On rates of occurrence of geomagnetic reversals

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Abstract

The magnetostratigraphic time scale provides a record of the occurrence of geomagnetic reversals. The temporal distribution of reversals may be modelled as the realization of an inhomogeneous renewal process; i.e., one in which the intensity, $\lambda(t)$, or reversal rate is a function of time. Variations in reversal rate occurring on time scales of tens of millions of years are believed to reflect changes in core-mantle boundary conditions influencing the structure of core flow and the field produced by the geodynamo. We present a new estimate for reversal rate variations as a function of time using nonparametric adaptive kernel density estimation and discuss the difficulties in making inferences on the basis of such estimates. Using a technique proposed by Hengartner and Stark (1992a; b; 1995), it is possible to compute confidence bounds on the temporal probability density function for geomagnetic reversals. The method allows the computation of a lower bound on the number of modes required by the observations, thus enabling a test of whether "bumps" are required features of the reversal rate function. Conservative 95% confidence intervals can then be calculated for the temporal location of a single mode or antimode of the probability density function. Using observations from the time interval 0–158 Ma, it is found that the derivative of the rate function must have changed sign at least once. The timing of this sign change is constrained to be between 152.56 and 22.46 Ma at the 95% confidence level. Confidence bounds are computed for the reversal rate under the assumption that the observed reversals are a realization of an inhomogeneous Poisson or other renewal process with an arbitrary monotonically increasing rate function from the end of the Cretaceous Normal Superchron (CNS) to the present, a zero rate during the CNS, and a monotonically decreasing rate function from M29R at 158 Ma to the onset of the CNS. It is unnecessary to invoke more than one sign change in the derivative of the rate function to fit the observations. There is no incompatibility between our results and a recent assertion that there is an asymmetry in average reversal rate prior to and after the CNS, when the CNS is assumed to be a period of zero reversal rate. Neither can we use our results to reject an alternative hypothesis that rates are essentially constant from 158 to 130 Ma, and from 25 Ma to the present, with an intermediate nonstationary segment. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Since the 1960s, there has been a steadily improving record of the times of occurrence of geomagnetic reversals during the last 160 Ma. Although these times are relatively well documented, the details of the geodynamo mechanism are still poorly understood, and it seems unlikely that geomagnetic reversals will ever be predictable in a deterministic sense. In the absence of deterministic models, we resort to a statistical characterization of the kind first proposed...
by Cox (1968). He noted the irregular lengths of stable polarity intervals and suggested (following the description of Parker, 1955 of the geodynamo) that one might model reversals as a renewal process triggered by instabilities in fluid motions in Earth’s outer core.

Under the Poisson model (the simplest example of a renewal process), one expects (assuming symmetry between the reverse and normal polarity states of the field) that although the lengths of individual stable polarity intervals are highly variable, there is a characteristic rate of reversal if the process is averaged over a sufficient time interval. The observed distribution of reversals exhibits two departures from the basic Poisson model. Firstly, there is a paucity of short intervals in the magnetostratigraphic record: this could be caused by censoring, that is our inability to detect very short intervals (e.g., McFadden, 1984a; McFadden and Merrill, 1984); alternatively, it might reflect an inhibition or inability of the core to generate new reversals during some short time interval immediately following a transition (McFadden and Merrill, 1993). A consequence of the lack of short intervals is that interval lengths are better fit by a gamma distribution, a model that has been widely exploited since its introduction by Naidu (1971). Merrill et al. (1996) provide a review of such analyses.

The second interesting feature is that the rate of occurrence of geomagnetic reversals appears to be time-dependent; this has been interpreted as a reflection of gradually changing conditions at the core–mantle boundary. The time constants associated with such a process should be consistent with those expected for mantle convection. It is of some interest to characterize the temporal variation in the reversal rate function; in particular, it has often been suggested that there are periodic or quasiperiodic fluctuations in the rate as a function of time. This has generated a lively debate in the literature regarding the significance of such features (see, e.g., Mazaud et al., 1983; Negi and Tiwari, 1983; McFadden, 1984b, 1987; Lutz, 1985, 1987; Raup, 1985; Stothers, 1986; Stügler, 1987; Lutz and Watson, 1988). The time constants typically attributed to these variations are short when compared with overturn times for the mantle, and one would therefore have to invoke short-term variations at the CMB if they were substantiated. More recently, longer-term changes in reversal rates have been linked to plume activity at the core–mantle boundary (Loper, 1992; Loper and McCartney, 1986; Courtillot and Besse, 1987; Larson and Olson, 1991), or the arrival at the CMB of cold material from a mantle flushing event (Gallet and Hulot, 1997). McFadden and Merrill (1997) have suggested an asymmetry in changes in reversal rate prior to and after the Cretaceous Normal Superchron (CNS), along with speculations about the kinds of changes in CMB conditions that might cause these differences. There are of course innumerable scenarios that can be invoked to explain the observations, many of which are essentially untestable. However, some progress has recently been made in investigating the effects of lateral variations in heat flux at the CMB, through varying the boundary conditions applied in recent geodynamo simulations (Bloxham, 1998; Roberts and Glatzmaier, 1998; Glatzmaier et al., 1999). Glatzmaier et al. (1999) show that more frequent reversals occur in their simulated geodynamo, when the thermal structure of the mantle constrains the pattern of heat flux from the core to differ significantly from that preferred by the rotationally dominated dynamics of the core. Although temporal variation in reversal rate has more usually been attributed to changing CMB conditions, it could also be a manifestation of changes at the inner-core/outer-core boundary. Hollerbach and Jones (1993a; b; 1995) have shown that the presence of a finitely conducting inner core is a crucial element in stabilising numerical dynamo models to produce more Earth-like behaviour. It is therefore perfectly plausible to suppose that growth of the inner core might in the past have had significant long-term effects on geomagnetic field structure and reversal rates (see, e.g., Kent and Smethurst, 1998). Even though the inner core is presumably only growing slowly at this stage of Earth’s history, we cannot rule out the possibility that a feature like the CNS might arise from the growth of the inner core past a critical size, so that the geometry of the outer core no longer favours the dynamics that generate instabilities in the geomagnetic field.

