Viscoelasticity and the Correspondence Principle

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1. Viscoelasticity

In an elastic material, the stress-strain constitutive relation in one dimension is given by Hooke’s law:

$$\sigma = 2\mu \epsilon,$$  \hspace{1cm} (1)

where $\mu$ is the shear modulus. Note that the stress is fully determined by the material’s present state of deformation; there is no time dependence. In contrast, a fluid is described by a viscous rheology,

$$\sigma = 2\eta \dot{\epsilon},$$  \hspace{1cm} (2)

where the dot represents the time derivative ($\dot{\epsilon} \equiv d\epsilon/dt$). In this case the stress has no dependence on the actual state of deformation, but rather is determined by the present strain rate.

A viscoelastic material is one whose rheology depends on both of these effects. We can derive the constitutive relations from first principles by assuming the stress (or strain) is determined by the material’s past history of strain (or stress) with some linear functional describing its ‘memory’ (see the Appendix for more). More commonly, the basic viscoelastic constitutive relations are simply stated; one may imagine taking a linear combination of the effects described by (1) and (2), represented graphically as elastic springs and viscous dashpots (Figure 1). Note that there are two unique ways we can combine the two elements: by adding the strains, or by adding the stresses.

The Maxwell rheology is represented by combining the two elements in series, so the strains (actually the strain rates) add:

$$\dot{\epsilon}_{\text{total}} = \dot{\epsilon}_{\text{elastic}} + \dot{\epsilon}_{\text{viscous}}$$

$$= \frac{\sigma}{2\mu} + \frac{\sigma}{2\eta}.$$  \hspace{1cm} (3)

This material responds instantly to a step change in stress (recoverable elastic strain), but if a step increase in strain is held constant the stress will decay to zero after a long time (nonrecoverable viscous strain). For this reason, the material is known as a Maxwell fluid – it has no ‘reference’ state and under a constant stress will deform indefinitely (although we are technically dealing only with infinitesimal strains in the linear theory).

In the Kelvin or Kelvin-Voigt rheology we add the elements in parallel, so the stresses add:

$$\sigma_{\text{total}} = 2\mu \epsilon + 2\eta \dot{\epsilon}.$$  \hspace{1cm} (4)

Because the compression of the spring cannot be relieved by the dashpot, the stress never decays to zero whenever some nonzero strain is present; the material therefore has a fixed reference state and is known as a Kelvin solid.
These rheologies represent only two of the infinite number of possible constitutive relations for a viscoelastic material; many others such as the Standard Linear Solid, Burgers rheology, etc. can be derived from more general spring/dashpot models using the standard rules of circuit diagrams, or alternatively via the more general method described in the Appendix.

2. Laplace Transform

The Laplace transform is defined as

$$\mathcal{L}\{f(t)\} \equiv \tilde{f}(s) = \int_0^\infty f(t)e^{-st}dt.$$  \hspace{1cm} (5)

This integral and others of its type were noted and investigated by Euler and Lagrange as a common form, at a time when mathematicians had to solve integrals and differential equations without the aid of a trusty handbook (or computer algebra system). However, it was Laplace in 1785 who first realized that the transform could be used to convert differential equations into a far more manageable algebraic form, turning a curiosity into an immensely useful mathematical tool. Some properties of the Laplace transform that will be useful below:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \frac{1}{s}$$  \hspace{1cm} (6)

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}dt = \frac{1}{s^2}$$  \hspace{1cm} (7)

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}$$  \hspace{1cm} (8)

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

$$= f(t)e^{-st}|_0^\infty + s\int_0^\infty f(t)e^{-st}dt$$

$$= f(0) + s\tilde{f}(s).$$  \hspace{1cm} (9)

The last result is derived using integration by parts; typically we assume that $f(0) = 0$ and use the simpler form $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\tilde{f}(s).$ This is the property which allows us to convert differential equations into algebraic ones, potentially making them much easier to solve, similar to our earlier use of the related Fourier transform.

