Chapter 3

Kinematics: Mostly Deformation

> We beat it out flat; we beat it back square; we battered it into every form known to geometry–but we could *not* make a hole in it. Then George went at it, and knocked it into a shape, so strange, so weird, so unearthly in its wild hideousness, that he got frightened and threw away the mast. Then we all three sat round it on the grass and looked at it.
>
> Jerome K. Jerome (1889) *Three Men in a Boat (to say nothing of the Dog)*

3.1 Introduction

We begin with the most general part of continuum mechanics: **kinematics**, or how to describe the motion of a continuum. The reason it is general is just because it has no physics in it; indeed, it should probably be regarded as a branch of geometry.[1] This generality makes the subject both powerful and limited: powerful because we can apply it to situations in which we know nothing of the physics involved (not an uncommon situation in the Earth), and limited because we cannot use it to distinguish between different types of physics that might be relevant. This limitation is very much a current concern in geophysics: between plate tectonics and satellite geodesy we know the current and past motions of the Earth's surface (the kinematics) far better than before, but how to translate this improved kinematics into better understanding of the physics remains challenging.

[1] Mathematically, indeed, it is a specialization of differential geometry.

3.2 Descriptions of Motion: Particles and Locations

In describing the kinematics of a continuous medium, we can choose among several mathematical descriptions even for our system of specifying locations in that medium; which one we choose depends on which is most convenient for the problem we are trying to solve. All these descriptions have in common an unchanging reference frame, usually fixed in inertial space, to which any description can be referenced.[2] Within this frame, we use Cartesian coordinates to describe positions, vector components, and so on–but as much as possible we will write vectors and tensors in coordinate-free form as s reminder that they are independent of the particular system of components we might choose for the coordinates.

To specify a particular location in a continuum – and more especially, how locations change with time – we can choose between two descriptions, each of which goes by several names, none particularly memorable:

- In the **material**, or **Lagrangian**, description we specify a location by specifying the associated "particle" of the continuum–a particle being an (infinitesimal) part of it. For example, to specify location $\mathbf{r}$ in our overall reference frame we write $\mathbf{r} = \mathbf{r}^L(\mathbf{x}, t)$ which makes the position in space, $\mathbf{r}$, a function both of time t and of of the particle labeled by the vector $\mathbf{x}$. The function $\mathbf{r}^L$ has time dependence because, without it, the continuum would not move, leaving us with nothing to analyze. The point to focus on is that $\mathbf{x}$, though a vector, is not a spatial location but a particle label. This raises the obvious question of whether or not it is legitimate to use a vector for this purpose. To make it valid, we define $\mathbf{x}$ to be the spatial location of the particle at $t = 0$; this is certainly a vector. Put slightly differently, if we take $t = 0$ to be the time at which $\mathbf{r}^L(\mathbf{x}, t) = \mathbf{x}$, then the label $\mathbf{x}$ is the same kind of thing as the function $\mathbf{r}^L$: a vector. We do need one more assumption to make this work, namely what is called the axiom of continuity,[3] which is that no finite volume of a continuum can become either zero or infinite. This means that, whatever the subsequent motion, no particle-label vectors can cease to exist.

[2] The reference frame does not have to be inertial; in the subject known as geophysical fluid dynamics it is usually taken to be rotating at a constant angular velocity.

[3] This is not the same as the equation of continuity, which we will derive in the next chapter.

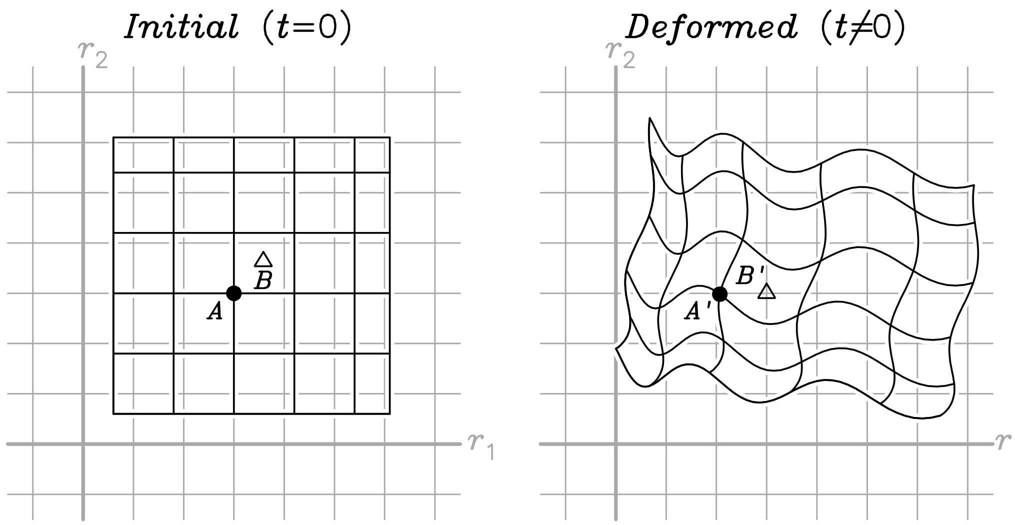

Figure 3.1: Deformation of a material, as shown by a grid (solid lines) in it, which at $t = 0$ is orthogonal, but at some other time has become deformed. The gray grid is the reference. In the notation used in the text, the black lines are lines of constant $\mathbf{x}$, and the gray lines are lines of constant $\mathbf{r}$.

- In the **spatial**, or **Eulerian**, description we write $\mathbf{x} = \mathbf{x}^E(\mathbf{r}, t)$; that is, the function $\mathbf{x}^E$ gives the label $\mathbf{x}$ of the particle that happens to be at the spatial position $\mathbf{r}$ at time t.

If this all seems abstract, it may help to think about what these functions mean if the argument is held constant. If $\mathbf{x}$ is constant, $\mathbf{r}^L(\mathbf{x}, t)$ would give the location, as a function of time, for the particle $\mathbf{x}$: we ride along with the particle, and follow its trajectory through space. If $\mathbf{r}$ is constant, we have a fixed spatial position, and a succession of particles moves through this fixed location, with the one at time t having the label $\mathbf{x}^E(\mathbf{r}, t)$.