In this paper, we adopt the statistical description for reversal occurrences and attempt to put bounds on the behaviour of the reversal rate function using methodology developed by Hengartner and Stark.
1992a; b) and first applied to the geomagnetic field by Constable (1993). Firstly, we construct new estimates of the geomagnetic reversal rate as a function of time, then we assess the reliability of such features as bumps in the reversal rate curve. Under the assumption that geomagnetic reversal times can be modelled by a renewal process with a time-dependent rate parameter, we are able to show that the observations are entirely consistent with a piecewise monotonically changing reversal rate, the rate decreasing in the early part of the record and increasing more recently. However, they are inconsistent with a rate function that remains constant or monotonic throughout the time interval 0–160 Ma. We place confidence limits on the time interval within which the derivative of the rate function changed sign. We compute bounds on the reversal rate probability function under the assumptions that: (i) the reversal times are samples from an inhomogeneous renewal process; and (ii) the rate function is monotonically decreasing prior to the CNS, zero during the CNS, and monotonically increasing since the end of the CNS. The procedure is applied to several different time scales. The conclusions about monotonicity hold in all cases, but the computed bounds on the rate function vary somewhat with the details of the chronology.

2. Nonparametric estimates of temporal variations in reversal rate

The time scales used in this study are those of Harland et al. (1990) and Cande and Kent (1995), and a hybrid generated from combining the two. The time scale provides us with a list of numbers, \( t_i \), \( i = 1, \ldots, N \), that are the approximate times at which the geomagnetic field polarity altered. Initially, we will regard those changes as instantaneous; they are generally thought to occur over time spans of the order of 1000 to 10 000 years (e.g., Bogue and Merrill, 1992), which is short compared with the interreversal time, and almost certainly better than the resolution of the magnetic anomaly time scale. Harland et al. (1990) estimate 20 000 years as the achieved resolution of their time scale. We label their time scale GTS89: it contains 292 transitions. (Cande and Kent, 1992, 1995) present two versions of their time scale, one in which they attempt to produce a scale with uniform resolution throughout; they believe this contains almost all polarity intervals longer than 30 000 years. We will designate this version CK95. In addition, they also document finer scale features from the magnetic anomaly record, that they call cryptochrons. The version including these other events will be referred to as CK95cc. Our version of CK95 differs in one instance from that published in Table 2 of Cande and Kent (1995): we have omitted the Reunion event, C2r.1n, because under the 1995 recalibration it became shorter than 30 000 years. We designate C2r.1n a cryptochron and include it in CK95cc. CK95 contains 181 transitions, while CK95cc has an additional 112 events for a total of 293. Some of the cryptochrons correspond to short events of stable magnetic polarity, while others may reflect changes in the paleointensity of the field. A distinction between these two causes cannot be made solely on the basis of marine magnetic anomaly data, other paleomagnetic observations are required to distinguish the cause of the observed anomalies. It is possible that in statistical terms the kind of event that causes a cryptochron may be the same as that which generates a reversal: the distinction is that one perturbs the status quo sufficiently that equilibrium cannot be restored and a reversal takes place. The ability to detect cryptochrons depends on a number of variables, including local magnetic mineralogy, spreading rate and water depth where the magnetic data are collected, so in addition to documenting different kinds of events, the resolution of CK95cc varies along its length. CK95 and GTS89 show substantial differences where they overlap, reflecting the use of different ages in the absolute calibration and improved magnetic anomaly data in CK95: one very obvious age difference is in the onset of the CNS itself which is placed at 124.32 Ma in GTS89 and 118 Ma in CK95. It is likely that the most robust current estimate for the time scale results from a hybrid, using CK95 from the present to 83 Ma, and GTS89 from 83 to 158 Ma, because Pick and Tauxe (1993) present arguments that favour the onset of the CNS being closer to 124 Ma than 118 Ma. We designate this time scale CK95.hybrid or CK95cc.hybrid depending on whether it includes cryptochrons and use it for our analyses. We will assume for the time
being that each time scale is complete in the sense that it contains every reversal that occurred, and that their times are known precisely; the implications of this assumption for our results will be discussed later.

