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THE TIME-DOMAIN BEHAVIOR OF POWER-LAW NOISES

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Abstract. The power spectra of many geophysical phenomena are well approximated by a power-law dependence on frequency or wavenumber. I derive a simple expression for the root-mean-square variability of a process with such a spectrum over an interval of time or space. The resulting expression yields the power-law time dependence characteristic of fractal processes, but can be generalized to give the temporal variability for more general spectral behaviors. The method is applied to spectra of crustal strain (to show what size of strain events can be detected over periods of months to seconds) and of sea level (to show the difficulty of extracting long-term rates from short records).

1. Introduction

Many types of geophysical data come from processes so complex that their outcome is best taken to be random, even if the underlying physics is not; the most efficient characterizations of such data are likely to be a statistical model. The aim of this paper is to develop a useful relation for a particular (but common) class of such models, and show several applications if it.

The particular type of model considered might be called the power-law process. This is a one-dimensional stochastic process whose behavior in the time domain (or space domain if appropriate) we denote by $x(t)$; this time-domain behavior is such that its power spectrum has the form

$$P_x(f) = P_0(f/f_0)^{\nu} \tag{1}$$

where f is spatial or temporal frequency, P_0 and f_0 are normalizing constants, and ν is the spectral index. The reason for adopting this form is that it is observed to be a good fit to the spectra of a wide variety of geophysical phenomena, often over many decades of frequency. The index ν often falls in the range -3 to -1, meaning that the energy at low frequencies exceeds that at high frequencies (a "red" spectrum). Spectra of this form have been found for bathymetry with $\nu = -2.3$ [Malinaverno, 1989]; fault and joint geometry, with $\nu = -2$ [Power *et al.*, 1987]; and crustal deformation, with $\nu = -2.7$ [Wyatt *et al.*, 1988].

Despite the ubiquity of stochastic processes with power-law spectra, they have received relatively little attention in the statistical literature. The only exception has been the case $\nu = -2$, which corresponds to a random walk (Brownian motion): this is the integral of white noise (for which $\nu = 0$), the spectral index being shifted to -2 by the operation of integration (and squaring to get power). Mandelbrot and Van Ness [1968] developed mathematical forms for processes that have power spectra close to (1), those with $-3 < \nu < -1$ being termed "fractional Brownian motions," and those with $-1 < \nu < 1$, "fractional Gaussian noises." Mandelbrot [1983] provides a general discussion of these, and Feder [1988] a readable introduction.

For clarity, it should be noted that the models introduced by Mandelbrot do not have exact power-law spectra [Graf 1983], and indeed are more usually discussed in terms of their behavior under changes in time scale (assuming them to represent a time series). In this view, the important parameter becomes a number H, called by Mandelbrot [1983] the Hurst exponent; as we will see below, $H = -\frac{1}{2}(\nu + 1)$. In practice, both Mandelbrot's models and the spectral form (1) are mathematical idealizations, and in different situations either one might be the better description of actual data. I have chosen to use (1) as a model because my main purpose is the interpretation of spectra, for which (1) can easily be generalized (Section 4). It is worth noting that recent statistical studies [Mohr, 1981; Graf, 1983] have shown that the best method for determining H is to compute the power spectrum, fit a function of the form (1) to it, and then find H from the ν so determined; in that view ν could be regarded as the more fundamental parameter.

However, it should be noted that (1) is not in general a complete specification of a process; the power spectrum is only a summary of second moments (variance versus frequency). Stochastic processes with identical power spectra can have very different appearances in the time domain [Press, 1978]. I address this point more fully below.

2. Time-Domain Variation

The question to be addressed here is how to go from the power spectrum (1) to the variation of the process over time T; that is, to the statistics of

$$y_T(t) = x(t+T) - x(t) \tag{2}$$

This quantity was introduced by Kolmogorov in studies of the theory of turbulence, and its second moment, under the name of structure function of x, has seen wide use in meteorology and elsewhere [Lindsay and Chie, 1976]. The attraction of looking at the variation of x over a fixed time T is that $y_T(t)$ is often stationary, and thus easily characterized statistically, even when $x(t)$ is nonstationary (as must be true for $\nu \le -1$). But the statistics of y_T may often be as of much interest as those of x; in particular, if we want to decide whether some recent fluctuation in $x(t)$ is consistent with its past behavior it is to the distribution of y_T that we must turn. Unfortunately, if we choose to summarize this past behavior as having a spectrum of the form (1), the fractional Brownian motions of Mandelbrot and Van Ness [1968] turn out to be very inconvenient, since the expression for the power spectrum of such processes is extremely complicated [Graf, 1983; Geweke and Porter-Hudake, 1983]. There is thus no simple relationship between the spectrum of such processes and their variation in the time domain.