Figure 3.1 shows the initial state of a body (at $t = 0$) and a deformed one, which can be thought of as being at another time t. For both states, the coordinate framework for $\mathbf{r}$ is shown in the background, in gray. In the initial state, lines of constant $\mathbf{x}$ form an orthogonal system that is aligned with $\mathbf{r}$, though for clarity I have spaced the lines at different distances. In the deformed state, constant values of $\mathbf{x}$ no longer form straight lines, or even an orthogonal system. The particle at A on the left deforms to A' on the right; the value of $\mathbf{x}$ for this particle remains what it was on the left, namely (3, 3)–but the value of $\mathbf{r}$ associated with it is now quite different, namely (2.07, 2.98). And if we look at $\mathbf{r} = (3,\ 3)$ on the right, we see that the particle labeled B' is at this spatial location; in the undeformed state it was at $\mathbf{r} = (3.58,\ 3.63)$, which is thus the value of $\mathbf{x}$ associated with this particle: that is, the label for B and B'.

An informal, but useful, analogy for these two descriptions is that the material description is the one you take of the other cars while you are driving on the freeway; the spatial description is that used by the Highway Patrol parked by the roadside and looking for speeders. This analogy brings out an important reason for having both descriptions. In describing the physics of a continuum, the material description is the one to use, because it describes things relative to the material, where any interactions take place. (Your major concern when driving is not to collide with the cars around you–where you, and they, all are is of less concern). It is also often true that our observations are made in a material framework: for example, a seismometer attached to the moving Earth or a drifter moving with the ocean.

But mathematically, the material system is more complicated and difficult than one fixed in space. If, as in Figure 3.1, we define a set of orthogonal coordinates attached to particles at $t = 0$, at other times the these coordinate axes will become distorted: not orthogonal, probably not even straight. We will no longer have any of the simplicity that Cartesian coordinates provided us with. If we want to take derivatives relative to the coordinate axes (as we often do), the derivative with respect to (say) the 1-axis cannot be defined without allowing for the motion of the material.

Our way out of this dilemma will be to focus on situations in which the change of axes does not complicate the mathematics, either because the motions are "small" or because they are specialized to forms for which the change of axes is easy to allow for.

3.3 Rigid-Body Motion

We are primarily interested in the **deformation** of the material: that is, its change in shape. A material that does not deform is said to behave as a **rigid body**; the mathematical definition of this is that for any two particles $\mathbf{x}_1$ and $\mathbf{x}_2$, the distance $|\mathbf{r}^L(\mathbf{x}_1, t) - \mathbf{r}^L(\mathbf{x}_2, t)|$ does not change. While rigid-body motion is not generally taken to be part of continuum mechanics, it has enough geophysical uses that we should discuss it briefly.

It is obvious that we can add a constant vector $\mathbf{a}$ to the position vectors $\mathbf{r}^L(\mathbf{x}_1)$ and $\mathbf{r}^L(\mathbf{x}_2)$ without affecting the difference $\mathbf{r}^L(\mathbf{x}_1) - \mathbf{r}^L(\mathbf{x}_2)$, not to mention its length. Such addition of a constant is equivalent to **rigid-body translation** of the material. It should also be obvious that rotating the reference frame (or equivalently the vector in the frame) does not change any vector lengths. The most general form of rigid-body motion is thus translation plus rotation; if one point in the body does not translate then the most general motion is rotation. It can be shown that any rotation, or combination of rotations, can be represented as a single rotation about some axis; so we can specify the rotation with three parameters, two to give the orientation of this axis, and one to give the amount of rotation.[4] This result is the Euler's theorem used in plate tectonics. For infinitesimal rotations the three rotation parameters can be just the rotations about each axis, which, in this case only, form a vector. For any larger rotations (usually called **finite rotations**) the axis and angle (called in plate tectonics the Euler pole and angle) are the visually most appealing description, but not the most convenient for analysis. A number of other choices are available, of which the best are the quaternions, now seeing increasing use after long neglect. As noted in Section 2.3, large rotations, unlike small ones, do not commute; quaternions do not either.[5]

3.4 Deformation

We now turn to more general types of deformation. Since what matters is the relative motion of nearby particles, we therefore consider motions of the continuum relative to some particular (though arbitrary) particle, called the **reference particle**, which we label $\mathbf{x}_R$. The change in relative position between this particle and some other particle (labeled $\mathbf{x}$) is given by:

$$\mathbf{u}(\mathbf{x}, t) = [\mathbf{r}^L(\mathbf{x}, t) - \mathbf{r}^L(\mathbf{x}_R, t)] - [\mathbf{r}^L(\mathbf{x}, 0) - \mathbf{r}^L(\mathbf{x}_R, 0)]$$

If $\mathbf{u}(\mathbf{x}, t) = 0$ for all $\mathbf{x}$ and all t, we are back to a pure translation, since any line between two particles has an unchanging length and direction.

We therefore want to develop descriptions for more general forms of $\mathbf{u}$. We focus on three cases which have very similar mathematical structure,

[4] Geodesists describe the mapping from one reference frame to another using rigid-body motions, which they call Helmert transformations: 3-parameter transformations for pure translation, and 6-parameter ones for translation and rotation.

[5] In two dimensions even finite rotations commute.

and are also not unreasonable approximations to many actual behaviors: these are homogeneous strain, small deformation, and rates of deformation.

3.5 Small Deformation

Define $\xi = \mathbf{x} - \mathbf{x}_R$: this is the vector from the reference particle to another, arbitrary, one. Our first assumption is to consider deformation in a small region only, so that ξ is infinitesimal, which we indicate by writing it as $d\xi$. Also, we look at $\mathbf{u}$ and related quantities for a fixed (but nonzero) value of t–that is, we consider the material in two configurations: one at $t = 0$ and one at some other time. We do not consider how the material gets from one configuration to another; in the terms usually employed, we are concerned with deformation but not motion. In this case, $\mathbf{x}$ describes the locations in the initial state, and $\mathbf{r}$ in the final (second) state–the material and spatial descriptions thus become the undeformed and deformed states respectively, though it is actually arbitrary which state we call "undeformed". By definition, when $d\xi = 0$ (that is $\mathbf{x} = \mathbf{x}_R$), the vector $\mathbf{u}$ is always zero: the reference particle is always itself.