We begin by constructing an estimate of a function that we will regard as a kind of probability density function of reversals as a function of time. At time, \( t \in [0,T] \), this function is denoted \( r(t) \) and \( r(t)\,dt \) provides a measure of how likely we are to find a reversal if we look in a time interval of length \( dt \) about the point \( t \), given that a total of \( N \) reversals took place over the interval \([0,T]\). If we regard reversals as a deterministic process whose occurrence times are predictable, then we can place a delta function at the point in time where each event occurs, and the resulting probability density function will be zero everywhere else. However, as was indicated earlier, we lack the ability to predict reversals, and at present, our best model for the generation of reversals is that of a renewal process triggered by random instabilities in fluid motions in Earth’s outer core. Given that this is the case, we can regard the observed sequence of geomagnetic reversals as a sample of size \( N \) from such a random process. We want to reconstruct the temporal distribution function from this sample; that is, we want to estimate the rate function for the underlying process at any given time in the interval \([0,T]\), without making assumptions about its parametric form. This is a standard problem in probability density function estimation; however, while there are many ways of constructing estimates of the density function (see, e.g., Silverman, 1986), it is difficult to calculate confidence bounds for them and thus assess the reliability of such estimates. Here, we construct a plausible estimate of \( r(t) \) and then use various forms of hypothesis testing to assess the reliability of some of its more interesting features.

We use a kernel estimate for the density function (Silverman, 1986). In the simplest form, the density of reversals is found by centering a Gaussian kernel on each observation time and adding up the contributions of all these Gaussian functions; i.e., we have:

\[
r(t) = \frac{1}{N} \sum_{i=1}^{N} g_i(t),
\]

where \( t_i \) is the occurrence time of the \( i \)th reversal, \( N \) is the total number of reversals and \( g_i \) is the normalized Gaussian probability density function:

\[
g_i(t) = \frac{1}{\sqrt{2\pi} \sigma_i} \exp\left[-\frac{1}{2} \left(\frac{t - t_i}{\sigma_i}\right)^2\right].
\]  

\( r(t) \) has the properties of a probability density function; it is positive, the total area under the curve is one, and the area under the curve between any two time points gives the fraction of reversals occurring in that time interval. It is clear that the amount of detail found in \( r(t) \) is dependent on how large we make \( \sigma_i \) for the Gaussian kernels. Large standard deviations will generate very smooth functions, while small standard deviations will generate functions with an implausible amount of structure. This illustrates a fundamental limitation in density estimation that is basically the same as in estimating power spectral density: there is a trade off between the twin evils of bias and variance. In addition, the density estimate is likely to be poor near the ends of the bounded interval \([0,T]\); the interval chosen is imposed purely by the temporal limitations of our dataset, and it is implausible to suppose that there are no reversals outside this designated time span. Silverman (Chap. 2) proposes several remedies for the bounded domain issue: the most appropriate for our purposes is the reflection technique, in which the data are augmented by adding the reflections of the data in each boundary so that, for example, beyond \( t = 0 \), we enlarge the dataset from \( \{t_1,t_2,\ldots,t_N\} \) to \( \{-t_N,\ldots,-t_2,-t_1,t_1,t_2,\ldots,t_N\} \). An estimate is constructed for the enhanced dataset of size \( 2N \), then rescaled to provide an estimate based on the original data. This is analogous to assuming that the density estimate should be considered stationary near the boundary, and will produce density estimates with zero slope at \( t = 0 \).

Silverman (1986), Chap. 3, discusses in some detail the choice of optimal kernel width, and shows that small values of \( \sigma_i \) are desirable to eliminate bias, but will result in very large values of integrated variance. The compromise is to choose kernels that minimize the mean integrated squared error, the sum of the variance and the squared bias. He also shows that the choice of the particular kind of kernel func-
tion does not greatly affect the efficiency of the estimation; for example, even a rectangular kernel has an efficiency of 0.9295, and the Gaussian 0.9512 relative to the optimal kernel. Unfortunately, the optimal kernel width that minimizes the mean integrated squared error depends on the unknown density that we wish to estimate, so there is no magic bullet that can be used to determine the \( \sigma_i \). Furthermore, it would seem sensible for \( \sigma_i \) to depend on the local density of data points with broader kernels in more sparsely populated regions. Some asymptotic results are available that provide guidelines for choosing kernel widths for symmetric unimodal density functions, and reasonable damage control in the face of long-tailed distributions such as the log-normal and t-distributions, along with mechanisms for adapting the width to the local density of data. Silverman’s suggestion of:

\[
\sigma_i = \sigma = 0.9 AN^{-1/3},
\]

with \( A = \min \) (standard deviation, interquartile range/1.34) for the dataset was used for an initial pilot estimate for the density function \( \tilde{r}(t) \). Then the estimate is refined by defining local bandwidth factors, \( \lambda_i \), for each data point derived from the pilot estimate by:

\[
\lambda_i = \left( \frac{\tilde{r}(t_i)}{h} \right)^{-\alpha},
\]

where \( h \) is the geometric mean of the \( \tilde{r}(t_i) \):

\[
\log h = N^{-1} \sum_t \log \tilde{r}(t_i),
\]

and \( \alpha \) is a parameter that determines the sensitivity to local structure \( 0 \leq \alpha \leq 1 \). Silverman outlines arguments in favour of using \( \alpha = 1/2 \). The adaptive kernel estimate \( \tilde{r}(t) \) is defined as:

\[
\tilde{r}(t) = N^{-1} \sum_{i=1}^{N} \lambda_i^{-1} g_i \left( \frac{t}{\lambda_i} \right).
\]

Fig. 1(b) shows the related function, \( \tilde{R}(t) = N \tilde{r}(t) \), for the CK95.hybrid and CK95cc.hybrid time scales, all computed with Gaussian kernels with local bandwidths defined according to the above recipe. For comparison, a reversal rate for CK95.hybrid computed using a conventional 50-point running mean is plotted as the plus signs. The running mean generates more structure in the reversal rate curve, but suffers from a number of disadvantages. There is an implicit assumption that the rate can be regarded as constant over the length of the window; rate estimates are unavailable near the ends of the interval, and near the boundaries of the CNS if we choose to consider that as a time of zero reversal rate separating two distinct regimes; furthermore, the individual estimates are clearly not independent because adjacent estimates have many points in common. It is difficult to know how much of the structure we should try and interpret. In contrast the adaptive density estimates are quite smooth, and almost entirely lacking in structure. We can assess visually the amount of bias by comparing the cumulative integral of \( \tilde{R}(t) \) up to time \( t \) with number of reversals occurring during the same time interval. This is shown in Fig. 1(a), where the symbols are the empirical cumulative distribution obtained from the age.

Fig. 1. Reversal rate function \( \tilde{R}(t) \) and cumulative number of reversals, \( N(t) \) as a function of time for CK95.hybrid (heavy black line) and CK95cc.hybrid (dashed black line). Rate functions are constructed using adaptive Gaussian kernels as described in the text. Gray symbols with linear interpolant in lower part of figure are 50 point sliding window estimates of reversal rate from CK95.hybrid.
scale (we will denote this by \(N_\tau(t)\)), and the smooth line \(C(t)\):

\[
C(t) = \int_{t_0}^{t} \hat{R}(t')dt'.
\]

(3)

We see that in several regions where \(N_\tau(t)\) has a lot of curvature, \(C(t)\) fits the data rather poorly, and between 120 and about 155 Ma, it is biased towards large values. One might plausibly argue that we would be justified in looking for more detail in the reversal rate function, particularly at times when the rate is high, and that the smoothness is a result of the approximately bimodal structure in the reversal rate function. The adaptive estimators of optimal kernel width are known to perform rather poorly for bimodal densities for which the modes are separated by more than a few standard deviations. In an attempt to mitigate this problem, Fig. 2 shows an alternative adaptive estimate for \(R(t)\) in which the record is split into two sections, one spanning the interval 160 to 83 Ma and the other from 83 Ma to the present. There is substantial additional structure with a corresponding reduction in misfit to the observations, but as indicated before we must expect that the estimates so derived have significantly larger variance than those in Fig. 1.

The differences between Figs. 1 and 2 highlight the difficulty in making reliable estimates of density functions, and without direct knowledge of the true density function it is difficult to put error bars on these estimates. However, we have so far avoided the necessity of making assumptions about the parametric form of the density function. We use these estimates as a guide to further hypothesis testing about the reversal process, keeping in mind that the actual data we can bring to bear on this problem are not the reversal rate estimates, but the observed occurrence times.

3. Renewal processes

Poisson processes are the simplest example of a renewal process. Geomagnetic reversals have been modelled as an example of a Poisson process (see, e.g., Cox, 1968; Phillips, 1977; McFadden, 1984a; McFadden and Merrill, 1984, 1993, 1997) with simple assumptions made about the way the rate function varies with time. The validity of this procedure must depend on the degree to which they satisfy the basic conditions for a Poisson process. Conditions for a Poisson process of rate \(\lambda\) are (e.g., Cox and Lewis, 1966) that as \(h \to 0\):

(i) the number of reversals in disjoint time intervals are independent, i.e., \((0, t]\) does not affect \((t, t + h]\);

(ii) chance of a reversal in an interval of length \(h\) is approximately proportional to \(h\) with a remainder that is \(o(h)\);