There is however a relatively simple method whereby we can relate the spectral level (1) to the distribution of $y_T(t)$, provided that we only aim to find only the second moment, or variance, of y_T (denoted by $\langle y_T^2 \rangle$), which, as noted above, is the structure function. This restriction is of course unavoidable given that our basic description is the power spectrum. Specifying only the second moment is adequate to define the distribution of y_T if it is Gaussian. This restriction will not usually be seriously violated for real data, but should be kept in mind before inferences about the complete distribution are made from the value of $\langle y_T^2 \rangle$.

To determine $\langle y_T^2 \rangle$ from the spectrum, we observe that $y(t)$ is derived from $x(t)$ by convolution; we can rewrite (2), in the notation of Bracewell [1965], as

$$y(t) = x(t) * [\delta(t-T) - \delta(t)] \tag{3}$$

Straightforward application of the results of Fourier theory then shows that the power spectrum of y_T is given by

$$P_y(f) = |G_T(f)|^2 P_x(f)$$

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where $G_T(f)$ is the frequency response of the convolution filter, used to produce $y(t)$ from $x(t)$ namely

$$|G_T(f)|^2 = 4\sin^2 \pi f T$$

Finally, since the variance of a random process is equal to the integral of its power spectrum we find

$$<y_T^2> = \int_0^\infty P_y(f)df = \int_0^\infty 4P_x(f)\sin^2 \pi f T \; df \tag{4}$$

Other methods than (2) of forming auxiliary series exist, and some have long been used in (for example) studies of oscillator stability. Rutman [1978] discusses many of these and shows how the transfer-function approach just discussed can be used to derive their behavior for different spectra; while he discusses the structure function, it is not treated very fully because of its limited usefulness in oscillator studies.

If we now specialize $P_x(f)$ to the form (1) and make the change of variable $u = \pi f T$, we find

$$<y_T^2> = \frac{4P_0}{f_0^\nu} \frac{T^{-(\nu+1)}}{\pi^{\nu+1}} \int_0^\infty u^\nu \sin^2 u \; du = C_\nu \frac{P_0}{f_0^\nu} T^{-(\nu+1)} \tag{5}$$

This immediately implies that the standard deviation of y_T, $\sigma_T = (<y_T^2>)^{1/2}$, is proportional to $T^{-(\nu+1)/2}$, or T^H in Mandelbrot's notation; for $\nu = -2$, we find σ_T proportional to $T^{1/2}$, the familiar result for Brownian motion. The definite integral in (5), and thus the coefficient C_ν, can be found in closed form:

$$C_\nu = \frac{-1}{2^{\nu+1}\pi^\nu \Gamma(-\nu)\cos(\nu\pi/2)}$$

which for $\nu = -2$ gives $C_\nu = 2\pi^2$. Figure 1 shows $C_\nu^{1/2}$ over the range $-3 < \nu < -1$, and illustrates how the expression (4) goes to infinity at both limits of this range because of the divergence in the integral in (5). These divergences occur at opposite limits of the integral; put crudely, as ν approaches -3, the low-frequency fluctuations in $x(t)$ become so large that $y(T)$ becomes nonstationary, while as ν approaches -1, the high-frequency fluctuations in $x(t)$ approach an ultraviolet catastrophe, with infinite variance at high frequencies. This latter divergence will not be a concern in practice, and could be eliminated in the theory by replacing the δ-functions in (3) with finite-width sampling functions.