Our second assumption is that the deformation is smooth enough that we can write $\mathbf{u}$ as a Taylor series in $d\xi$. Remembering that $d\xi$ is a vector between particles (that is, it depends on particle labels $\mathbf{x}$), this Taylor series is written, pretending for the moment that the $\mathbf{x}$ axes define Cartesian coordinates:

$$u_i(d\xi) = \frac{\partial u_i}{\partial x_j} d\xi_j + \text{higher-order terms} \tag{3.1}$$

where we are using, once again, the summation convention for repeated indices.

Our third assumption is that all we need from the Taylor series is the first term; this assumption amounts to requiring that the gradients in equation 3.1 be much less than one. And our fourth assumption is that the motions are small enough that the axes for $\mathbf{r}$ and $\mathbf{x}$ will locally coincide: this is a separate requirement from the one for small gradients. Remember that for $t = 0$, the reference state, these axes do coincide. This last assumption means that we may take $\mathbf{u}$ to be a function of $\mathbf{r}$ rather than of $\mathbf{x}$. The last two assumptions, combined, allow us to write the displacement $\mathbf{u}$ as

$$u_i = \frac{\partial u_i}{\partial r_j} d\xi_j \tag{3.2}$$

$$= \tfrac{1}{2}\left[\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right] d\xi_j + \tfrac{1}{2}\left[\frac{\partial u_i}{\partial r_j} - \frac{\partial u_j}{\partial r_i}\right] d\xi_j$$

$$\stackrel{\text{def}}{=} E_{ij} d\xi_j + \Omega_{ij} d\xi_j$$

The first expression gives the displacement in terms of the **displacement gradient**, which in coordinate-free terms is the dyad $\nabla\mathbf{u}$; on the next line, we add, subtract, and regroup terms to get particular combinations of these gradients. In coordinate-free form equation 3.2 becomes

$$\mathbf{u} = (\nabla\mathbf{u})d\xi = (\tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)d\xi + \tfrac{1}{2}(\nabla\mathbf{u} - \mathbf{u}\nabla)d\xi \stackrel{\text{def}}{=} \mathbf{E}d\xi + \boldsymbol{\Omega}d\xi$$

where we have defined new quantities $\mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u}+\mathbf{u}\nabla)$ and $\boldsymbol{\Omega} = \tfrac{1}{2}(\nabla\mathbf{u} - \mathbf{u}\nabla)$. From equation 3.2 it is clear that $\mathbf{E}$ is symmetric and $\boldsymbol{\Omega}$ antisymmetric.

3.6 Small Deformation in Two Dimensions

To clarify the meaning of different terms of $\mathbf{E}$ and $\boldsymbol{\Omega}$, we consider these expressions in two dimensions. The Cartesian components of $\mathbf{E}$ and $\boldsymbol{\Omega}$ are then:

$$\mathbf{E} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} \partial_1 u_1 & \tfrac{1}{2}(\partial_1 u_2 + \partial_2 u_1) \\ \tfrac{1}{2}(\partial_2 u_1 + \partial_1 u_2) & \partial_2 u_2 \end{pmatrix}$$

$$= \begin{pmatrix} u_{1,1} & \tfrac{1}{2}(u_{1,2} + u_{2,1}) \\ \tfrac{1}{2}(u_{2,1} + u_{1,2}) & u_{2,2} \end{pmatrix}$$

where we have again used the convenient contractions (equation 2.13)

$$u_{i,j} \stackrel{\text{def}}{=} \partial_j u_i \stackrel{\text{def}}{=} \frac{\partial u_i}{\partial r_j}$$

(Remember, again, that in general a spatial derivative might be $\partial/\partial\mathbf{x}$ or $\partial/\partial\mathbf{r}$; only for small deformations does this difference not matter). Now suppose the only nonzero term is E_{11}; then

$$u_1 = E_{11} d\xi_1$$

describes the displacement field: there is only displacement in the u_1 direction. If $E_{11} > 0$, this displacement increases as we move away from $d\xi_1 = 0$. This deformation is called a **uniaxial extension**. If $E_{11} < 0$, we have displacement, also increasing away from $d\xi_1 = 0$, but towards the 2-axis:

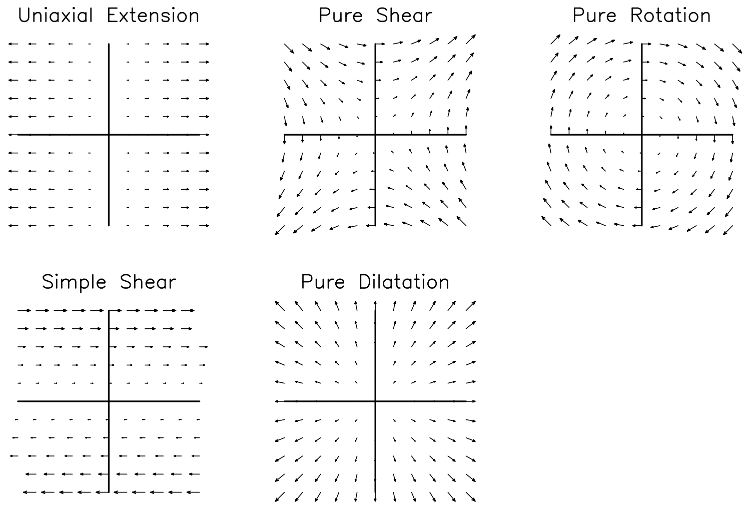

Figure 3.2: A variety of displacement fields, producing different kinds of strain. All these strains are finite, but homogeneous, so they are adequate representations of infinitesimal strains.

this is **uniaxial contraction**. Together these are termed **uniaxial strain**. Of course, E_{22} gives the same kind of deformation field in the orthogonal direction.

Next consider E_{12} nonzero; then

$$u_1 = E_{12} d\xi_2 \qquad u_2 = E_{12} d\xi_1$$

This is called a **pure shear**, and will cause the axes to move so that the angle between them changes from $\pi/2$ to $\pi/2 - 2E_{12}$. This suggests that shear may be specified as an angle change $\gamma = 2E_{12}$; γ is called **engineering shear** (to be distinguished from the **tensor shear** component E_{12}).