(iii) the chance of more than one reversal in an interval of length \(h\) is \(o(h)\).

\[
[ g(h) = o(h) \text{ if } \lim_{h \to 0} g(h)/h = 0 ].
\]

Suppose reversals are generated by an inhomogeneous Poisson process with intensity (or rate) \(\lambda(t)\), then:

\[
\lambda(t) = \lim_{h \to 0^+} \frac{N(t + h) - N(t)}{h} = \frac{dN}{dt},
\]

(4)

where \(N(t)\) is the number of reversals that have occurred by time \(t\). \(N(t)\) is a random variable if reversals are generated by a Poisson process.
It is now widely accepted (see Merrill et al., 1996) that even if we allow for a temporal variation in the rate function, condition (i) above is not satisfied. Firstly, the reversal process itself is not instantaneous, but may take as long as ten or twenty thousand years when the accompanying variations in field intensity are considered. We obviously cannot consider the probability of a reversal occurring during this time interval to be the same as during full polarity intervals. McFadden and Merrill (1993) have also suggested that there is a recovery period immediately following a reversal during which the probability of generating another polarity flip is substantially diminished. The observational support for their arguments seems somewhat weak in the sense that where detailed records of reversals are available they often seem to exhibit multiple flips back and forth between polarity states. Some reversals such as the Brunhes exhibit precursory events (e.g., Hartl and Tauxe, 1996) suggesting an enhanced rather than diminished propensity for reversals during times of low geomagnetic field intensity.

Aside from the complications of describing a physical system by an idealized statistical model, there is the additional problem that observations of marine magnetic anomalies, from which the bulk of the time scale is derived, have a finite (and variable) resolution so that short polarity intervals may be concatenated together. McFadden (1984a) showed that concatenation of intervals generated from a Poisson process will result in polarity intervals that are well modelled by a gamma distribution, with probability density function for polarity interval lengths of:

$$g(t) = \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \quad t \geq 0. \quad (5)$$

A Poisson process is the special case of a gamma distribution for which $k = 1$ and interval lengths are exponentially distributed. The parameter $k$ is called the shape parameter for the gamma distribution, while $\lambda$ is a scale parameter. Varying $k$ changes the shape of the density, whereas varying $\lambda$ corresponds to a change in the units of measurement. In the reversal process, we can ascribe physical interpretations to these parameters: we can think of $\lambda$ as the long-term reversal rate after the recovery phase is over far from a reversal, while $k$ controls the shape of the probability function immediately following a polarity change. Values of $k < 1$ correspond to an enhanced probability (relative to a Poisson process with the same value of $\lambda$) of another transition immediately following one that has just occurred. Values of $k > 1$ indicate a diminished probability of another transition immediately following any given one. (McFadden and Merrill, 1984; McFadden and Merrill, 1993) have modelled observed transition times for the last 160 Ma by a gamma distribution, with piecewise linear trends (from 84 to 0 Ma and from 160 to 118 Ma) in $\lambda$ as a function of time. The results of this analysis generate values of $k$ ranging between 1 and 2 for the Cenozoic and between 1 and 4 for the less well-constrained Albian/Aptian part of the sequence. McFadden and Merrill (1993) note that at times with high reversal rates $k$, and thus, in their interpretation inhibition, tends to be greatest. An alternative possibility is that censoring or inability to detect short events has a greater influence when reversal rate is high, because there will be more short intervals to miss. The resolution of whether censoring or inhibition is what influences estimates of $k$ here is unlikely to come from the timescale data alone: here, we simply note that resolution at time intervals much less than 20 ka is unlikely to be uniform throughout the time scale, which might well lead to variations in estimates of $k$ with time.

More recently, Gallet and Hulot (1997), noting the notorious difficulty in making accurate estimates of changing parameters for a gamma process from the comparatively small number of reversals available, have suggested that the available evidence might support a model in which the rate is stationary prior to 130 Ma, and from 25 Ma to the present. The intermediate time is seen as a period with steadily increasing rate following a rapid decrease into the CNS: it is suggested that the nonstationarity might be caused by the arrival and subsequent acclimatization of a substantial amount of cold material at the CMB.

In order to proceed with our analysis, we will follow the model espoused by McFadden and Merrill (see, e.g., McFadden and Merrill, 1997) over the last decade or more, and also used by Gallet and Hulot (1997), which supposes that the generation of reversals is "essentially Poisson", and investigate what
constraints we can infer on the temporal dependence of the probability density for occurrence of reversals.