3. Correspondence Principle

The viscoelastic correspondence principle states that the Laplace transform of the solution to any deformation problem with a linear rheology can be expressed in the form of the corresponding elastic solution, given a suitable choice of Laplace-transformed effective shear modulus. The transformed version of Hooke’s law,

$$\tilde{\sigma} = 2\tilde{\mu}\tilde{\epsilon}$$  \hspace{1cm} (10)

thus holds for any linear rheology, with the caveat that the deformation must be instantaneously applied (during an earthquake or other sudden event), so that we can drop the initial value term in the Laplace transform property (9).
Figure 1: Creep and Relaxation functions for Maxwell and Kelvin rheologies: response to step changes in stress (top) and strain (bottom).
Consider a general viscoelastic rheology (in one dimension) of the form

\[ \sum_{n=0}^{N} p_n D^{(n)} \sigma(t) = \sum_{n=0}^{N} q_n D^{(n)} \epsilon(t), \]  

(11)

where \( D \equiv d/dt \). The solution to this equation for a set of specified forces and boundary conditions is time-dependent and may more generally vary in all three spatial dimensions; we may expect that exact solutions to realistic problems of this type will be rare and hard-fought. Fortunately there is an easier way, thanks to the Laplace transform. Suppose we have obtained (by whatever means) an exact solution to the elastic problem with the same forces and boundary conditions, but with the simpler constitutive relation \( \sigma = 2\mu \epsilon \). The displacement \( u \) for this case may be written as a function of the shear modulus:

\[ u(x) = f(x, \mu). \]  

(12)

Now consider the Laplace transform of (11):

\[ \tilde{\sigma}(s) \sum_{n=0}^{N} p_n s^n = \tilde{\epsilon}(s) \sum_{n=0}^{N} q_n s^n. \]  

(13)

We may rewrite this in the form of Hooke’s law (10):

\[ \tilde{\sigma}(s) = \left( \frac{\sum_{n=0}^{N} q_n s^n}{\sum_{n=0}^{N} p_n s^n} \right) \tilde{\epsilon}(s). \]  

(14)

This is an algebraic equation, and holds for viscoelastic and elastic rheologies alike (any choice of \( p_n \) and \( q_n \)). Thus the Laplace transform of (12) is the solution in both cases,

\[ \tilde{u}(x) = \tilde{f}(x, \tilde{\mu}), \quad \text{where} \quad 2\tilde{\mu} = \frac{\sum_{n=0}^{N} q_n s^n}{\sum_{n=0}^{N} p_n s^n}. \]  

(15)

So we have only to perform the inverse transform using the new \( \tilde{\mu} \), and we have the solution for our viscoelastic rheology.

**Examples: 1-D shear.** Consider the case of simple shear in one dimension due to a constant force \( \sigma_0 \): from (1), the elastic strain is \( \epsilon = \sigma_0/(2\mu) \). Integrating once to obtain displacement, we have

\[ u(x) = \frac{\sigma_0}{2\mu} x, \]  

(16)

the Laplace transform of which is

\[ \tilde{u}(x, s) = \frac{\sigma_0}{2\mu s} x. \]  

(17)

We may now implement any viscoelastic rheology for which we know the corresponding effective shear modulus; for example consider the Maxwell constitutive relation (3). The Laplace transform of (3) is

\[ s\tilde{\epsilon}(s) = \frac{1}{2} \left( s \frac{1}{\mu} + \frac{1}{\eta} \right) \tilde{\sigma}(s), \]  

(18)
so the effective shear modulus is
\[ \tilde{\mu} = \frac{\mu s}{s + \mu/\eta}. \]  
(19)

Substituting this into (17), we have
\[ \tilde{u}(x, s) = \frac{\sigma_0(s + \mu/\eta)}{2\mu s^2} x \tilde{\mu} = \frac{\sigma_0}{2\mu} x \left( \frac{1}{s} + \frac{\mu}{\eta s^2} \right). \]  
(20)

Looking at properties (6) and (7), we can easily write down the inverse transform:
\[ u(x, t) = \frac{\sigma_0(s + \mu/\eta)}{2\mu s^2} x \tilde{\mu} = \frac{\sigma_0}{2\mu} x \left( 1 + \frac{t\mu}{\eta} \right). \]  
(21)

This solution has been diagrammed in Figure 1: the material undergoes an instantaneous elastic effect followed by a constant viscous shear.

