Chapter 2: Classical Electrodynamics in Geomagnetism

1. Overview

As we turn to the geomagnetic part of this course we will apply many of the same mathematical tools as are used in studying Earth's gravitational potential. However, instead of Newton's law, the fundamental physics are described by the equations of classical electrodynamics. This chapter starts with Helmholtz's theorem, Maxwell's equations, and Ohm's law in a moving medium, and motivates the equations that are used in static geomagnetic field modeling. Much of the material covered here is to be found in Chapter 2 of *Foundations*; a less advanced treatment is given in Chapters 2 and 4 of Blakely's book on Potential Theory. Those needing a refresher on magnetic and electric fields could also consult the background materials listed on the class website. For the basic physics of magnetic fields, Chapters 5-7, of Introduction to Electrodynamics by David J. Griffiths, (4th Edition, Cambridge University Press, 2017). There are several excellent video tutorials on div, curl by Grant Sanderson on his Youtube channel 3blue1brown, (https://www.youtube.com/watch?v=rB83DpBJQsE&feature=youtu.be), and on Laplace's equation amongst other things in Khan Academy e.g., Laplacian intuition, Harmonic functions,... https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/laplacian/v/laplacian-intuition

1:1 Helmholtz's Theorem and Maxwell's Equations

The universe of classical electrodynamics begins with a vacuum containing matter solely in the form of electric charges, possibly in motion, and electric and magnetic fields. We can detect the presence of these fields by the forces they exert on a moving point charge q. If the charge q is located at position **r** at time t and moves with velocity **v** relative to an inertial frame, then

$$\mathbf{f} = q[\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)]. \tag{3}$$

This expression allows us in principle to measure the electric and magnetic fields using a moving charge as a detector in an inertial reference frame.

Maxwell's equations provide the curl and divergence of the electric fields and magnetic fields in terms of other things. The reason this is useful is that Helmholtz's Theorem tells us that if we know the curl and the divergence of a vector field, we can explicitly calculate the field itself, and furthermore, the curl and the divergence represent sources for the field, essentially creating the field.

Here is Helmholtz's theorem. A vector field **F** in R^3 which is continuously differentiable (except for jump discontinuities across certain surfaces) is uniquely determined by its divergence, its curl and jump discontinuities if it approaches 0 at infinity. The field can be written as the sum of two parts

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A} \tag{4}$$

where V is called a scalar potential and **A** a vector potential. These two potentials can be explicitly computed from the following two integrals:

$$\text{Scalar potential} \qquad V(\mathbf{r}) = \frac{1}{4\pi} \int d^3\mathbf{s}\, \frac{\nabla \cdot \mathbf{F}(\mathbf{s})}{|\mathbf{r} - \mathbf{s}|} \tag{5}$$

$$\text{Vector potential} \qquad \mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int d^3\mathbf{s}\, \frac{\nabla \times \mathbf{F}(\mathbf{s})}{|\mathbf{r} - \mathbf{s}|}. \tag{6}$$

What these two equations state is that the field **F** is generated by two kinds of sources: one is the divergence of **F**, the other its curl. The actual specifications of the divergence and curl depend on basic laws of physics.

For example, recall that in classical gravity, the gravitational potential V is generated by matter density ρ and we see this through:

$$V(\mathbf{r}) = -G \int d^3\mathbf{s} \frac{\rho(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}. \tag{6a}$$

But this is just equation (5), because of Poisson's equation, $\nabla^2 V = 4\pi G\rho$ and the fact that here $\mathbf{F} = \mathbf{g} = -\nabla V$. Helmholtz tells us that if there is no divergence or curl anywhere in space, then the vector field $\mathbf{F}$ must vanish, again confirming that $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ are the sources of the field.

Now let us turn to Maxwell's equations in a vacuum. We are (for the time being at least) operating in a universe that comprises an infinite vacuum containing electrical charges, represented by a local density ρ, which may be moving, and hence generating electric current, represented by a local current density $\mathbf{J}$. Here are Maxwell's equations:

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \qquad \text{Faraday's Law} \tag{7}$$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \text{Coulomb's Law or Gauss' } - \text{ electric fields} \tag{8}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \partial_t \mathbf{E}) \qquad \text{Ampere's Law} \tag{9}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Gauss's law} - \text{magnetism} \tag{10}$$

where ρ is charge density (in SI units coulombs/m^3), $\mathbf{J}$ is current density (amperes/m^2), μ_0 is permeability of vacuum ($4\pi \times 10^{-7}$ henries/m), ϵ_0 is capacitivity of vacuum ($10^7/4\pi c^2$ farads/m); $\mathbf{B}$ is in teslas, and $\mathbf{E}$ is in volts/m. The quantities μ_0 and ϵ_0 used to be considered exactly defined constants of the SI measurement conventions but μ_0 has changed very slightly under the new SI definition of the kg ; the quantity c is the velocity of light in a vacuum, which is also an exact number in SI. Notice the somewhat unconventional streamlined notation for time derivative: $\partial/\partial t = \partial_t$.