3.1. A temporal distribution function for poisson generated reversals

We look first at the case where the reversal rate is constant in time, i.e., \( \lambda(t) = \lambda \). Then the probability of observing \( n(T) \) events in an arbitrary interval of length \( T \) has a Poisson distribution with:

\[
\text{prob} \{ N(T) = n \} = \frac{(\lambda T)^n e^{-\lambda T}}{n!}, \quad n = 0, 1, \ldots,
\]

and the number of reversals expected in the interval \( T \) is just given by the expected value for \( N(T) \):

\[ E[N(T)] = \lambda T. \]  

3.2. Nonstationary process

If we now allow temporal variation in \( \lambda \), we have:

\[
\text{prob} \{ N(T) = n \} = \frac{[\Lambda(T)]^n e^{-\Lambda(T)}}{n!}, \quad n = 0, 1, \ldots,
\]

where:

\[ \Lambda(T) = \int_0^T \lambda(t) \, dt, \]

and the expected value for the random variable \( N(T) \) is

\[ E[N(T)] = \sum_{j=0}^{\infty} j \frac{\Lambda(j)}{j!} e^{-\Lambda(T)} = \Lambda(T). \]

We want to estimate \( \lambda(t) \) from the occurrence times of reversals, which requires that we relate \( \lambda(t) \) to a probability density conditional on \( N \). For a stationary Poisson process, reversals are just uniformly distributed in time. For a nonstationary process, we can also derive the expected temporal distribution as follows.

Suppose we observe \( n \) reversals by time \( T \), (i.e., \( N(T) = n \)), at times \( t_i \), \( i = 1, \ldots, n \). Then what we want is the conditional probability distribution for the distribution of reversals given that \( N(T) = n \):

\[
F_n(t) = \text{prob} \{ t_j \leq t | N(T) = n \} = \frac{\Lambda(t)}{\Lambda(T)} , \quad 0 \leq t \leq T.
\]

The probability density function for reversal times is \( dF_n(t)/dt \):

\[
f(t) = \frac{\lambda(t)}{\Lambda(T)} , \quad 0 \leq t \leq T.
\]

This relates \( f(t) \) the temporal distribution of reversals to reversal rate \( \lambda(t) \). Note that \( f(t) \) and \( \lambda(t) \) will have the same shape although we do not actually know \( \Lambda(T) \), the total number of reversals to be expected; in particular, when \( \lambda(t) \) is monotonic so is \( f(t) \).

4. A confidence region for the temporal distribution function for reversals

The actual data on geomagnetic reversals are the occurrence times, and they provide us with an empirical estimate \( \hat{F}_n(t) \) for \( F_n(t) \) defined in Eq. (11) as the distribution of reversal times over the interval \((0,T)\). We have:

\[
\hat{F}_n(t) = \frac{1}{n} \sum_{j=1}^{n} 1_{t \geq t_j},
\]

where:

\[ 1_{t \geq t_j} = \begin{cases} 1, & t \geq t_j \\ 0, & t < t_j \end{cases} \]

What we would like to do is use the information provided in \( \hat{F}_n(t) \) to put bounds on the behaviour of \( f(t) \) as defined in Eq. (12). To do this, we make use of methodology described by Hengartner and Stark (1992a; b; 1995). The Kolmogorov–Smirnov distance between \( F \), the true distribution from which our data are drawn, and \( \hat{F} \), our estimate is the maximum absolute distance between them:

\[
\| F - \hat{F} \| = \sup \{ |F(t) - \hat{F}(t)| \}.
\]
We can make a probabilistic statement about \( F - \hat{F} \) (Massart, 1990). If \( F \) is the true probability distribution from which the data are drawn, then:

\[
\text{prob}\{\| F - \hat{F} \| \leq \chi \} \geq 1 - \alpha ,
\]

where:

\[
\chi = \chi_\alpha(\alpha) = \sqrt{\frac{2}{\alpha} \ln \frac{\alpha}{2n}} .
\]

So \( \{ F, \| F - \hat{F} \| \leq \chi \} \) is a \( 1 - \alpha \) confidence region for \( F \). Fig. 3 shows the 95% confidence regions for CK95 and CK95cc.hybrid. We see from Fig. 3 that we can think of the empirical \( \hat{F}(t) \) as data with confidence limits on them. Predictions from any candidate probability density function \( f(t) \) must lie within the specified confidence limits for it to be considered a viable model for reversal rates.

### 4.1. Confidence envelopes for monotonic \( f(t) \)

If we are prepared to assume that \( f(t) \) is monotonically increasing or decreasing, then Hengartner and Stark (1992b; 1995) show that it is possible to obtain confidence envelopes for the reversal rate density \( f(t) \). We find largest and smallest values at a point \( t \) of monotonic density functions \( f(t) \) that are consistent with the confidence region defined for \( F \).

Let:

\[
w_j = t_{j+1} - t_j, \quad j = 1, \ldots, n - 1
\]

with \( t_j \) the times of reversals. Hengartner and Stark (1992a; b; 1995) show that the end points of a \( 1 - \alpha \) confidence region for \( f(t) \) are found by the two linear programs — (1) max \( \beta_{k-1} \), (2) min \( \beta_k \) — subject to the constraints:

1. \( \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n \geq 0 \) monotonicity and positivity,
2. \( \sum_{j=1}^n w_j \beta_j = 1 \) density integrates to 1,
3. \(-\chi + m/n \leq \sum_{j=1}^m \beta_j w_j \leq \chi + (m - 1)/n, \quad m = 1, \ldots, n \) density lies in confidence set.

