SIOG 231: GEOMAGNETISM AND ELECTROMAGNETISM

Chapter 3: Lorentz Force, Diffusion Equations, Electromagnetic Sounding

1. The Electromagnetic Diffusion Equation

Here we are going to show that in a uniform conductor, time varying electric and magnetic field decay in a way described by the diffusion equation. We are going to need a couple of vector calculus identities introduced in the last lecture, so here they are again:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{I1}$$

$$\nabla \times (\nabla s) = 0 \tag{I2}$$

$$\nabla(st) = s\nabla t + t\nabla s \tag{I3}$$

$$\nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s\nabla \cdot \mathbf{A} \tag{I4}$$

$$\nabla \times (s\mathbf{A}) = s\nabla \times \mathbf{A} + \nabla s \times \mathbf{A} \tag{I5}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A} \tag{I6}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{I7}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{I8}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{I9}$$

We introduced the use of ∇ in Lecture 2, but it will be worth reminding ourselves of the definition of the cross product:

$$\mathbf{A} \times \mathbf{B} = [A_yB_z - A_zB_y \ , \ A_zB_x - A_xB_z \ , \ A_xB_y - A_yB_x]$$

Recall from Lecture 2 that if we are not dealing with magnetizable or polarizable media, and we can neglect the displacement term at the frequencies and conductivities we generally deal with, we are left with what we call the pre-Maxwell's equations:

Faraday's Law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Coulomb's (Gauss') Law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$$

Ampère's Law:

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}$$

Gauss' Law of magnetism (no monopoles allowed!):

$$\nabla \cdot \mathbf{B} = 0$$

The only constitutive equation we need is Ohm's Law:

$$\mathbf{J} = \sigma \mathbf{E}$$

Electrical conductivity, σ, has units of S/m, where the S is the siemen. Broken down, conductivity has units of $A^2.s^3/(kg.m^3)$, or amps per volt per m. The reciprocal of conductivity is electrical resistivity, ρ (not to be confused with charge density above or density in gravity), which has units of volts per amp times meters, or Ωm.

If we substitute Ohm's Law into Ampère's Law we get :

$$\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E}$$

We take the curl of this

$$\nabla \times \nabla \times \mathbf{B} = \mu_o \sigma \nabla \times \mathbf{E}$$

and use Faraday's Law to substitute for $\nabla \times \mathbf{E}$ to get

$$\nabla \times \nabla \times \mathbf{B} = -\mu_o \sigma \frac{\partial \mathbf{B}}{\partial t} \quad .$$

Now we need the vector identity number I9 ($\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$) to get

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\mu_o \sigma \frac{\partial \mathbf{B}}{\partial t} \quad .$$

But, the no-monopoles law (Gauss' Law for magnetism) says that $\nabla \cdot \mathbf{B} = 0$, so ...

$$\nabla^2 \mathbf{B} = \mu_o \sigma \frac{\partial \mathbf{B}}{\partial t} \quad .$$

Similarly, we can take the curl of Faraday's Law and substitute Ampère's and Ohm's Laws to get

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \quad \rightarrow \quad \nabla \times \nabla \times \mathbf{E} = -\mu_o \sigma \frac{\partial \mathbf{E}}{\partial t} \quad .$$

To pull the same vector identity trick we need to take the divergence of both sides of Ampère's Law and use vector identity I1 ($\nabla \cdot (\nabla \times \mathbf{A}) = 0$) to get

$$\nabla \cdot \mathbf{J} = 0 \quad .$$

Up until now everything is general, but we need to assume that conductivity is constant σ_o in order to go from the divergence of current density to divergence of electric field:

$$\nabla \cdot \mathbf{E} = 0$$

giving

$$\nabla^2 \mathbf{E} = \mu_o \sigma_o \frac{\partial \mathbf{E}}{\partial t} \quad .$$

Although this last result only holds in regions of constant conductivity, this is not a big limitation, since regions with varying conductivity can be divided into subregions of constant conductivity with boundary conditions applied between them.