A major assumption has been passed over in using equation (1) to go from equation (4) to equation (5); namely, that while (4) presupposes $x(t)$ to possess a power spectrum, any process with an apparent spectral index less than -1 must be nonstationary, so that its spectrum does not exist: a contradiction more apparent than real. It is true that (1), with $\nu < -1$, cannot describe the spectrum of a stationary process. However, if we suppose the

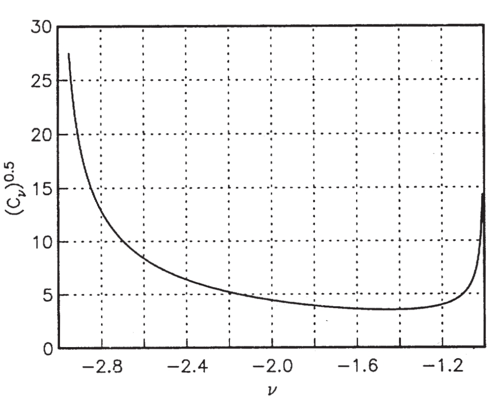

Fig. 1. The coefficient C_ν for the time-domain wander of a pure power-law noise (equation 5), as a function of the spectral index ν.

$P_x(f)$ to be described by (1) for $f > f_b$, and to be (say) constant at $P_0(f_b/f_0)^\nu$ for $f < f_b$, $x(t)$ will be a stationary process, the operations leading from (4) to (5) will be valid, and, the result in (5) will be essentially unchanged as long as $T \ll f_b^{-1}$. While introducing a cutoff frequency is in one sense arbitrary, it must exist for any actual process (see, for example, Keshner [1982] for such a model for $1/f$ noise). Since the finite span of our observations will always render us incapable of observing such a cutoff, there seems to be no reason to avoid introducing it to avoid the difficulties into which a too-strict adherence to an ideal mathematical model would otherwise lead us [Slepian, 1978].

An important generalization is to note that (4) applies for a general spectral shape $P_x(f)$, provided that at low frequencies $P_x(f)$ increases less rapidly than f^{-3}, and at high frequencies decreases more rapidly than f^{-1}. Of course, we then will not usually be able to find a closed-form expression for $<y_T^2>$, but must calculate it numerically. This allows us to proceed even in the face of the departures from power-law (or fractal) behavior noted, for example, by Gilbert [1989]. A simple spectral shape which fits many spectra quite well is a piecewise power-law form; for i=1, . . .

$$P_x(f) = P_i(f/f_i)^{\nu_i} \quad f_{i-1} < f \le f_i \tag{6}$$

with f_0 being set to zero. The integral (4) then becomes the sum of integrals over each frequency interval. Because of the oscillatory nature of the integrand and the wide range in frequency, the numerical integration must be done with some care. For $\pi f T \ll 1$, we may approximate $\sin \pi f T$ by $\pi f T$; for large values of this quantity we use $\sin^2 \pi f T = \frac{1}{2}(1 - \cos 2\pi f T)$, which leads to two integrals, one of which can be done analytically, the other of which, though still requiring numerical integration, becomes small for $\pi f T$ large.

3. Comparison with Spectral Crossover Approach

Agnew [1987] described another problem relating to power-law processes: how to compare a record whose errors are of this form with measurements with independent error σ made at regular time intervals Δ. (This characterizes the problem of comparing crustal deformation measurements made using strainmeters and tiltmeters with those made by geodetic methods.) For a spectral index less than -1, there will be some frequency at which the fluctuations in the power-law errors equal the error gotten by averaging the independent measurements. At a higher frequency the power-law errors will be smaller and using records with such errors will give a better result; at lower frequency the independent errors, suitably averaged, will be superior.

This problem is easily solved if cast in spectral form (Figure 2). The spectrum of the independent-error measurements must be constant, with level P_m, from 0 to the Nyquist frequency, $(2\Delta)^{-1}$. Since the integral of the spectrum is the variance, $P_m = 2\sigma^2\Delta$. This equals the power-law spectrum (1) at a crossover frequency

$$f_c = \left[\frac{2\sigma^2 \Delta f_0^\nu}{P_0}\right]^{1/\nu} \tag{7}$$

which thus sets the boundary between one or another process having a lower level. The crossover frequency can be equally easily obtained graphically for a spectrum of more general shape, though a closed-form expression becomes cumbersome. To give a concrete example, we may compare the strain spectrum shown in Figure 3 with repeated distance measurements with a σ of 10^{-7}. If these were made weekly, the equivalent spectral level would be 1.2×10^{-8} ε^2/Hz, or -79 dB, giving a crossover frequency corresponding to a period of 300 days; if they were made daily, this period becomes 200 days.