For the Kelvin rheology, we may follow the same procedure. The Laplace transform of (4) is
\[ \sigma(s) = 2(\mu + \eta s)\tilde{\epsilon}(s) \]  
(22)

so the effective shear modulus in this case is
\[ \tilde{\mu} = \mu + \eta s. \]  
(23)

Substituting this into (17), we have
\[ \tilde{u}(x, s) = \frac{\sigma_0}{2s(\mu + \eta s)} x \]  
(24)

In this case the corresponding time-domain function is not so easy to identify; in these cases one must either solve a complex line integral using Cauchy’s theorem of residues, or look up the answer in a table or computer algebra system. The result is
\[ u(x, t) = \frac{\sigma_0}{2\mu} x \left( 1 - e^{-t\mu/\eta} \right), \]  
(25)

from which we again recognize the function diagrammed in Figure 1.

4. Application: Vertical Point load in on a Maxwell viscoelastic half space

Love (1944) gives the following analytic solution for the surface deformation resulting from a point load on a homogeneous half space, with \( z \) positive down:
\[ u(x, y) = \frac{-P}{4\pi(\lambda + \mu)} \frac{x}{r^2} \]  
(26)
\[ v(x, y) = \frac{-P}{4\pi(\lambda + \mu)} \frac{y}{r^2} \]  
(27)
\[ w(x, y) = \frac{P(\lambda + 2\mu)}{4\pi\mu(\lambda + \mu)} \frac{1}{r} \]  
(28)

The corresponding solutions in the Laplace domain have an extra factor of \( 1/s \).
To implement a Maxwell rheology, we need the full 3-D tensor form corresponding to (3).
Assuming the volumetric strain is elastic only, this is [Segall, 2010]:
\[
\ddot{\sigma}_{ij} + \frac{\mu}{\eta} \left( \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij} \right) = 2\mu \dot{\epsilon}_{ij} + \lambda \dot{\epsilon}_{kk} \delta_{ij},
\] (29)
which leads to the same form (25) for the effective shear modulus, and additionally
\[
\tilde{\lambda} = \frac{s\lambda + K(\mu/\eta)}{s + \mu/\eta},
\] (30)
where \( K = \lambda + 2\mu/3 \) is the bulk modulus. This is a somewhat contrived example, since nobody seriously suggests that the Earth is well represented by a homogeneous viscoelastic half space. A layered elastic over viscoelastic model would be better, but this simpler case will serve for the purpose of demonstration.

Making the substitutions (25) and (30) for a Maxwell rheology, after some simplifications we have
\[
\tilde{u}(x, y, s) = \frac{-3P(\mu + s\eta)}{4\pi s(K\mu + (\lambda + \mu)s\eta)} \frac{x}{r^2}
\] (31)
\[
\tilde{v}(x, y, s) = \frac{-3P(\mu + s\eta)}{4\pi s(K\mu + (\lambda + \mu)s\eta)} \frac{y}{r^2}
\] (32)
\[
\tilde{w}(x, y, s) = \frac{P(\mu + s\eta)(K\mu + (\lambda + 2\mu)s\eta)}{4\pi s^2 \mu \eta (K\mu + (\lambda + \mu)s\eta)} \frac{1}{r}
\] (33)
Takaing the inverse Laplace transform of these functions is not trivial, but we can get some help from the computer, via Mathematica’s *InverseLaplaceTransform[]*. The result is:
\[
u(x, y, t) = \frac{-Px}{4\pi K r^2} \left( 1 - \frac{\mu e^{-\frac{Et}{3\eta}}}{3(\lambda + \mu)} \right) \] (34)
\[
v(x, y, t) = \frac{-Py}{4\pi K r^2} \left( 1 - \frac{\mu e^{-\frac{Et}{3\eta}}}{3(\lambda + \mu)} \right) \] (35)
\[
w(x, y, s) = \frac{P}{4\pi K r} \left( \frac{3\lambda + 5\mu}{3\mu} - \frac{\mu e^{-\frac{Et}{3\eta}}}{3(\lambda + \mu)} + K t \right) \] (36)
where \( E = \mu(3\lambda + 2\mu)/(\lambda + \mu) \) is Young’s modulus. The third term in (36) attests to the unphysicality of our example; after the transient effects have dissipated, we are left with a linearly increasing subsidence in the area around the point load at a rate of \( P/(4\pi \eta r) \). Interestingly, the horizontal displacements don’t contain this secular component, which comes from an extra term proportional to \( 1/s^2 \) in the equation for \( \tilde{w} \).