These two equations in **B** and **E**

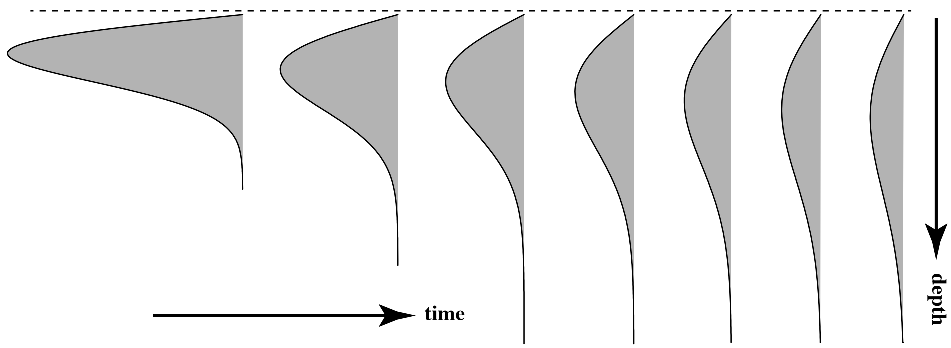
$$\nabla^2 \mathbf{E} = \mu_o \sigma_o \frac{\partial \mathbf{E}}{\partial t} \qquad\qquad \nabla^2 \mathbf{B} = \mu_o \sigma_o \frac{\partial \mathbf{B}}{\partial t}$$

are **diffusion equations**. (Wave propagation went away with the displacement current.) That is, we could write them in the same way as the heat equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$$

where $\eta = 1/(\mu_o \sigma_o)$ is the magnetic diffusivity, which depends on conductivity. Magnetic diffusivity will become important when we look at the physics of Earth's core, but it also describes electromagnetic induction since the electric field works in exactly the same way.

Consider the following figure (courtesy Bob Parker):



It shows the effect of a sudden magnetic field pulse propagating into a uniformly conductive Earth at seven equally spaced time steps $t = 1 - 7$ from left to right (the pulse originates at time zero). Depth increases downwards. The amplitude decays as t^{-1}, and the depth to the peak amplitude grows as $t^{1/2}$.

Because electromagnetic induction methods are governed by the diffusion equation, they can only detect conductivity variations that are of comparable size to the depth of burial. The idea that EM induction is a bit like heat flow might seem a bit grim, but it is not so bad. Unlike heat flow, we can control the geometry of the source field, and by choice of frequency we can control the decay rate. We will consider the latter now.

2. Skin Depth

Now it is time to consider a single frequency $\omega = 2\pi f$, so

$$\mathbf{B} = \mathbf{B}(t) = \mathbf{B_o} e^{i\omega t} \quad \text{and so} \quad \frac{\partial \mathbf{B}}{\partial t} = i\omega \mathbf{B}$$

and the same for $\mathbf{E}$, so our diffusion equations become

$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma_o\mathbf{E} \qquad\qquad \nabla^2 \mathbf{B} = i\omega\mu_o\sigma_o\mathbf{B} \quad .$$

Let us consider an external source of magnetic field at Earth's surface, $\mathbf{B}$, which is purely horizontal and uniform with frequency ω. Remember, we have had to assume that conductivity σ is constant, so in this case the earth is what we call a half-space. (We don't need to consider a vertical source field because, since no significant electric current can flow in the atmosphere, we are essentially above a uniform current sheet if there are no lateral variations in conductivity, and a vertical field cannot exist.) So we have at $z = 0$

$$\mathbf{B} = B_o e^{iwt}$$

(usually z is considered to be positive downwards). Going back to our definition of the Laplacian, we have that

$$\nabla^2 \mathbf{B} = \frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2}$$

but for a uniform horizontal field the first two derivatives in x and y must be zero, and **B** can only vary with z, so our diffusion equation becomes

$$\frac{d^2B}{dz^2} = i\omega\mu_o\sigma_o B(z) \quad .$$

The next step will be simpler if we define a **complex wavenumber** $k^2 = i\omega\mu_o\sigma_o$ so that

$$\frac{d^2B}{dz^2} = k^2 B(z) \quad .$$

This is a second order linear ODE with solutions of the form

$$B(z) = c_1 e^{kz} + c_2 e^{-kz} \quad .$$

The first term grows with depth so c_1 must equal 0, and considering $z = 0$ we can infer that $c_2 = B_o e^{i\omega t}$. We can write k as