Viewed from the perspective of Helmholtz's Theorem we see that the Maxwell equations (7) and (8) tells how the electric field is generated (by changing magnetic fields — Faraday's law) or by the presence of electric charges (Coulomb's law); and equation (9) says we can generate a magnetic field by a combination of moving charges (Ampere's law) and by changing the electric field in time (Maxwell's discovery, which does not have the word law associated with it). Equation (10) says there are no isolated magnetic charges, that is, no magnetic monopoles.

1:2 The Static Case for Geomagnetic Field Modeling

We can make use of Helmholtz's theorem, Maxwell's equations and the appropriate constitutive relations in describing any electromagnetic problem in geophysics. For some purposes we can neglect time variation in geomagnetic processes and imagine a system of stationary charges and steady current flows. Many geomagnetic phenomena take place over long time scales and certainly for the purposes of modeling the present internal geomagnetic field this seems like a reasonable approximation. It would allow us to set all time derivatives in Maxwell"s equations to zero. Thus in (7), the first of the Maxwell equations, we would set $\partial_t \mathbf{B} = 0$; then the curl of the electric field vanishes. Making use of this in (4) and (6) we find that the electric field may be written as the gradient of a scalar ϕ (the electric potential). Thus

$$\mathbf{E} = -\nabla\phi. \tag{11}$$

Putting this together with (8) we get $\nabla^2\phi = -\rho/\epsilon_0$ (Poisson's equation again, but notice the sign!) and from (5) we then get

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{s} \frac{\rho(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}. \tag{12}$$

For the magnetic field it follows from $\nabla \cdot \mathbf{B} = 0$ and Helmholtz theorem (5),(4) that we can always write $\mathbf{B} = \nabla \times \mathbf{A}$. (Physicists who worry about these things may like to note that we are implicitly invoking the Coulomb gauge under which $\nabla \cdot \mathbf{A} = \mathbf{0}$.) The vector field $\mathbf{A}$ is known as the magnetic vector potential. Now if we specialize to the static case with $\partial_t \mathbf{E} = 0$, we find from (9), and (6) that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3\mathbf{s} \frac{\mathbf{J}(\mathbf{s})}{|\mathbf{r} - \mathbf{s}|}. \tag{13}$$

Again from (9) we have that $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ and taking the divergence yields

$$\nabla \cdot \mathbf{J} = 0. \tag{14}$$

1:3 Constitutive Relations

Maxwell's equations as written in (7)-(10) apply to a vacuum. When we need equations describing the behavior of electromagnetic fields inside a material we require some mechanism for spatial averaging of the charge and current distributions due to the atoms making up the material. This question is considered in most courses on electromagnetism, and in *Foundations*. These lead us to a form of Maxwell's equations capable of describing field within various materials

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{16}$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho \; - \; \nabla \cdot \mathbf{P} \tag{17}$$

$$\nabla \times \mathbf{B}/\mu_0 = (\mathbf{J} \; + \; \partial_t \mathbf{P} \; + \; \nabla \times \mathbf{M} \; + \; \epsilon_0 \partial_t \mathbf{E}) \tag{18}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{19}$$

where $\mathbf{P}$ and $\mathbf{M}$ are the electric polarization per unit volume and the magnetization, or magnetic polarization per unit volume of the material. Physically what happens is that the presence of an electric field (for simplicity) polarizes the material, causing charge separation. This introduces a large number of tiny electric dipoles into the medium, quantified by the term $\mathbf{P}$ – this is simply the density of electric dipole moment present in the material. If the dipole density were precisely constant, there would be no effect on $\mathbf{E}$, because the dipole fields would cancel on average (except at the ends of the specimen, where charges would accumulate). But variations in the dipole density do cause electric fields – this is seen in the fact that the term in the modified equations is $\nabla \cdot \mathbf{P}$. The magnetic effect is similar, but more complicated because electrons' intrinsic magnetic moments and their motions within atoms cause magnetic fields.

The solution of Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$ in a material thus requires knowledge of $\mathbf{J}$, $\mathbf{P}$ and $\mathbf{M}$ and these in turn depend on the way the material responds to the fields. These are called the *constitutive relations* for the material and are often determined by $\mathbf{E}$ and $\mathbf{B}$ themselves. They are not fundamental like Maxwell's equations, but are the result of empirical observations and experiments done on different materials. The simplest possible behavior is linear. For example for many materials over a wide range of field values, we find

$$\mathbf{J} = \sigma \mathbf{E} \tag{20}$$

$$\mathbf{P} = \epsilon_0 \chi_E \mathbf{E} \tag{21}$$

$$\mathbf{M} = \beta \mathbf{B}/\mu_0 \tag{22}$$

where σ, χ_E and β are constants. Of course, we recognize σ as the electrical conductivity (so that (20) is a statement of Ohm's law), χ_E is the electrical susceptibility, and β a kind of magnetic susceptibility.

