SIO 223B–Problem Set 1

1.1 A simple way to compute the derivative of a function sampled at equal intervals Δ is to take first differences. Write down the formula for the frequency response of a first-differencer, assuming the samples are taken at times $t = n\Delta$, n from $-\infty$ to ∞. Compare this to the frequency response for ideal differentiation, in both amplitude and phase. What if the samples are taken at $t = (n + \frac{1}{2})\Delta$?

1.2 The differential equation for the position q of the mass of a seismometer subject to an acceleration a is

$$\ddot{q} + 2\gamma\omega_0\dot{q} + \omega_0^2 q = a$$

where γ is the fraction of critical damping and ω_0 the natural frequency (in rad/s). Suppose that a varies sinusoidally with radian frequency ω (which is related to the frequency f by $\omega = 2\pi f$), solve this equation to get q. What is the frequency response of this system? Compute it and plot the response (in amplitude and phase) for γ equal to .0001 (nearly undamped), .1 (underdamped), 1 (critically damped) and 10 (overdamped). (You will find this most instructive if you plot the amplitude and its log, and plot against both a linear and logarithmic frequency). Which of these would you expect to give the highest-fidelity record of the acceleration, assuming that we can record q directly? Will the record a better representation of the acceleration at low or high frequencies? What does the frequency response look like (for γ = 1) if the input is taken to be displacement? (Hint: how is acceleration related to displacement for sinusoidal variation?). In what frequency range does the motion of the mass best reflect the displacement?

1.3 The equation for a 1-D propagating wave in an attenuating medium is

$$e^{i(\omega t - kx)}e^{-\frac{\pi x}{\lambda Q}}$$

where λ is the wavelength, ω is the radian frequency, and k is the radian wavenumber; Q is a measure of the attenuation. How are ω and k related to λ and to the velocity of propagation c? Considering this as a linear system (input at $x = 0$, output at x, what are the amplitude and phase of the frequency response? Suppose c is 8 km/s and Q is 100; plot the amplitude and phase of the frequency response for x equal to 1, 100, and 10,000 km. Use dB for the amplitude, and a log frequency scale.

1.4 Suppose we feed a real sinusoidal input into a system that is

- Slightly nonlinear: $y = x + \varepsilon x^2$, with ε being small compared to typical values of x.
- Slightly time-varying: $y = x(1 + \varepsilon \cos 2\pi f_0 t)$; ε is << 1. Describe the output in each case, as seen in the frequency domain. (A transform should not be needed—and you may find it better, for this problem, to use a cosine, *not* complex exponentials).

1.5 What is $\int_{-\infty}^{\infty} \frac{\sin(\pi m(x-y))\sin(y)}{(x-y)}\,dy$, for arbitrary m? (No integrals needed: use the Fourier theorems).

1.6 Using the Fourier theorems (no integrals if you can help it):

- Show that the Fourier transform of $\frac{\sin \pi t}{\pi(t-1)}$ is $-e^{-2\pi i f}\Pi(f)$.
- Find the convolution of e^{-at^2} with e^{-bt^2}.
- If $X(f) = |\mathrm{sinc}(f)|$, what is the autocorrelation of $x(t)$? (Note that you do not need to find $x(t)$, which is *not* $\Pi(t)$.)

1.7 Let $E[x]$ and $O[x]$ be the even and odd parts of $x(t)$, and $\mathbf{F}$ be the Fourier transform operation.

- Show that

$$\mathbf{F}[x] = 2\int_0^{\infty} E[x]\cos 2\pi ft\,dt - 2i\int_0^{\infty} O[x]\sin 2\pi ft\,dt$$

- How is $\mathbf{F}[\mathbf{F}[x]]$ related to x?
- Show that $\mathbf{F}[\mathbf{F}[\mathbf{F}[\mathbf{F}[x]]]] = x$.

1.8 What is $\int_{-\infty}^{\infty} \delta(at)dt$?

1.9 A seismic body-wave pulse in displacement is described by a half-cycle cosine:

$$x(t) = \begin{cases} \cos(\pi t/\tau) & |t| \le .5\tau \\ 0 & \text{otherwise} \end{cases}$$

Find the transform of this pulse. Define the "corner frequency" to be the frequency at which the zero-frequency value of the transform crosses the high-frequency "asymptotic envelope" (this is the line [on a log-log plot] that successive maxima of the absolute value of the transform touch, as f becomes large). What is this frequency? (You may find it helpful to make a log-log plot, since then it will be clearer that we are fitting two aymptotes, the zero-frequency value being one; but the result can be found analytically).