Figure 3.2 shows some of these simple two-dimensional deformations; of course for clarity we have to make them finite rather than infinitesimal, but since we have made them homogeneous they remain accurate. The tails of the arrows form a regular grid in the undeformed material; their heads show the positions of these particles after the deformation, so the arrows themselves show the displacement field $\mathbf{u}$. In addition to the two strain types already described, we also show a **pure rotation**, in which $\mathbf{E}$ is zero and $\boldsymbol{\Omega}$ is not; this is a rigid-body motion. If we add this rotation to the pure shear, we get the type of deformation known as **simple shear** (shear parallel to one axis), which thus includes both strain and rotation. Simple shear has a special place in geodynamics, as being the kind of deformation that takes place across diffuse plate boundaries when the motion is parallel to the boundary: crustal deformation in Southern California is one place where this is a good first approximation. Finally, Figure 3.2 shows the case in which $E_{11} = E_{22}$: this is often called **areal strain**; in the Earth, this kind of deformation is most typically found in volcanic areas. Note, in all these drawings, that there is nothing special about the point in the center; if we took displacements relative to some location on the edge, we would get the same sort of picture.

In two dimensions, the antisymmetric part $\mathbf{\Omega}$ is

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1/2(\partial_2 u_1 - \partial_1 u_2) \\ 1/2(\partial_1 u_2 - \partial_2 u_1) & 0 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}$$

so there is only one component, Ω. To see what motion this gives, take a vector of length $d\xi$ which initially is at an angle θ to the 1-axis and is rotated by another (small) angle ω, as in the left panel of Figure 3.3. The end of the vector is displaced by

$$u_1 = d\xi[\cos(\theta + \omega) - \cos\theta]$$

$$u_2 = d\xi[\sin(\theta + \omega) - \sin\theta]$$

which can be written in matrix form as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\omega - 1 & -\sin\omega \\ \sin\omega & 1 - \cos\omega \end{pmatrix} \begin{pmatrix} d\xi\cos\theta \\ d\xi\sin\theta \end{pmatrix}$$

For $\omega \ll 1$ this becomes

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} d\xi_1 \\ d\xi_2 \end{pmatrix}$$

which shows that $\mathbf{\Omega}$ describes just a rigid-body rotation–though this rotation has to be small for this description to work. As we saw in Section 3.3, larger rotations cannot be described by this simple means.

It is instructive to work though how the Cartesian components of $\mathbf{E}$ and $\mathbf{\Omega}$ change in two dimensions as we change the direction of the coordinate axes–although the algebra is slightly tedious, we end up with the result which justifies calling $\mathbf{E}$ a tensor. Take, as before, a vector $d\xi$ at an angle θ to the 1-axis; as a result of the deformation it becomes a new vector $d\xi'$.

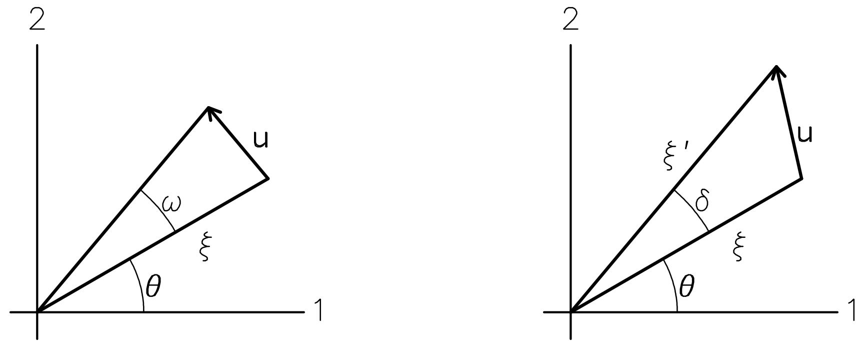

Figure 3.3: The left-hand diagram shows the motion of a vector under rotation; the right-hand one shows a vector that undergoes both rotation and change in length. See text for details.

(In figure 3.3 we show these, and label them, as finite vectors). We define the **stretch** to be the ratio of lengths of these vectors, and the **extension** to be this minus one:

$$e \stackrel{\text{def}}{=} \frac{|d\xi'|}{|d\xi|} - 1$$

Since $d\xi' = d\xi + \mathbf{u}$, we can write a quantity related to e

$$\frac{(d\xi_1 + u_1)^2 + (d\xi_1 + u_2)^2 - (d\xi_1^2 + d\xi_2^2)}{d\xi_1^2 + d\xi_2^2} \tag{3.3}$$

$$= \frac{((1+e)|d\xi|)^2 - |d\xi|^2}{|d\xi|^2} = (1+e)^2 - 1 \approx 2e$$

where the last approximation is valid for e small. We next write the displacements u_i in terms of the displacement gradients; for example, for u_1,

$$u_1 = \frac{\partial u_1}{\partial r_1} d\xi_1 + \frac{\partial u_1}{\partial r_2} d\xi_2$$

which means that

$$(u_1 + d\xi_1)^2 - d\xi_1^2 = 2\frac{\partial u_1}{\partial r_1} d\xi_1^2 + 2\frac{\partial u_1}{\partial r_2} d\xi_1 d\xi_2 \tag{3.4}$$

plus higher-order terms involving products of displacement gradients. Substituting expressions like equation 3.4 into 3.3 we find that

$$2e = 2\frac{\partial u_1}{\partial r_1}\left(\frac{d\xi_1}{|d\xi|}\right)^2 + 2\frac{\partial u_2}{\partial r_2}\left(\frac{d\xi_2}{|d\xi|}\right)^2 + 2\left(\frac{\partial u_1}{\partial r_2} + \frac{\partial u_2}{\partial r_1}\right)\frac{d\xi_1 d\xi_2}{|d\xi|^2}$$

which implies

$$e(\theta) = E_{11}\cos^2\theta + E_{22}\sin^2\theta + 2E_{12}\sin\theta\cos\theta \tag{3.5}$$

since, for example, $d\xi_1 = |d\xi|\cos\theta$. Since e is just a uniaxial strain, equation 3.5 is also an expression for $E_{11}(\theta)$, the 11 component of strain in a coordinate system rotated counterclockwise by θ from the original. For example, equation 3.5 implies that for $\theta = 90°$ we will get $E_{11}(90°) = E_{22}$–which is obviously the case. For pure shear, with $E_{11} = E_{22} = 0$ and $E_{12} \neq 0$, $E_{11}(\theta)$ has a four-lobed pattern, with two of the lobes being negative. This behavior is apparent if we look at the pure shear shown in Figure 3.2; at 45° to the axes there is just uniaxial extension and contraction.

