]}$$

($\sinh(-x) = -\sinh(x)$ and $\cosh(-x) = \cosh(x)$) which, by dividing everything by $B\sinh(k_n h_n)$ becomes

$$c_n = \frac{-(A/B)\coth(k_n h_n) + 1}{k_n[-A/B + \coth(k_n h_n)]} \quad .$$

Finally, substituting $A/B = -k_j c_{n+1}$ we get

$$c_n = \frac{c_{n+1}\coth(k_n h_n) + 1/k_n}{k_n c_{n+1} + \coth(k_n h_n)} \quad .$$

Because $E_y$ is parallel to the layer interfaces, $E$ and $E'$ are continuous across the interfaces and the admittance at the bottom of layer $n$, $c_{n+1}$, is the same as the admittance at the top of layer $n+1$. So, we have derived an expression for the admittance at the top of each layer in terms of the admittance at the top of the layer immediately beneath, along with the thickness and conductivity of the layer. This is actually quite profound. The MT source fields are propagating downwards from outside Earth, so it should be obvious that the electric and magnetic fields at any given depth depend on the conductivity structure above that depth. Yet, the MT response at any given depth *only depends on structure below that depth.* One consequence is that in a one-dimensional world, the MT response on the seafloor is the same as the response you would observe if the ocean were not there.

If we know the admittance at the top of the last layer, $c_N$, then we can recursively compute $c$ up to $c_1$, which is just the MT response at the surface and what we want. If we are terminating our model with a perfect conductor, then $c_N = 0$, but it is usually more efficient to terminate the stack of layers at a shallower depth by terminating the model with some sensible conductivity. Then, $c_N$ is just the admittance at the top of a uniform half-space, which we have already computed as

$$c = \frac{1}{i\omega}\frac{E}{B} = \frac{1}{i}\sqrt{\frac{1}{\omega\mu_o\sigma_o}} \quad .$$

Since $c$ is simply related to $E/B$, it is easy enough to re-write the recursion for MT impedance. This was done by Schmucker (1970, pp. 61–65). The MT ratio of electric field $E$ to magnetic field $B$ at the surface can be written as

$$\frac{E}{B} = \frac{i\omega}{k_1 G_1} \quad .$$

where $G$ is a dimensionless quantity that is defined for the top of each layer of thickness $h_i$, and can be obtained from the recurrence relation

$$G_i = \frac{k_{i+1}G_{i+1} + k_i\tanh(k_i h_i)}{k_i + k_{i+1}G_{i+1}\tanh(k_i h_i)} \quad ,$$

started by setting $G_N = 1$. The familiar MT apparent resistivity and phase are related to $G$ by

$$\rho_a = \frac{\omega\mu_o}{|k_1 G_1|^2} \qquad \phi = \arctan\left(\frac{\mathrm{Imag}(k_1 G_1)}{\mathrm{Real}(k_1 G_1)}\right) \quad .$$

The similarity to the recurrence formula derived above is obvious.

However, this still isn't quite the same as the MT impedance, $Z$. The recurrence relation you will see most often nowadays is attributed to Ward and Hohmann (1988). This recurrence relation for a stack of layers is

$$Z_j = L_j \frac{Z_{j+1} + L_j \tanh(ik_j h_j)}{L_j + Z_{j+1} \tanh(ik_j h_j)}$$

where we have defined an intrinsic layer impedance $L_j$ given by

$$L_j = \frac{\omega \mu_o}{k_j}$$

and the wavenumber is now $k_j = \sqrt{-i\omega\mu_o\sigma_j}$. Note the minus sign on $i$. The starting value, $Z_N$, is just the intrinsic layer impedance of the terminating half-space. The relationship between $G$ and $Z$ is messy:

$$G_j = \sqrt{\frac{-\omega\mu_o}{i\sigma_j}} \frac{1}{Z_j}$$

It is sometimes useful to compute the EM fields within the 1D model. Again, this is particularly handy for marine MT, since although the MT response doesn't depend on the seawater, the fields do. Schmucker's formulation allows one to write the ratio of the electric field at the bottom of any layer $E_{j+1}$ to the electric field at the top of that layer $E_j$ as

$$\frac{E_{j+1}}{E_j} = \cosh(k_j h_j) - G_j \sinh(k_j h_j)$$

and the magnetic field ratio as

$$\frac{B_{j+1}}{B_j} = \frac{k_{j+1} G_{j+1}}{k_j G_j} \big(\cosh(k_j h_j) - G_j \sinh(k_j h_j)\big) \quad .$$