The theory developed in Section 2 gives another way of looking at this problem. At a period $T_c = f_c^{-1}$, the rms fluctuation in the power-law process will be (from (5) and (7))

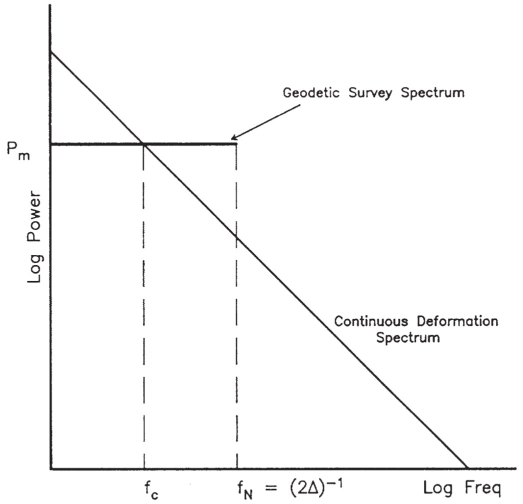

Fig. 2. Cartoon of the relationship between the power spectrum of a sampled measurement with independent errors, and a continuous measurement subject to low-frequency divergence. See Section 3 for details.

$$<y_{T_c}^2>^{1/2} = C_v^{1/2} \left[\frac{P_0}{f_0^v}\right]^{-v/2} 2^{(v+1)/(2v)} \sigma^{(1+v)/v} \Delta^{(1+v)/(2v)}$$

The error in the independent-measurement series, suitably averaged, will be $N^{-1/2}\sigma$ where N is the number of measurements; obviously for regular sampling $N \approx T_c/\Delta$. Again using (7), we find

$$\frac{\sigma N^{-1/2}}{<y_{T_c}^2>^{1/2}} = \frac{1}{(2C_v)^{1/2}}$$

which is always less than one. This is as it should be, since at the crossover frequency the independent measurements should be capable of resolving the fluctuations of the power-law series.

4. Applications

The top panel of Figure 3 shows the spectrum of earth strain at Piñon Flat Observatory. The peak at high frequencies (0.1 Hz) is caused by microseisms; the narrow peaks around multiples of 1 cycle/day (1.16×10^{-5} Hz) are caused by earth tides and thermal effects. Except for these narrow peaks the spectrum is very well fit by equation (6), the piecewise power-law model. The lower panel of Figure 3 shows the value of $<y_T^2>^{1/2}$ computed from this model, and also for one in which the spectrum is assumed to fall off as f^{-2} above 0.1 Hz, as would be true if the data were highpassed to remove microseismic energy. At periods from 10 to 100 seconds, the wander is less than 0.1 $n\varepsilon$, which may be taken to be the resolution limit of this data for rapid changes such as the coseismic offsets discussed by Wyatt [1988]. At times longer than a few days the fluctuations increase steadily.

Figure 4 shows similar results for sea level. In this case the high-frequency peak is caused by ocean swell, and narrow peaks are caused by the tides, but the spectrum as a whole can be fit by a few power-laws. If we compute the wander from this spectrum, we obtain a nearly constant value (dotted line in the lower panel): the variation caused by swell dominates over all other changes out to periods of years (since we have removed the tides). If we filter out the swell, we obtain a wander of about 1 cm for periods from minutes to hours, increasing at longer times. However, sea level is not usually evaluated as a point measurement; rather, the usual approach is to take mean values over longer times. We can easily use the techniques of Section 2 to find the wander of two adjoining means, averaged over the same length of time as separates them: the variance of this quantity (twice the two-