Because of the monotonicity assumption, the limits acquired at times \( t_j \) can be interpolated conservatively to provide a confidence envelope for the density \( f(t) \) at all times. If no feasible solution can be found to the linear program, then this indicates that the observations are inconsistent with the assumption of monotonicity — a constant or steadily decreasing \( f(t) \) cannot fit the observations to within the prescribed confidence limits. This is a powerful means of deciding whether the lumps and bumps observed in the reversal rate functions of Figs. 1 and 2 are actually required by the observations.

![Fig. 3. A confidence region for the temporal distribution function for reversals derived from CK95 and CK95cc.hybrid. Dashed lines are the confidence region for the empirical function (solid). Gray lines are predicted from the adaptive estimates of Fig. 2 (for CK95) and Fig. 1 (CK95cc.hybrid).](image)
5. Testing hypotheses about reversal rates

As we noted earlier, a wide variety of interesting behaviour has been attributed to the reversal rate function. In this section, we endeavour to assess the realism of some of these claims, while making the minimum number of assumptions. We begin with the simplest question, namely are the data consistent with a constant or increasing (or decreasing) rate with time? We apply it to CK95.hybrid, CK95cc.hybrid. Note that this places no constraints on how rapidly the rate changes, the only requirement is that the sign of the rate derivative \( \frac{d\lambda}{dt} \) should not change. The results are perhaps unsurprising: although CK95 and CK95cc are both compatible with a monotonically increasing rate since the beginning of the CNS (or even earlier), there is no monotone rate function that will satisfy the data over the longer time interval 0–158 Ma. There is no feasible solution to the algorithm described in Section 4.1 for 0–158 Ma.

Thus we can say with some confidence that a Poisson process with monotonic rate (either increasing or decreasing) is not a viable model for the reversal process over the past 158 Ma: at some point in the interval 0–158 Ma, the rate derivative is required to change sign. It might seem obvious to the reader that we should choose the CNS as the place where this occurs. Since no reversals occurred the rate is apparently zero during that time. However, we need not make this assumption, and indeed we ought not to with this statistical model, since a different realization of the reversal process would produce a different record of reversal times. We can instead use the observations to find a 95% confidence interval for the time during which \( \frac{d\lambda}{dt} \) must have changed sign. This is once again accomplished by solving linear programs described by Hengartner and Stark (1995), but with the monotonicity constraints modified from those described in Section 4.1. Instead of using \( \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n \geq 0 \), we successively relax the monotonicity constraint (but not positivity) beginning with \( \beta_n \) until we find a feasible solution with the constraint:

\[
\beta_1 \geq \beta_2 \geq \ldots \geq \beta_j \leq \beta_{j+1} \ldots \leq \beta_n.
\]

The linear program is modified in the monotonically increasing region to find (1) max \( \beta_{k+1} \) and (2) min \( \beta_k \) so that the confidence envelopes can be constructed in the same way as before. The smallest value of \( j \) with a feasible solution provides the upper 95% confidence limit for the time at which \( \frac{d\lambda}{dt} \) must have changed sign. We can do a similar exercise at the lower end of the interval starting with the constraint that:

\[
\beta_1 \leq \beta_2 \leq \ldots \leq \beta_j \leq \beta_{j+1} \ldots \leq \beta_n,
\]

and successively relaxing the less than or equality constraint and altering it to a greater than or equality constraint, beginning with \( \beta_1 \), and adding in \( \beta_2 \ldots \beta_n \). The value of \( j \) that provides the first feasible solution then allows us to identify the lower bound on the position of the antimode for the density function. The result of this exercise is a confidence interval for the position of the antimode under the assumption that the density function is piecewise monotonic, with decreasing rate from the present to the antimode interval and increasing prior to that time. The 95% confidence interval for the sign change in \( \frac{d\lambda}{dt} \) is 22.46–152.56 Ma. This interval may seem disappointingly wide, however, it reflects the difficulty in putting confidence intervals on pointwise estimates of the rate at times when very few events are occurring. It serves to emphasize that although we believe there are no reversals during the CNS, we cannot infer from this that the Poisson process rate \( \lambda(t) \) was zero during this interval, only that \( \frac{d\lambda}{dt} \) must have changed sign at some point, and under the piecewise monotonicity assumption, we can, with 95% confidence, place this sign change between 22.46 and 152.56 Ma. We cannot use these bounds to rule out the scenario proposed by Gallet and Hulot (1997), in which it is suggested the onset of nonstationary field behaviour occurs at 130 Ma; the data are entirely compatible with a statistical model in which the intrinsic reversal rate has steadily increased since 152.56 Ma. One consequence of this is that if we wish to retain the stochastic model for reversals, and seek correlations between observed reversal times (or lack thereof) with other geomagnetic or geological phenomena, then we need to widen the temporal bounds within which such correlations might be manifest.