We can simplify Maxwell's equations by defining new fields **H** and **D**,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{23}$$

$$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}. \tag{24}$$

D is called the electric *displacement vector*. **H** has traditionally been called the magnetic field vector, while **B** was called the flux intensity or magnetic induction. The two are often confused. In view of its primary place in the theory we shall call **B** the *magnetic field* vector and by analogy with **D**, **H** will be the *magnetic displacement* vector. You should be aware these names are not yet standard, but they ought to be. With these definitions in place we achieve a form of Maxwell's equations for the second and third relations:

$$\nabla \cdot \mathbf{D} = \rho \tag{25}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \partial_t \mathbf{D}. \tag{26}$$

The last term is called the *displacement current* and is a way of generating magnetic fields without any charges having to move. This is how electromagnetic waves propagate in a vacuum. Notice that the first and last of Maxwell's equations remain unchanged from their vacuum forms, (7) and (10). In fact we rarely use **D** in geomagnetism; one reason is that most Earth materials are not highly polarizable, and another is that we almost always drop the term involving **D** in (26) as we shall see next.

1:4 Application to the Geomagnetic Field

A reasonable approximation in geophysical problems is to neglect the displacement current $\partial_t \mathbf{D}$ in (26). This can be shown by a crude dimensional analysis as follows. (For more details see *Foundations,* Section 2.4) Take the time derivative of (26), and insert Ohm's law (20); for simplicity assume χ_E and β in (21)-(22) are negligible; then (26) becomes:

$$\nabla \times \partial_t \mathbf{B}/\mu_0 = \sigma \partial_t \mathbf{E} + \epsilon_0 \partial_t^2 \mathbf{E}. \tag{27}$$

Now we use $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ (16) and rearrange slightly

$$\nabla \times \nabla \times \mathbf{E} + \mu_0 \sigma \partial_t \mathbf{E} + \mu_0 \epsilon_0 \partial_t^2 \mathbf{E} = 0. \tag{28}$$

We would like to estimate the approximate size of each of the three terms in (28). If we assume length scales of variation are L or larger and time scales T or larger, very roughly we can replace space derivatives by $1/L$ and time derivatives by $1/T$; then

$$0 = [1 + \mu_0 \sigma (L^2/T) + \mu_0 \epsilon_0 (L/T)^2] |\mathbf{E}|. \tag{29}$$

Because $\mu_0 \epsilon_0 = 1/c^2$, where c is the speed of light, the last term represents the ratio of typical speeds in the system over c squared. In geomagnetism scales are typically many thousands of kilometers, and time scales can be as low as minutes, but may be years: even for $L = 10^3$ km and $T = 10$ s, the last term in (29) is 10^{-5}. The term with conductivity is much larger than this in the interior, say $\sigma \approx 10^{-3}$S/m, then the second term is roughly $4\pi \times 10^{-7} \times 10^{-3} \times 10^{12}/10$ or about 120. So displacement current is unimportant and the balance is between the first two terms. The four equations (7)-(10), or the set valid within material (7), (10), (25), (26), in which the displacement current is neglected ($\partial_t \mathbf{E}$ or $\partial_t \mathbf{D}$) are sometimes referred to as the *pre-Maxwell equations*.

But in the atmosphere σ is so small, the second term is small too. When this happens we see the simple-minded analysis breaks down and the discover that the size of $\nabla \times \nabla \times \mathbf{E}$ cannot be $|\mathbf{E}|/L^2$ – terms in the spatial derivative cancel among themselves and the corresponding term in (27) vanishes by itself.

1:4.1 When is Laplace's equation applicable?

As we already saw the magnetic field can **always** be written as the curl of a vector potential (because of Helmholtz's Theorem, (4)-(6), and $\nabla \cdot \mathbf{B} = 0$). In **certain circumstances** there is an alternative representation in terms of a scalar potential for **B**. In our application to the geomagnetic field we will usually make the approximation that Earth's atmosphere is an insulator with no electrical currents. Is this reasonable? Yes, actually $\sigma \approx 10^{-13}$ S/m close to the ground so $\mathbf{J} = 0$ does seem like a reasonable approximation. The atmosphere is also only very slightly polarizable magnetically so we can set $\mathbf{M} = 0$. Thus within the atmospheric cavity we find the essential content of (26) is

$$\nabla \times \mathbf{B} = 0. \tag{30}$$

Recalling that $\nabla \times \nabla = 0$, this tells us that **B** can be written as the gradient of a scalar because when

$$\mathbf{B} = -\nabla \phi \tag{31}$$

(30) is automatically satisfied. Since **B** is also solenoidal (divergence free) from (19), the scalar potential ϕ is harmonic:

$$\nabla^2 \phi = 0$$

This is Laplace's equation and explains why in potential theory we can use so much of the machinery used in gravity analyses in geomagnetism. A primary example is spherical harmonics.