This shows how the extension, and the extensional components, transform for a rotation of the axes. To derive how the shear strain will change for a rotation, we look first at the change in orientation of a single vector, which is described by the angle given, for small displacements, by (Figure 3.3):

$$\delta = \frac{u_1(-\sin\theta) + u_2\cos\theta}{|d\xi|}$$

that is, the projection of $\mathbf{u}$ onto a direction perpendicular to $d\xi$, divided by $|d\xi|$. Again using the expression for u_1 and u_2 in terms of displacement gradients, and that $d\xi_1 = |d\xi|\cos\theta$, we get that, for small displacements,

$$\delta = \frac{\partial u_1}{\partial r_1}(-\sin\theta\cos\theta) + \frac{\partial u_1}{\partial r_2}(-\sin 2\theta) + \frac{\partial u_2}{\partial r_2}(\sin\theta\cos\theta) + \frac{\partial u_2}{\partial r_1}cos^2\theta$$

$$= (E_{22} - E_{11})(\sin\theta\cos\theta) + (E_{12} + \Omega)(-\sin^2\theta) + (E_{12} - \Omega)(\cos^2\theta)$$

$$= (E_{22} - E_{11})\sin\theta\cos\theta + E_{12}(\cos^2\theta - \sin^2\theta) - \Omega$$

so the change in direction depends on $\mathbf{E}$ and $\mathbf{\Omega}$; reasonably, there is no dependence on θ for the part involving $\mathbf{\Omega}$, as expected for a rigid-body rotation. If we add 90° to θ, we get the change in orientation for an orthogonal line, which is:

$$-(E_{22} - E_{11})\sin\theta\cos\theta - E_{12}(\cos^2\theta - \sin^2\theta) - \Omega$$

Subtracting these two expressions to get the change in angle between the (originally orthogonal) lines and dividing by two to get the tensor shear gives

$$E_{12}(\theta) = (E_{22} - E_{11})\sin\theta\cos\theta + E_{12}(\cos^2\theta - \sin^2\theta) \tag{3.6}$$

which shows, among other things, that $E_{11} - E_{22}$ is just as much a shear as E_{12}.

The full transformation matrix for the Cartesian components of strain in two dimensions is therefore:

$$\begin{pmatrix} E'_{11} \\ E'_{12} \\ E'_{22} \end{pmatrix} = \begin{pmatrix} \cos^2\theta & -\sin\theta\cos\theta & \sin^2\theta \\ \sin^2\theta & \sin\theta\cos\theta & \cos^2\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \end{pmatrix} \begin{pmatrix} E_{11} E_{12} E_{22} \end{pmatrix}$$

Because the Cartesian components of $\mathbf{E}$ transform in this way, we know that $\mathbf{E}$ is indeed a tensor, since obedience to such a transformation rule defines the tensor character of an entity.

This transformation in turn can be used to find additional ways of parameterizing the strain into other components; we describe these for the two-dimensional case, but the results carry over to three dimensions as well. If we consider $\Delta_A = E_{11} + E_{22}$, we can see from equation 3.6 that this will not vary with θ at all–that is, this quantity is invariant, or more properly **an invariant of E**. (There are other invariants, but they involve powers of the components). Δ_A is called the **areal strain** and amounts (for small strains) to the ratio of areas in the deformed and undeformed states, minus one.

In three dimensions, the equivalent ($\Delta_V = E_{11} + E_{22} + E_{33}$) is called the **dilatation**, and is equivalently related to the change in volume. If we subtract the dilatation from the strain tensor to form $\mathbf{E}^D = \mathbf{E} - \Delta_A \mathbf{I}$, we have the **deviatoric strain** $\mathbf{E}^D$, which can also be written as $\mathbf{E}^D = \mathbf{E} - \mathrm{Tr}(\mathbf{E})\mathbf{I}$, where $\mathbf{I}$ is the identity tensor, and $\mathrm{Tr}(\mathbf{E})$ is the trace of the strain tensor: for Cartesian components the sum of the diagonal terms.

From equation 3.6, we can see that at 45° to our original (and arbitrary) choice of axes the shear strain would be $^1\!/\!_2(E_{22} - E_{11})$: as noted above, this is thus just as much a shear as E_{12}. We thus can express the strain as:

$$[^1\!/\!_2\Delta_A, {}^1\!/\!_2(E_{22} - E_{11}), E_{12}] \stackrel{\text{def}}{=} [^1\!/\!_2\Delta_A, {}^1\!/\!_2\gamma_1, {}^1\!/\!_2\gamma_2)$$

where γ_1 and γ_2 are the **engineering shear strains**. This is often a useful representation because the material may respond differently to shear than to change in area: for example, in rocks shear leads to failure, but dilatation does not, at least in compression.

Alternatively, we can see from equation 3.6 that there is an orientation of coordinate axes that will make $E_{12} = 0$. Let the angle of the axes (relative to the original set) be $\theta = \theta_p$; then equation 3.6 shows that to make the shear zero we have to have

$$\tfrac{1}{2}(E_{22} - E_{11}) \sin 2\theta_p + E_{12} \cos 2\theta_p = 0$$

which means that the angle is given by

$$\theta_p = \tfrac{1}{2}\arctan\left[\frac{2E_{12}}{E_{11} - E_{22}}\right] \tag{3.7}$$

For axes that make this angle to the original axes, only E_{11} and E_{22} are nonzero, so yet another way to express strain is as

$$[\theta_p, E_{11}(\theta_p), E_{22}(\theta_p)]$$

These are termed the **principal axis strains**: the principal axes are those for which $E_{12} = 0$, which is to say, the axes for which $\mathbf{E}$ is diagonal (in Cartesian coordinates). While this set of numbers does not directly transform to other coordinates, it is very often useful to look at strains in this way.