We return now to the problem of putting confidence bounds on reversal rate over the time interval 0–158 Ma, and solve for these bounds using one
Fig. 4. Confidence limits on the reversal rate function for CK95.hybrid (thin black line). Heavy dashed line is the adaptive estimate from Fig. 2, gray symbols from sliding window omitting that part of the sliding window estimate influenced by the CNS. Note the assumption that the rate is zero during the CNS.

Further assumption. We suppose that the reversal rate is zero between 124.3 and 83.0 Ma and add this as an additional constraint. Fig. 4 shows the 95% confidence limits for the reversal rate, under the assumption that it decreased monotonically from 158 to 124 Ma, was zero throughout the CNS, and monotonically increased from 83 Ma to the present. The adaptive kernel density estimate lies well within these bounds except during the CNS when the rate is forced to be zero. Without the assumption that this rate is zero, the calculated bounds are quite similar except that the upper bounds sit at around 2 Ma⁻¹ across the CNS. The adaptive kernel estimates during the CNS are quite plausible in that they decay toward zero several million years after the onset of the CNS, and start to increase before the end of it. Even though the reversal process may have turned off during the CNS, there is nothing that requires the timing to coincide with that of the last reversal beforehand or the first afterwards: it may have taken several million years for the final changes to take place that apparently altered the core dynamics so that reversals no longer occurred. Although we cannot reject the reversal rate estimates derived from the 50-point sliding window, the adaptive kernel estimates have the advantage that we do not have to assume anything about how the rate changes as a function of time, and also provide us with estimates throughout the time interval for which observations are available.

Differences in the results between CK95.hybrid and CK95cc.hybrid allow us to make some assessment of how inadequate knowledge of the time scale may affect the conclusions reached so far. The basic conclusion is independent of the details of the time scale. Regardless of whether we use CK95.hybrid or CK95cc.hybrid, we find that the data are compatible with a monotonically increasing rate from the onset of the CNS to the present, and with a monotonically decreasing rate from 158 to 83 Ma. The confidence bounds for CK95cc.hybrid are shown in Fig. 5. Again the adaptive kernel estimate lies well within the calculated limits, but in this case, the sliding window estimate falls significantly outside in two intervals. There is no inconsistency in this result: it arises from the fact that the sliding window estimates are not required to satisfy the data bounds like those shown in Fig. 3. The fact that CK95cc cannot be distinguished from the other time scales on the basis of its statistical behaviour leaves open the question of whether the cryptochrons should be regarded as a

Fig. 5. Same as Fig. 4, but for CK95cc.hybrid.
different kind of event from that required to generate a full polarity reversal. A related but distinct question is whether there are cryptochrons during the CNS.

6. Concluding remarks

Under the assumption that geomagnetic reversals are generated by a renewal process with a time varying rate parameter, we have been able to show that none of the time scales studied here require the existence of bumps in the reversal rate curve since 158 Ma. Thus, we can say with some force that earlier discussions of periodic changes in reversal rate center on features that are not required by the observations. However, the hybrid timescales do require the existence of a single antimode: we can say with some confidence that the derivative of the rate function has changed sign at some time between 158 Ma and the present. The algorithm of Hengartner and Stark has been used to show that at the 95% confidence level, the sign change is constrained to occur between 22.46 and 152.56 Ma.

The model presented by McFadden and Merrill (1984), in which reversal rate is modelled by two intervals prior to and after the CNS, each with linearly varying rate changes, is seen to be entirely compatible with the observations. Also is the more recent model of Gallet and Hulot (1997), which proposes nonstationary rates between 130 and 25 Ma. We should also comment on a recent claim by McFadden and Merrill (1997) of confirmation that the rate of decrease of $l$ on entering the CNS is greater than that on leaving. The confidence envelopes derived here for rates in the two time intervals have significant overlap, showing that these pointwise estimates of reversal rate cannot be used to distinguish reversal rates at 125 Ma from those at 82 Ma, for example. McFadden and Merrill (1997) concluded that the times taken for 50 reversals to occur over each of two intervals prior to and after the CNS were extremely unlikely to have been drawn from Poisson processes with the same rate parameter. In each case, the rate was higher before than after the CNS, suggesting a faster decrease than subsequent increase. Again, there is no incompatibility with the model found in this work. The strength of their conclusion relies on the assumption of constant rates over a 50-point sliding window, and the fact that they only attempt to define a difference in average reversal rates over windows ranging in length from 15 to 46 Ma.

The 95% confidence envelopes for the rate function typically have a width of about three or four reversals per million years, except near the ends of the intervals where they are much wider, because the monotonicity assumption provides little constraint on the upper bound outside the time interval for which we have measurements. More definitive results concerning the details of reversal rate changes require that we consider average properties over longer time intervals.

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References