For a more realistic model, we should use a different viscoelastic rheology or include an overlying elastic layer, and perhaps also the effects of gravity. The resulting equations are unwieldy and
best solved on the computer; a description is found in [Smith and Sandwell, 2004]; further details and a Fortran implementation may be found at http://topex.ucsd.edu/body_force.

Other resources: The derivation of viscoelastic rheologies found in the Appendix is largely based on [Christensen, 1982], who provides many further details. The useful case of an antiplane dislocation in an elastic layer over a viscoelastic halfspace is described in convenient notation by [Segall, 2010]. Finally, the tensor forms of additional constitutive laws and their Laplace-transformed effective moduli may be found in [Peltier, 1982].

References


Appendix: Viscoelastic Rheology.

Integral form. In an elastic rheology, the stress in a material is linearly related to the strain; expressed in tensor form this is
\[ \sigma_{ij} = C_{ijkl} \epsilon_{kl}. \] (37)

Note that the stress is fully determined by the present state of the material’s deformation. A useful question to ask is: what happens if the stress is also sensitive to the material’s prior history of deformation? In this case, the relationship becomes an integral:
\[ \sigma_{ij}(t) = \int_0^\infty \epsilon_{kl}(t-s) dG_{ijkl}(s), \] (38)
where \( G_{ijkl} \) is the (time-varying) tensor of material properties relating stress to strain. Following Christensen (1982), if we let \( \epsilon(t < 0) = 0 \) and assume that \( G \) is continuous up to first order, then we can re-write this as
\[ \sigma_{ij}(t) = \epsilon_{kl}(t) dG_{ijkl}(0) + \int_0^t \epsilon_{kl}(t-s) \frac{dG_{ijkl}}{ds} ds. \] (39)

Integrating by parts, and letting \( \tau = t-s \), we find the initial terms cancel and we are left with
\[ \sigma_{ij}(t) = \int_0^t G_{ijkl}(\tau) \frac{d\epsilon_{kl}(t-\tau)}{d\tau} d\tau. \] (40)

This is a generally applicable, linear constitutive relation for a material with the property of ‘memory.’ Note that we could have followed the same derivation while assuming that strain is a function of stress; the resulting relation is:
\[ \epsilon_{ij}(t) = \int_0^t J_{ijkl}(\tau) \frac{d\sigma_{kl}(t-\tau)}{d\tau} d\tau. \] (41)

What is the relationship between \( G \) and \( J \)? In the elastic case, we had simply \( G = J^{-1} \). In the viscoelastic case, it’s not so simple – we can’t simply invert the integral equations. However, taking the Laplace transforms of (40) and (41) does permit a straightforward comparison of the transformed quantities. We have:
\[ \tilde{\sigma}_{ij} = s \tilde{G}_{ijkl} \tilde{\epsilon}_{kl}, \] (42)
\[ \tilde{\epsilon}_{ij} = s \tilde{J}_{ijkl} \tilde{\sigma}_{kl}. \] (43)

Thus, we find \( \tilde{G} = (s^2 \tilde{J})^{-1} \). This will be useful in a moment.