A final way of looking at two-dimensional strains is the **Mohr's circle** construction, a geometrical way of expressing the transformation 3.6 and of showing a particular state of strain. From equation 3.7 we can see that

$$\frac{\sin 2\theta_p}{\cos 2\theta_p} = \frac{E_{12}}{\tfrac{1}{2}(E_{11} - E_{22})}$$

This expression in turn suggests reparameterizing these two shear strain components as

$$E_{12} = R \sin 2\theta_p \qquad \tfrac{1}{2}(E_{11} - E_{22}) = R \cos 2\theta_p$$

where $R^2 = E_{12}^2 + (E_{11} - E_{22})^2/4$. Substituting these expressions into equation 3.6 we get

$$E_{12}(\theta) = -R \cos 2\theta_p \sin 2\theta + R \sin 2\theta_p \cos 2\theta = R \sin 2(\theta_p - \theta) \tag{3.8}$$

Similarly, we can write equation 3.5 as

$$E_{11}(\theta) = \tfrac{1}{2}E_{11}(1 + \cos 2\theta) + \tfrac{1}{2}E_{22}(1 - \cos 2\theta) + E_{12} \sin 2\theta$$

$$= \tfrac{1}{2}(E_{11} + E_{22}) + \tfrac{1}{2}(E_{11} - E_{22}) \cos 2\theta + E_{12} \sin 2\theta$$

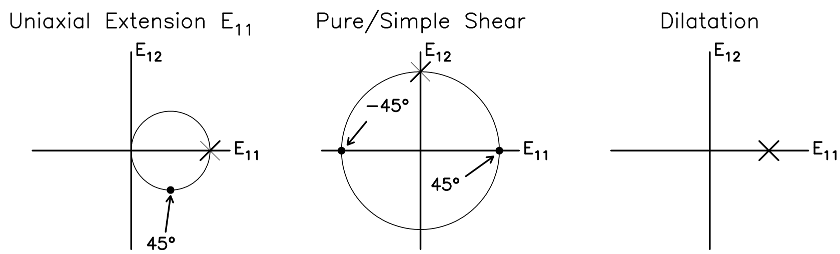

Figure 3.4: Mohr's circle representation of three states of strain.

$$= \tfrac{1}{2}(E_{11} + E_{22}) + R\cos 2(\theta_p - \theta) \tag{3.9}$$

If we plot E_{11} and E_{12} along two axes as a function of θ, using equations 3.8 and 3.9, the curve defined by these equations is a circle of radius R, centered at $\tfrac{1}{2}(E_{11} + E_{22})$. This is called the **Mohr's circle for two dimensions** (there is a more elaborate version for three dimensions), and it is a useful way to display strain because it makes clear the possible range of both extension and shear. For example, it at once shows that for $\theta = -\theta_p$ the shear is zero, with the two extensional strains being $E_{11}(\theta_p)$ and $E_{22}(\theta_p)$, given by where the circle cuts the E_{11} axis. (These points are 180° apart in terms of angle measured around the Mohr circle, and so 90° apart in terms of orientations of the coordinate axes, because of the factor of two in equations 10 and 11). Figure 3.4 shows three of the strain states from Figure 3.2 drawn in this way; in each diagram, the cross corresponds to the strain state for unrotated axes, and the dots to rotations of axes by the amount they are labeled by. Thus, at 45° to a uniaxial extension the shear is maximal; at ±45° to the coordinates for pure shear, we have uniaxial extension and contraction. And since there is no shear for dilatation, there is no Mohr's circle: just a point away from the origin. Note that the Mohr's circle construction, and indeed all the parameterizations we have considered, hold, not just for **E** in two dimensions, but for any two-dimensional symmetric tensor. We will see later how it can be applied to stress–which is, indeed, where it is more usually met with.

3.7 Small Deformation: Three-Dimensional Results

We now return to our general result 3.2 and 3.3, and look first at the tensor $\mathbf{\Omega}$. The Cartesian components can be written as a matrix

$$\mathbf{\Omega} = \begin{pmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{pmatrix}$$

where the indexing of the "vector" ν (which we have not yet demonstrated to be a vector) is done so that we can write

$$\Omega_{ik} = \epsilon_{ikm}\nu_m$$

where ϵ is the permutation symbol. But since we have (equation 2.6)

$$\epsilon_{ikp}\epsilon_{ikm} = \epsilon_{pik}\epsilon_{mik} = 2\delta_{pm}$$

we can write

$$\epsilon_{ikp}\Omega_{ik} = \epsilon_{ikp}\epsilon_{ikm}\nu_m = 2\delta_{pm}\nu_m = 2\nu_p$$

giving, finally, $\nu_m = {}^1\!/\!{}_2\epsilon_{ijm}\Omega_{ij}$ for ν given $\mathbf{\Omega}$. But, substituting into this the definition of Ω_{ij} gives

$$\nu_m = {}^1\!/\!{}_2\epsilon_{mij}\partial_i u_j$$

which is, in coordinate-free form, $\nu = {}^1\!/\!{}_2\nabla \times \mathbf{u}$; ν is the (polar) vector that is the curl of displacement. This corresponds to a small rigid-body rotation about the ν axis, by an amount $|\nu|$. $\mathbf{\Omega}$ is thus a (small) rotation tensor, while $\mathbf{E}$ describes the deformation. Note that if $\mathbf{E}$ is zero everywhere the material deforms as a rigid body, in which case the only possible solution for $\mathbf{\Omega}$ is that it is everywhere constant: different rotations in different places are not possible without some nonzero strain.

The other result is for $\mathbf{E}$, and is that the expression in terms of principal strains, which we demonstrated for two dimensions, extends to three. The result comes from equation 2.11, that a symmetric matrix, can be decomposed into the product of an orthogonal matrix U and a diagonal matrix D:

$$\mathbf{E} = UDU^T$$

which means that there is some set of axes in which the components of **E** are

$$\begin{pmatrix} E_I & 0 & 0 \\ 0 & E_{II} & 0 \\ 0 & 0 & E_{III} \end{pmatrix}$$

The axes in this coordinate system are the **principal axes** of the strain tensor; in this particular coordinate system there are no shears, only extensions (or contractions), something we have already seen for the two-dimensional case.

3.8 Motion

We get results very similar to those for small strains if we consider the instantaneous motion of a continuum – which, for geophysics, is often the more relevant description. For example, if we are trying to relate current seismicity to deformation, we want the instantaneous rate of deformation, since the few decades (or at most few centuries) of geophysical measurement are instantaneous compared to geologic time.