$$k = \sqrt{i\omega\mu_o\sigma_o} = (1+i)\sqrt{\frac{\omega\mu_o\sigma_o}{2}} = \frac{(1+i)}{z_o}$$

where

$$z_o = \sqrt{\frac{2}{\omega\mu_o\sigma_o}}$$

is called the **skin depth**. We finally have

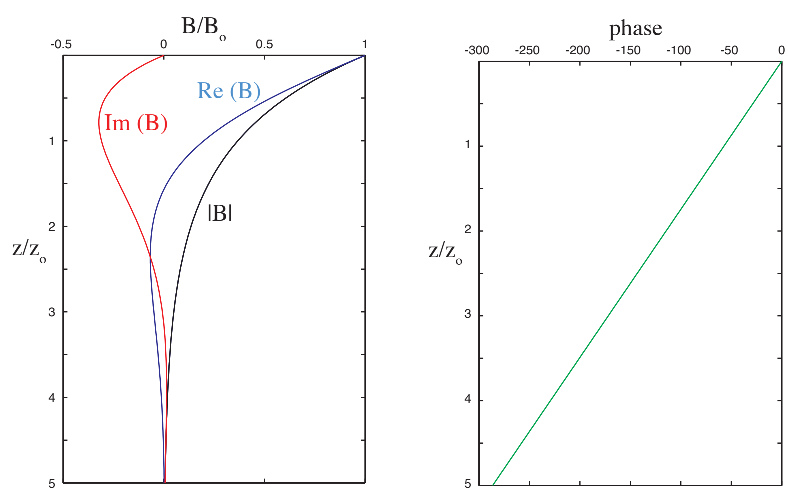
$$B(z) \quad = \quad B_o e^{i\omega t} e^{-(1+i)z/z_o} \quad = \quad B_o e^{-z/z_o}\big(\cos(\omega t - \frac{z}{z_o}) + i\sin(\omega t - \frac{z}{z_o})\big) \quad .$$

So $B(z)$ falls off exponentially with a characteristic distance of z_o, and with a phase shift of one radian or 57° every multiple of z_o.

The skin depth of EM energy determines how deeply it will penetrate the rocks, since at every skin depth the field has decayed to $1/e$ (≈37%) times its previous value. An easy way to remember the skin depth relationship is to convert conductivity to resistivity ρ and angular frequency to period in seconds T:

$$\text{skin depth} = \sqrt{\frac{2\rho}{\omega\mu_o}} = \sqrt{\frac{2\rho}{2\pi f\mu_o}} \approx 500\sqrt{\rho T} \quad \text{metres.}$$

The following plots show the amplitude and phase of a magnetic field in a uniform conductor over a depth of five skin depths. Assuming that the magnetic field at the surface is purely real (or in-phase), then we can also see how the imaginary component (out-of-phase) builds and then decays.

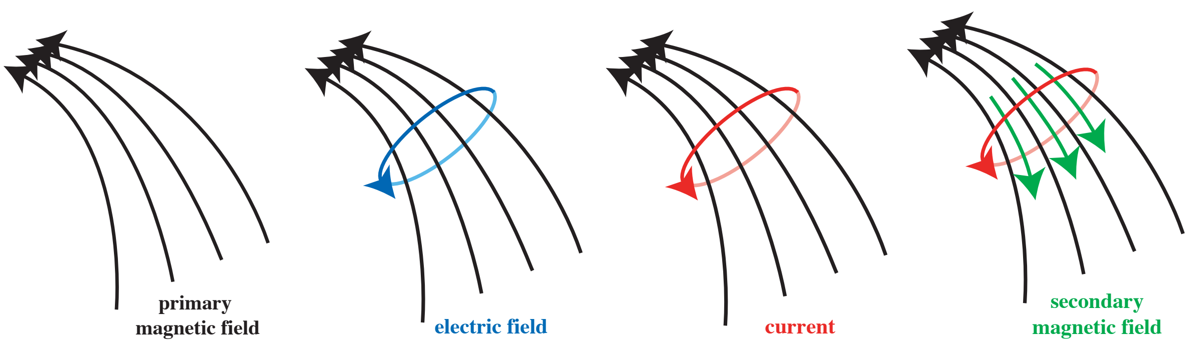


The skin depth is important in all types of EM prospecting, because it determines the depth of penetration that can be expected at a given frequency. The higher the frequency and the higher the conductivity, the shallower the induced currents. The lower the frequency and higher the resistivity, the deeper. Substituting a few numbers into the equation shows that skin depths cover all geophysically useful length scales from less than a meter for conductive rocks and kilohertz frequencies to thousands of kilometers in mantle rocks and periods of days.