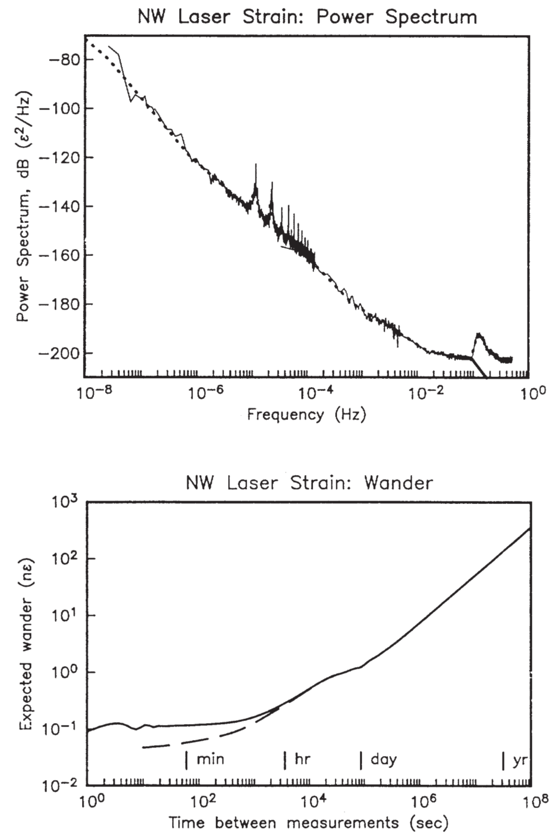

Fig. 3. The upper panel shows the power spectrum of earth strain at Piñon Flat Observatory in southern California, as measured with the NW-SE laser strainmeter there [Wyatt, 1988]. The spectrum from 3.3×10^{-5} to 0.5 Hz is based on 1-second data collected between day 31:17 and 33:14 of 1990; this has been processed with two different section lengths to improve the stability of the spectrum estimate at the higher frequencies. The spectrum from 2.6×10^{-8} to 1.5×10^{-4} Hz is based on hourly data collected between 1985:001 and 1990:003, with the larger gravity and thermal tides removed. The dotted line shows the spectral model used to compute the wander shown in the lower panel. In the lower panel, the solid line is the wander computed from the full spectrum and the dotted line that computed if the microseism peak is filtered out (as indicated by the solid line in the top panel).

sample Allan variance described by Rutman [1978]) is given by an integral similar to equation (4), but with $\sin^4 \pi f t / f^2$ replacing $\sin^2 \pi f t$. The dashed line in the lower panel of Figure 3 shows this quantity, and indicates that the standard deviation of (for example) annual means of sea level is 4 cm. In this case what may be of more interest is the rms rate of change over time T, gotten by dividing the wander by T. Neither version of wander in Figure 4 increases faster than T, so the rms rate of change will diminish with increasing T. While this lends some confidence to attempts to use these data to determine true secular increases in sea level (e.g., Barnett [1983]), it is nevertheless clear that short records are basically useless for this purpose; any rates determined from them will be dominated by random changes from various sources [Sturges, 1987].

The wander in strain shown in Figure 3 increases with larger T more rapidly than the sea level wander does, but still at a rate less than T, and in fact about as $T^{1/2}$. If we looked at rates, we would again see that the longer the time interval over which we compare strains, the slower the apparent rate. It has been a frequent observation of geologists that many rates of deformation, examined

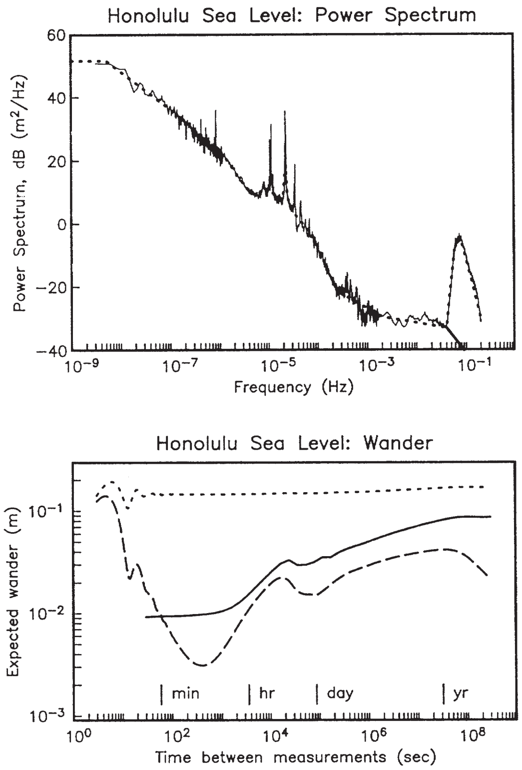

Fig. 4. The upper panel shows the power spectrum of sea level at Honolulu. The spectrum from 0.001 to 0.2 Hz is based on 2-second averages of sea-bottom pressure records, collected between 1964:216 and 1964:226; from 3.5×10^{-5} to 0.0017 Hz it is based on a similar record with 5-minute sampling collected between 1964:246 and 1965:29 [Snodgrass *et al.*, 1966; Snodgrass 1964]. The spectra at longer frequencies are based on hourly tidal heights measured with a conventional float gauge with the largest tides removed [Munk and Cartwright 1966]; from 10^{-6} to 3.5×10^{-5} Hz the data span was 1905:001 to 1947:158, and from 3×10^{-9} to 10^{-6} Hz the data span was 1905:007 to 1973:214. The dotted line shows the spectral model used to compute the wander shown in the lower panel. In the lower panel, the dotted line is the wander computed from the full spectrum and the solid line that computed if the high-frequency peak is filtered out; the dashed line is the wander for adjacent averages over an interval equal to the time between measurements.