Formally, though nonrigorously, we can see that over an infinitesimal time dt the displacement must be small. Then locally we may, as we did before, take only the first term of a Taylor series expansion of the relative velocity (instead of displacement). We can also ignore differences between axes, since over an infinitesimal time they will not change. Then we have

$$\mathbf{dv}_i = \frac{\partial v_i}{\partial r_k}\mathbf{dr}_k = \frac{\partial v_i}{\partial x_k} d\xi_k$$

where $\mathbf{dv}$ is the relative velocity between a reference point (or particle) and one $d\xi$ away. These partial derivatives define the **velocity gradient tensor** $\mathbf{L} = \partial_k v_i = \nabla \mathbf{v}$; unlike the case for the small-strain tensor (Section 2.1), these gradients need not themselves be infinitesimal. We can make exactly the same decomposition as we did for small strain, though we give the tensors different symbols:

$$\mathbf{L} = \mathbf{D} + \mathbf{W}$$

where $\mathbf{W} = {}^1\!/_2(\nabla \mathbf{v} - \mathbf{v}\nabla)$ is the **spin tensor** and $\mathbf{D} = {}^1\!/_2(\nabla \mathbf{v} + \mathbf{v}\nabla)$ is the **rate-of-deformation tensor**–names we shall now proceed to justify.

By exactly the same procedure as we applied to $\mathbf{\Omega}$, we can see that the components of $\mathbf{W}$ can be written as a vector, given by $\omega = {}^1\!/_2\nabla \times \mathbf{v}$; this is half the **vorticity vector** $\nabla \times \mathbf{v}$ as defined in fluid mechanics (confusingly, $\mathbf{W}$ is sometimes called the vorticity tensor). This quantity, again, need not be infinitesimal, unlike the small-rotation tensor $\mathbf{\Omega}$.

To show the meaning of $\mathbf{W}$, consider a small circular area (disk) of diameter ϵ_h and area S, centered on the reference particle. The mean velocity around the edge of this disk, which is a line we denote by Γ, is

$$\bar{v} = \frac{1}{2\pi\epsilon_h}\int_\Gamma \mathbf{v}\cdot d\mathbf{l} = \frac{1}{2\pi\epsilon_h}\int_S \nabla\times\mathbf{v}d\mathbf{A}$$

by Stokes' theorem. For $\nabla \times \mathbf{v}$ constant across the surface S, the mean velocity around the edge is

$$\bar{v} = \frac{1}{2\pi\epsilon_h}\pi\epsilon_h^2\nabla\times\mathbf{v} = \epsilon_h \cdot {}^1\!/\!_2(\nabla\times\mathbf{v})$$

which is just what we would expect if the disk rotated as a rigid body with angular velocity $\omega = {}^1\!/\!_2(\nabla \times \mathbf{v})$. $\mathbf{W}$ thus describes the local angular velocity, or spin, of the material, including both rigid-body rotation and any spin caused by local deformation.

To better see the meaning of $\mathbf{D}$, consider an infinitesimal line $d\xi$, which in spatial coordinates is $d\mathbf{r}(t)$. The length of this is the dot product $d\mathbf{r}\cdot d\mathbf{r}$, and the rate of change of length is thus

$$\frac{d}{dt}(d\mathbf{r}\cdot d\mathbf{r}) = 2\cdot d\mathbf{r}\cdot\frac{d}{dt}(d\mathbf{r}) \tag{3.10}$$

But we can write, at any time, $d\mathbf{r}$ as

$$d\mathbf{r} = (\mathbf{r}\nabla)\cdot d\xi \qquad \text{i.e.} \qquad dr_i = \frac{\partial r_i}{\partial x_j}d\xi_j$$

which implies that

$$\frac{d}{dt}(d\mathbf{r}) = \frac{d}{dt}(\mathbf{r}\nabla)\cdot d\xi + (\mathbf{r}\nabla)\cdot\frac{d\xi}{dt} = \frac{d}{dt}(\mathbf{r}\nabla)\cdot d\xi$$

because $d\xi$ is constant (it always includes the same particles). But, we can write this as

$$\frac{d}{dt}(d\mathbf{r}) = \frac{d\mathbf{r}}{dt}\nabla\cdot d\xi = \mathbf{v}\nabla\cdot d\xi$$

where $\mathbf{v}\nabla$ is, in Cartesian components, $\partial v_i/\partial x_j$. This is a gradient with respect to material coordinates (particles), not spatial coordinates. However, it is the case that

$$dv_i = \frac{\partial v_i}{\partial x_j}d\xi_j \qquad \text{and} \qquad dv_i = \frac{\partial v_i}{\partial r_j}dr_j$$

which means that we can write

$$\frac{d}{dt}(d\mathbf{r}) = \mathbf{L}d\mathbf{r}$$

Finally, substituting this into equation 3.10, we get

$$\frac{d}{dt}(d\mathbf{r} \cdot d\mathbf{r}) = 2 \cdot d\mathbf{r} \cdot \mathbf{L} \cdot d\mathbf{r} = 2d\mathbf{r} \cdot \mathbf{D} \cdot d\mathbf{r}$$

where the antisymmetric part of **L** has canceled in the dot product. Thus **D** is associated with rate of change of length; if $\mathbf{D} = 0$ then there is no change of length in any direction, so that where this is true the motion is that of a rigid body. **D**, being symmetric, has exactly the same properties as **E** does: we can define rates of extension, rates of shear, and principal axes just as we did above.

It is important to realize that, as in the small-strain case, **D** and **W** are not completely independent. Consider two different motions that have the same **D**. If we subtract these, the motion that is the difference between them has to have $\mathbf{D} = 0$ everywhere–but this means that this difference is a rigid-body motion, for which **W** is constant. Knowing **D** everywhere thus means that we know **W** to within a constant amount: knowledge of the rate of deformation also determines (nearly) the local spin.

3.9 Homogeneous Strain

The last category of deformation we turn to is **finite homogeneous strain**, which is of interest for two reasons. First, it can usually be used as a local approximation for the much more complicated case of large strains that are spatially variable. We have will see an example of this in the next section. Another application of homogeneous finite strain is in the analysis of deformed structures (such as fossils) in deformed rocks; even though the deformation of these can be very complicated, on a small scale it can be spatially uniform enough for homogeneous strain to be a valid approximation. The second reason for studying this type of deformation is that it shows, yet again, the kind of mathematical structure we have already encountered for small strains and rates of deformations, though with a few differences. We have in fact already relied on this common structure, since the strains in Figure 3.2, though used to illustrate small strains, are themselves not small.