The equivalent expressions in $Z$ are

$$\frac{E_{j+1}}{E_j} = \cosh(ik_j h_j) - \frac{L_j}{Z_j} \sinh(ik_j h_j)$$

and similarly

$$\frac{H_{j+1}}{H_j} = \frac{B_{j+1}}{B_j} = \frac{Z_j}{Z_{j+1}} \cosh(ik_j h_j) - \frac{L_j}{Z_{j+1}} \sinh(ik_j h_j)$$

You can see that Schmucker's expressions are slightly more elegant.

These formulas can be used to create figures such as the one in the last lecture, showing the magnitude of the fields inside a layered model. Current density is just obtained from electric fields and Ohm's Law. By setting magnetic field at the surface equal to 1, everything else can be computed.

**Weidelt's Transformation**

Analytical solutions exists for a layered, spherically symmetric Earth conductivity model (Srivastava, 1966; see also Parkinson, 1983, p. 313). However, given the greater simplicity of the layered magnetotelluric solution, and the various inverse solutions available for the flat-earth approximation, it is often desirable to analyze global data using flat-earth solutions. Weidelt (1972) provides a compact transformation between the two geometries. If we interpret a GDS c-response $c(\omega)$ obtained for spherical harmonic degree $n$ using a layered model to get a conductivity–depth relationship $\bar{\sigma}(z)$, this can be transformed into a conductivity profile in a spherical Earth $\sigma(r)$ of radius $R$ using

$$\sigma(r) = f^{-4}(r/R)\bar{\sigma}\left(R\frac{(r/R)^{-n} - (r/R)^{n+1}}{(2n+1)f(r/R)}\right)$$

where

$$f(r/R) = \frac{(n+1)(r/R)^{-n} + n(r/R)^{n+1}}{2n+1}$$

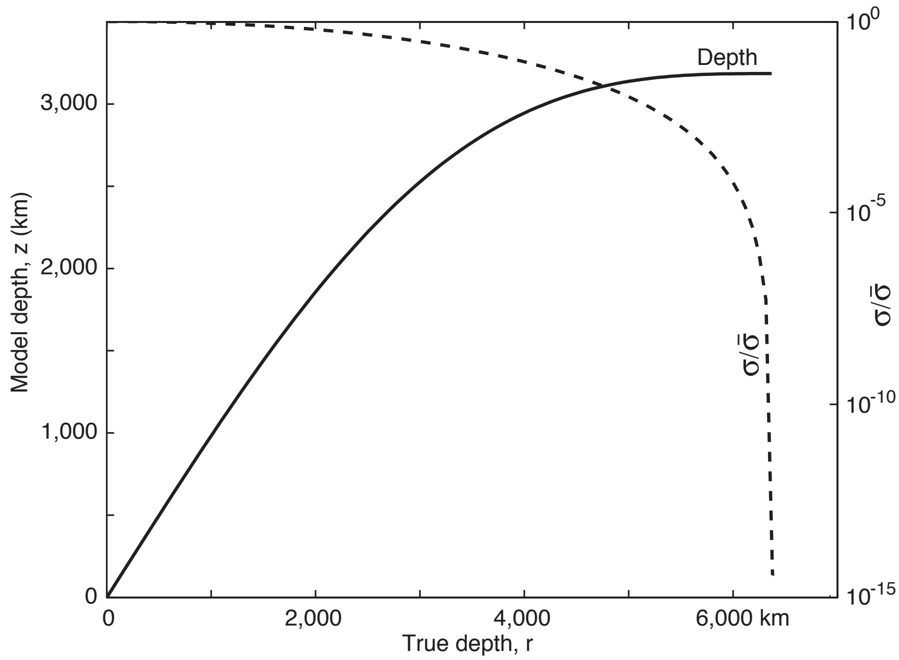
which for $n = 1$ reduces to

$$\sigma(r) = f^{-4}(r/R)\bar{\sigma}\left(R\frac{(R/r) - (r/R)^2}{3f(r/R)}\right)$$

and

$$f(r/R) = \frac{2R/r + (r/R)^2}{3} \quad .$$

We see from the figure that the correction is only significant below depths of about 2,000 km or, for typical mantle resistivities, periods of about one day or longer.





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