over long times, appear to be much less than those determined over shorter times (for example, geodetically). But such behavior is exactly what would be expected if the deformations being considered, like those of shorter period shown in Figure 3, had a power-law spectrum with index between −3 and −1. Such a stochastic model allows us to reconcile apparent changes of rate with uniformitarianism of rates (the "null" hypothesis of Gilluly [1949]): the present will always appear to be the most active period, whenever it happens to be.

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References

Agnew, D. C., On noise levels in deformation measurements: comparison in the frequency domain, *U.S. Geol. Surv. Open-File Report, 87-591*, 839-844, 1987.

Barnett, T.P., Possible changes in global sea level and their causes, *Climate Change 5*, 15-38. 1983.

Bracewell, R., *The Fourier Transform and its Applications*, 381 pp., McGraw-Hill, New York, 1965.

Feder, J.V., *Fractals*, (Plenum Press, New York), 1988.

Geweke, J., and S. Porter-Hudake, The estimation and application of long memory time series models, *J. Time Ser. Analy., 4*, 222-238, 1983.

Gilbert, L. E., Are topographic data sets fractal?, *Pure Appl. Geophys., 131*, 241-254, 1989.

Gilluly, J., Distribution of mountain-building in geologic time, *Geol. Soc. Amer. Bull., 60*, 561-590, 1949.

Graf, H.-P, Long-range correlations and estimation of the self-similarity parameter Spectra of fractals, Ph.D. thesis, 205 pp., Eidegenoissche Technische Hochschule, Zurich, 1983.

Keshner, M.S., 1/f noise, *Proc. IEEE, 70*, 212-218, 1982.

Lindsay, W., and C. Chie, Theory of oscillator instability based upon structure functions, *Proc. IEEE, 64*, 1652-1666, 1976.

Malinaverno, A., Testing linear models of sea-floor topography, *Pure Appl. Geophys., 131*, 139-155, 1989.

Mandelbrot, B., *The Fractal Geometry of Nature*, 466 pp., W. H. Freeman, San Francisco, 1983.

Mandelbrot, B., and J. Van Ness, Fractional Brownian motions, fractional noises, and applications, *SIAM Rev., 10*, 422-439, 1968.

Mohr, D.L., Modeling data as a fractional Gaussian noise, Ph.D. thesis, 150 pp., Princeton University, 1981.

Munk, W., and D. Cartwright, Tidal spectroscopy and prediction, *Phil. Trans. Roy. Soc., Ser. A, 259*, 533-581, 1966.

Power, W.L., T.E. Tullis, S.R. Brown, G.W. Boitnott, and C.H. Scholz, Roughness of natural fault surfaces *Geophys.. Res. Lett., 14*, 29-32, 1987.

Press, W.H., Flicker noises in astronomy and elsewhere, *Comments. Astrophys. 7*, 103-119, 1978.

Rutman, J., Characterization of phase and frequency instabilities in precision frequency sources: fifteen years of progress, *Proc. IEEE, 66*, 1048-1075, 1978.

Slepian, D., On bandwidth, *Proc. IEEE, 64*, 292-300, 1976.

Snodgrass, F. E., Precision digital tide gauge, *Science, 146*, 198-208, 1964.

Snodgrass, F. E., G. W. Groves, K. F. Hasselmann, G. R. Miller, W. H. Munk, and W. H. Powers, Propagation of ocean swell across the Pacific, *Phil. Trans. Roy. Soc., Ser. A, 259*, 431-497, 1966.

Sturges, W., Large-scale coherence of sea level at very low frequencies, *J. Phys. Oceanog., 17*, 2084-2094, 1987.

Wyatt, F., Measurements of coseismic deformation in southern California: 1972–1982, *J. Geophys. Res., 93*, 7923-7942, 1988.

Wyatt, F., S.-T. Morrissey, and D. C. Agnew, Shallow borehole tilt–A reprise, *J. Geophys. Res., 93*, 9197-9201, 1988.

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