Homogeneous strain can be defined in geometrical terms as any deformation in which all straight lines in the undeformed material remain straight in the deformed one. The equation for a straight line is $c_i x_i = d$ where the c_i's and d are arbitrary constants and we have used the summation convention; we can write this in vector form as $\mathbf{c} \bullet \mathbf{x} = d$, though we need to remember that $\mathbf{c}$ is not a geometrical vector, but just a triplet of numbers. Then we can write the geometrical constraint as

$$\mathbf{c} \bullet \mathbf{x} = d \qquad \text{and} \qquad \mathbf{c}' \bullet \mathbf{r} = d'$$

where $\mathbf{x}$ is, as usual, the coordinates in the undeformed configuration, and $\mathbf{r}$ the spatial coordinates (in either one). Algebraically this means that the relation between these two sets of coordinates must be

$$r_i = A_{ij} x_j$$

where A is an arbitrary 3×3 real-valued matrix.

It is a standard result from linear algebra (the **polar decomposition theorem**) that any real matrix can be decomposed into a product of two others:

$$A = RB_1 = B_2 R$$

where the matrix R is orthogonal and B_1 and B_2 are symmetric (though not the same). In three dimensions R corresponds to a rotation; B_1 and B_2, being symmetric, can be decomposed into another rotation matrix U and a diagonal matrix D:

$$A = RU_1 D_1 U_1^T = U_2 D_2 U_2^T R$$

What this means is that any finite homogeneous strain can be produced in two ways:

A The RB_1 decomposition corresponds to first deforming the material in a way given by B_1, which can be expressed as different extensions along the directions of the principal axes, followed by a rotation given by R.

B The $B_2 R$ decomposition corresponds to first rotating the material in a way given by R and then deforming it a way given by B_2–again, this is expressible in terms of (different) extensions along (different) principal axes.

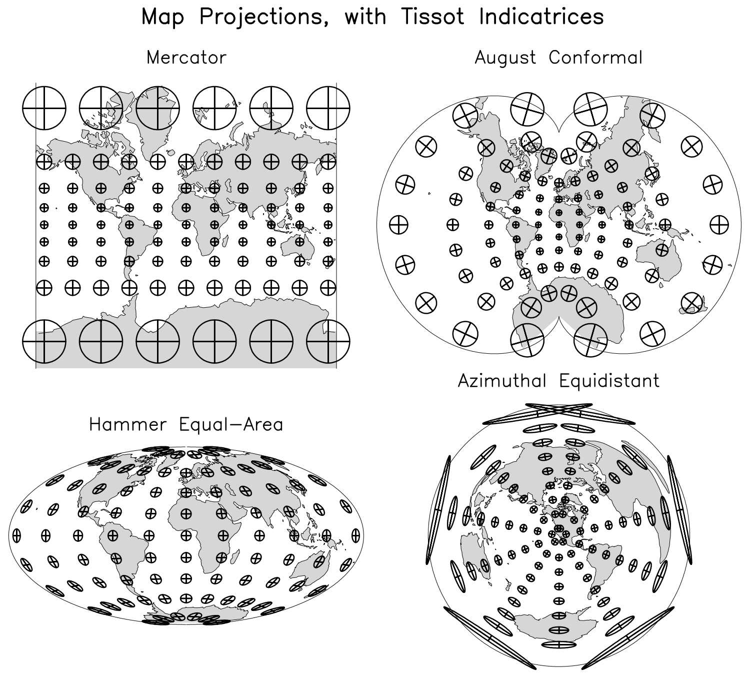

Figure 3.5: Map projections with Tissot indicatrices. The indicatrices were computed by projecting 5-km circles around particular latitude-longitude points onto the map plane, and then magnifying their scale by 100 to make them visible. The "August" projection maps the sphere into a two-cusped epicycloid. The land-sea distribution is shown in the normal aspect, except for the azimuthal equidistant projection, for which the map is centered on SIO.

A Cartographic Example of Finite Strain: Tissot's Indicatrix

An interesting application of the results of the previous section is the use of strain ellipses to show the deformations inherent in different map projections. This use appears to have been an independent invention, by Tissot, developed some time after the strain results had been worked out by Cauchy, and it is rare for the cartographic literature to reference continuum mechanics rather than Tissot's work.

For our purposes a map projection can be considered a function that maps[6] the surface of a sphere onto a plane. We could put this into vector form by saying that a projection considering a three-dimensional unit vector $\mathbf{e}$, which must end on the surface of the sphere, and saying that a projection maps this to a two-dimensional vector $\mathbf{p}$: $\mathbf{p} = \mathbf{P}(\mathbf{e})$. However, it is probably more useful to view a projection as a mapping from one pair of coordinates (the latitude and longitude ϕ and λ) to another pair, the Cartesian coordinates of $\mathbf{p}$: $\mathbf{p} = \mathbf{P}(\phi, \lambda)$. Note that while $\mathbf{p}$ is a vector in two dimensions, the spherical coordinates are not a vector in the sense we have been using.

A spherical surface cannot be mapped to a plane without some distortion. We can treat this distortion exactly as we do that of a material: to make this concrete, imagine that we have a globe made from a sheet of rubber, and flatten this sheet into a plane. This will distort the sheet (indeed tear it), and these distortions will be both large and nonuniform, making this a case of finite inhomogeneous strain. However, locally we can regard the distortion as nearly constant, and treat it using the results of the previous section for finite homogeneous strain: each local distortion can be described as a combination of rotation and stretching along two orthogonal axes. This means that a circle traced around some point on the sphere will be mapped into an ellipse (perhaps rotated) in the plane that the sphere is mapped into. The cartographic term for this strain ellipse is **Tissot's indicatrix**.

Figure 3.5 shows four examples of map projections, with the indicatrices shown on a grid of latitude-longitude points. The top two projections are **conformal**, meaning that they produce, locally, no change of shape (or equivalently, no changes in angles). The indicatrices therefore retain their original circular shape, but their size changes because of the scale variations that must occur in any conformal projection from the sphere to the plane. The upper left map is the familiar Mercator projection, showing the equally familiar exaggeration of scale close to the poles. Note that for this map there is no rotation, which is what makes the Mercator projection so useful for many purposes. The upper right shows a (deservedly) less familiar conformal projection, to illustrate that conformality can coexist with relative rotations. The lower left map is an **equal-area** projection, so the indicatrices all have the same area, but are increasingly distorted away from the center. Finally, the lower right map is an azimuthal equidistant projec-

[6] Unfortunately there is no easy way to avoid the dual use of "map" in both its specifically cartographic and more general mathematical sense.

tion, which has the specialized property that lines from the center show true great-circle distance and direction from that point; as the varying shapes and sizes of the indicatrices show, this projection is neither conformal nor equal-area.