		Period						
material	σ, S/m	1 year	1 month	1 day	1 hour	1 min	1 s	1 ms
outer core	10^5	8.9 km	2.6 km	470 m	95 m	12 m	1.6 m	50 mm
lower mantle	10	890 km	260 km	47 km	9.5 km	1.2 km	160 m	5 m
seawater, basaltic lava	3	1.6 Mm	470 km	85 km	17 km	2.3 km	290 m	9 m
marine sediments	1	2.8 Mm	820 km	150 km	30 km	3.9 km	500 m	16 m
cont. sediments	0.1	8.9 Mm	2.6 Mm	470 km	95 km	12 km	1.6 km	50 m
warm upper mantle	10^{-3}	89 Mm	26 Mm	4.7 Mm	950 km	120 km	16 km	500 m
cool mantle, granite	10^{-5}	890 Mm	260 Mm	47 Mm	9.5 Mm	1.2 Mm	160 km	5 km

Thus if one where prospecting for ore bodies in a basement which was buried beneath 100 m of 10 Ωm sediment, no matter what sort of EM method was being employed, a frequency lower than 250 Hz (which has a skin depth in the sediment of 100 m) would have to be used. Operating at 1 kHz would fail to discover anything in the basement rocks. On the other hand, using energy at a period of 1,000 seconds on a granite batholith of resistivity 1,000 Ωm will allow energy to penetrate more than 500 km into Earth (although the granite would give way to more conductive mantle rocks long before this distance was reached).

What is physically going on to attenuate the magnetic field?



We started with a time varying primary magnetic field a Earth's surface. Faraday's Law tells us that the time variations in the magnetic field will generate an electric field. Inside the conductive Earth, Ohm's Law says that the electric field will drive current flow. Ampère's Law says that the electric current will generate a magnetic field. We call this the *secondary magnetic field* and the consequence of the minus sign in Faraday's Law is that this secondary magnetic field opposes the primary field, weakening it. (This can be referred to as Lenz's Law, but Lenz's Law is just a statement about the sign of Faraday's Law.)

The magnetotelluric method uses measurements of both the electric and magnetic fields at Earth's surface, and to understand this better we need to do some more math.

3. The Magnetotelluric Method

For the magnetotelluric method, natural variations in Earth's magnetic field are used as the source. These variations are a consequence of the interaction of the solar wind with Earth's internal magnetic field (0.0001 Hz to 10 Hz), along with higher frequency energy excited by lightning in the atmosphere (10–100 Hz). At even higher frequencies man-made radio signals can be used. We will look at these phenomena in more detail in Chapter 10. In every case the magnetic source field can be considered horizontal, just as we have above in deriving the skin depth formula. Induced electromagnetic fields are measured by grounded electrodes making an electric field measurement. We will consider instrumentation in the next chapter.

Recall that substituting Ohm's Law into Ampere's Law we got

$$\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E} \quad .$$

From the definition of the curl given in Lecture 1:

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

we can see that if the horizontal magnetic field as derived above varies only in the x direction and is uniform in the y direction, the only component of the curl operator that isn't zero is $\partial B_x / \partial z$, which appears in the y-component of the curl, so

$$E_y = \frac{1}{\mu_o \sigma_o} \frac{dB_x}{dz} \quad .$$

We showed above that

$$B(z) = B_o e^{i\omega t - (1+i)z/z_o}$$

so we can calculate the derivative of B_x as

$$\frac{dB_x}{dz} = -\frac{1+i}{z_o} B_x$$

giving

$$E_y = -\frac{1+i}{\mu_o \sigma_o z_o} B_x = -\frac{k}{\mu_o \sigma_o} B_x \quad .$$