Differential Form. We have in (40) and (41) a set of constitutive relationships describing our material, but their usefulness is limited unless we can put them into the form of a differential equation. For ease of notation, we will drop the tensor indices on \( \sigma \) and \( \epsilon \) and work in one dimension, but the resulting relations can be easily generalized to the full 3-D case. Consider a general linear differential equation of the form
\[ \sum_{n=0}^N p_n D^{(n)} \sigma(t) = \sum_{n=0}^N q_n D^{(n)} \epsilon(t), \] (44)
where the operator $D \equiv d/dt$. The Laplace transform of (44) is

$$\tilde{\sigma}(s) \sum_{k=0}^{N} p_k s^k = \tilde{\varepsilon}(s) \sum_{k=0}^{N} q_k s^k. \quad (45)$$

Comparing this to (42), we see that equation 44 matches our constitutive relation whenever

$$s\tilde{G}(s) = \sum_{n=0}^{N} q_n s^n / \sum_{n=0}^{N} p_n s^n. \quad (46)$$

In the 3-D case, $G$, $p$, and $q$ are 4th-order tensors.

**Examples: Maxwell and Kelvin materials.** Thus far, we have assumed arbitrary material properties represented by the tensors $G_{ijkl}$ and $J_{ijkl}$. For a more concrete example, let’s consider shear in one dimension, and assume a simple time-dependent relaxation function described by

$$G(t) = G_0 e^{-t/t_0}. \quad (47)$$

The Laplace transform is $\tilde{G}(s) = G_0/(s + 1/t_0)$, recalling property (8). Considering the sums up to $N = 2$ in (46) we now have:

$$\frac{sG_0}{s + 1/t_0} = \frac{q_0 + sq_1}{p_0 + sp_1}
\quad sG_0 p_0 + s^2 G_0 p_1 = q_0/t_0 + s(q_0 + q_1/t_0) + s^2 q_1. \quad (48)$$

Equating powers of $s$, this equation is satisfied when $q_0 = 0$, $q_1 = G_0$, $p_0 = 1/t_0$, and $p_1 = 1$. Thus, the differential constitutive relation corresponding to (47) is

$$\frac{1}{t_0} \sigma + \dot{\sigma} = G_0 \dot{\varepsilon}, \quad (49)$$

which is equivalent to the Maxwell rheology (3) if $G_0 = 2\mu$ and $t_0 = \eta/\mu$.

What if we instead specify the creep function $J$? In this case, assume

$$J(t) = J_0(1 - e^{-t/t_0}). \quad (50)$$

The Laplace transform is

$$\tilde{J}(s) = J_0 \int_{0}^{\infty} (1 - e^{-t/t_0})e^{-st}dt \quad = \frac{J_0}{s} - \frac{J_0}{s + 1/t_0} \quad = \frac{J_0}{s^2 t_0 + s}. \quad (51)$$

Now recalling $\tilde{G} = (s^2 \tilde{J})^{-1}$, we have from (46) with $N = 2$

$$\frac{1 + st_0}{J_0} = \frac{q_0 + sq_1}{p_0 + sp_1}
\quad p_0 + s(t_0 p_0 + p_1) + s^2 t_0 p_1 = J_0 q_0 + J_0 sq_1. \quad (52)$$
which is satisfied when $q_0 = 1$, $q_1 = t_0$, $p_0 = J_0$, and $p_1 = 0$. Thus, our constitutive relation is

$$J_0 \sigma = \epsilon + t_0 \dot{\epsilon},$$  \hspace{1cm} (53)

which is equivalent to the Kelvin rheology (4) when $J_0 = 1/(2\mu)$ and $t_0 = \eta/\mu$.

**Generalization to higher orders.** One advantage of the integral formulations (40) and (41) for the constitutive relations is their ease of generalization. Consider a relaxation function described by

$$G(t) = \sum_{n=1}^{N} G_n e^{-t/t_n}. \hspace{1cm} (54)$$

This represents $N$ Maxwell elements in parallel, with different shear moduli and time constants. Using the method outlined above for a single element, we can now derive the constitutive relation for any number of elements. Note that this is also equivalent to the common form of a Prony series if we add an extra constant term $G_\infty$, representing a purely elastic element.

The generalization of the Kelvin material is

$$J(t) = \sum_{n=1}^{N} J_n (1 - e^{-t/t_n}), \hspace{1cm} (55)$$

which represents $N$ Kelvin elements in series; one may use this to derive the constitutive relation for an arbitrary number of these.