Similarly,

$$E_x = \frac{1}{\mu_o \sigma_o} \frac{-dB_y}{dz} = \frac{1+i}{\mu_o \sigma_o z_o} B_y = \frac{k}{\mu_o \sigma_o} B_y$$

noting that the x-component of the curl is $-\partial B_y / \partial z$. This equation is valid for any depth z, but in practice we are only interested in the surface where $z = 0$ and $B_x = B_o e^{iwt}$. We can take the ratio of the electric to magnetic field at any particular frequency to obtain an expression for half-space resistivity:

$$\left| \frac{E_y}{B_x} \right|^2 = \left(\frac{k}{\mu_o \sigma_o} \right)^2 = \frac{\omega \mu_o \sigma_o}{(\mu_o \sigma_o)^2} = \frac{\omega}{\mu_o \sigma_o}$$

$$\rho = \frac{\mu_o}{\omega} \left| \frac{E_y}{B_x} \right|^2 \quad .$$

This is the MT equation made famous in Cagniard's 1953 paper (or nearly so – Cagniard used H rather than B). The phase between E_x and B_y is given by the $(1+i)$ term, which is 45°. For E_y and B_x there is also a 180° phase difference associated with the minus sign, which means the phase differnce is -135° (but only for a half-space).

Even though the earth is not a half-space, we make the resistivity calculation and call it *apparent resistivity*, the resistivity the earth appears to have based on a single measurement at a single frequency. We can in any case compute the phase between E and B, which may well not be 45°. Because the earth may in fact be 3 dimensional (or if 2D may not line up with the measurement directions x and y), there may be cross coupling between E_x and B_x (and the y components), so in practice a 2×2 *impedance* matrix is calculated:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \end{pmatrix} \quad .$$

Note that many practitioners use H, not B, for the impedance matrix, so the apparent resistivity and phase formulas now become

$$\rho_a = \frac{1}{\omega\mu_o}\left|\frac{E}{H}\right|^2 \qquad \phi = \tan^{-1}\left(\frac{E}{H}\right) \quad .$$

Remember, all these values are functions of frequency, and in practice the elements of **Z** are obtained by taking cross-spectra between the magnetic and electric field measurements. We will learn more about this later.

Another representation of the transfer function between E and B is called the c-response (Weidelt, 1972) (also called admittance by Parker, 1994), given by

$$c = -\frac{1}{i\omega}\frac{E_y}{B_x} = \frac{1}{i\omega}\frac{E_x}{B_y} \quad .$$

Note that c is complex, and for causal systems the real part of c is positive, and the imaginary part is always negative. It can be seen that

$$\rho = \omega\mu_o|c|^2$$

and of course the phase is the same as for the MT response. If you do a units analysis, with E being V/m and B being V.s/m^2 (and of course $1/\omega$ being seconds), you see that c has units of meters, and so is sometimes called the inductive scale length.

4. Lorentz Force.

If we recall the Lorentz force on a moving charged particle:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

we can infer that the electromotive force (EMF) generated by a conductor moving with velocity **v** in a magnetic field acts as a sort of electric field

$$\mathbf{E}' = \mathbf{v} \times \mathbf{B}$$

which allows us to write the time derivative of the magnetic field in a moving conductor as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta\nabla^2\mathbf{B}$$

where **B** is the magnetic field, **v** is the velocity field, η is the magnetic diffusivity $(1/\mu\sigma)$. This approach is useful when considering the geodynamo in Earth's core, where the physical meaning of the first term on the right hand side represents the interaction of the magnetic field with the velocity field, and the second term represents the diffusion of magnetic field through the material.

The Lorentz force is also applicable when conductive seawater moves through Earth's magnetic field, which generates an electric current given by

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad .$$

For some simplifying assumptions (horizontal ocean currents, constant water conductivity, flat seafloor with no coastlines, low frequencies), the conduction term can be ignored and

$$\mathbf{E}_h = C(\mathbf{v}_h \times \mathbf{B}_z)$$

where C = [0, 1] is a correction factor associated with seafloor conductivity and can be taken as nearly equal to 1.