SIOG 231: GEOMAGNETISM AND ELECTROMAGNETISM

Chapter 12: Toroidal and Poloidal Fields, Magnetohydrodynamics, and the Geodynamo

1. Introduction

Now we turn to a brief discussion of a fundamental limitation of observations made outside the magnetic source region in Earth's core. Before we start, let us assemble two useful vector calculus identities, numbers 5 and 9 in our list:

$$\nabla \times (\mathbf{A}s) = (\nabla \times \mathbf{A})s \; - \; \mathbf{A} \times \nabla s \tag{90}$$

$$\nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A} \; + \; \nabla(\nabla \cdot \mathbf{A}). \tag{91}$$

1:1 Toroidal and Poloidal Fields

Our magnetic field modeling so far has only dealt with solutions of Laplace's equation in regions free of sources. In an earlier lecture we showed that the magnetic field can always be written as the curl of the magnetic vector potential $\mathbf{A}$, *i.e.*, $\mathbf{B} = \nabla \times \mathbf{A}$.

Now we consider a sphere of radius c (Earth's core) surrounded by an insulator and divide the vector potential into parts parallel to and perpendicular to $\mathbf{r}$ by writing

$$\mathbf{A} = T\mathbf{r} \; + \; \nabla P \times \mathbf{r} = T\mathbf{r} \; + \; \nabla \times (P\mathbf{r}) \tag{92}$$

where T and P are scalar functions of $\mathbf{r}$, known as the defining *scalars of the toroidal and poloidal fields*. That this is always possible is shown in detail in *Foundations*, Chapter 5. To find $\mathbf{B}$ we take the curl:

$$\begin{aligned} \mathbf{B} &= \nabla \times (T\mathbf{r}) \; + \; \nabla \times \nabla \times (P\mathbf{r}) \\ &= \mathbf{B}_T \; + \; \mathbf{B}_P \end{aligned} \tag{93}$$

and $\mathbf{B}_T$ is called the *toroidal part* of $\mathbf{B}$, while $\mathbf{B}_P$ is the *poloidal part.* This decomposition for $\mathbf{B}$ is unique and can always be done for all solenoidal vector fields (those with $\nabla \cdot \mathbf{F} = 0$). We are very familiar with the idea of a potential for $\mathbf{B}$ in an insulator; when $\mathbf{J}$ does not vanish, we need two scalars, not one to describe $\mathbf{B}$ completely. Conventionally, the scalars are always restricted to a class of functions whose average value over every sphere is zero, that is

$$0 = \int_{S(r)} T(r\hat{\mathbf{r}}) \, d^2\hat{\mathbf{r}} = \int_{S(r)} P(r\hat{\mathbf{r}}) \, d^2\hat{\mathbf{r}} \tag{94}$$

With this property, the scalars become unique in (92), which means that if $\mathbf{B}_T$ vanishes, then $T = 0$, and similarly for $\mathbf{B}_P$.

Using the vector identity (90) on the toroidal part of the field, we see

$$\begin{aligned} \mathbf{B}_T &= (\nabla \times \mathbf{r})T \; - \; \mathbf{r} \times \nabla T \\ &= - \, \mathbf{r} \times \nabla T \end{aligned} \tag{95}$$

because $\nabla \times \mathbf{r} = 0$; hence $\hat{\mathbf{r}} \cdot \mathbf{B}_T = 0$ always; in other words, the toroidal magnetic field has no radial component – it is a tangent vector field on every concentric spherical surface. The lines of force lie on spherical surfaces and are thus confined to the interior of the conducting sphere. If we think of the sphere as Earth's core, outside the core we have (for the sake of argument) $\mathbf{J} = 0$ and

$$\mathbf{B} = -\nabla\Psi, \quad \text{with } \nabla^2\Psi = 0. \tag{96}$$

Now $\mathbf{B}$ is continuous at $S(c)$ and since $\hat{\mathbf{r}} \cdot \mathbf{B}_T = 0$ just inside we conclude that $\hat{\mathbf{r}} \cdot \mathbf{B}_T$ is also zero just outside the core. But we showed in Part I section 8, that any harmonic function with internal sources and vanishing radial component on $S(c)$ is identically zero outside. Therefore $\mathbf{B}_T$ vanishes outside the conducting sphere. Hence the toroidal part of $\mathbf{B}$ in Earth's core is invisible outside the core and only the poloidal part, $\mathbf{B}_P$, has any detectable influence at the Earth's surface.

When we neglect the displacement current term in Maxwell's equations we can make a similar decomposition for the current flow in core, as we now show:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{97}$$

As we saw earlier, taking the divergence yields that $\mathbf{J}$ is solenoidal

$$\nabla \cdot \mathbf{J} = 0 \tag{98}$$

and thus we can also find a unique decomposition into toroidal and poloidal parts for the current density

$$\mathbf{J} = \mathbf{J}_T + \mathbf{J}_P. \tag{99}$$

By substituting for $\mathbf{B}$ in terms of $\mathbf{B}_T$ and $\mathbf{B}_P$ in (93) we will show that $\mathbf{B}_P$ comes from $\mathbf{J}_T$, that is, poloidal fields are generated by toroidal currents and similarly that $\mathbf{B}_T$ comes from $\mathbf{J}_P$, so toroidal fields are generated by poloidal current systems.

Here is the proof: take the curl of (93)

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} = \nabla \times \nabla \times (T\mathbf{r}) + \nabla \times \nabla \times \nabla \times (P\mathbf{r}). \tag{100}$$

It is obvious from the first term on the right that the toroidal part of $\mathbf{B}$ is associated with a poloidal current, whose scalar is just T/μ_0. To show the last term, with three curls, is in fact toroidal, we analyze $\mathbf{S} = \nabla \times \nabla \times (P\mathbf{r})$. From the identity (91)

$$\mathbf{S} = -\nabla^2(\mathbf{r}P) + \nabla(\nabla \cdot (\mathbf{r}P)). \tag{101}$$

We expand the first term in (101) with the Einstein notation:

$$\nabla^2(\mathbf{r}P) = \partial_j\partial_j(x_k P) = \partial_j((\partial_j x_k)P + x_k\partial_j P) = \partial_j(\delta_{jk}P + x_k\partial_j P) \tag{102}$$

$$\begin{aligned} &= \partial_k P + \partial_j(x_k\partial_j P) = \partial_k P + (\partial_j x_k)\partial_j P + x_k\partial_j\partial_j P \\ &= \partial_k P + \delta_{jk}\partial_j P + x_k\partial_j\partial_j P = 2\partial_j P + x_k\partial_j\partial_j P \\ &= 2\nabla P + \mathbf{r}\nabla^2 P. \end{aligned} \tag{103}$$

Then (101) becomes

$$\begin{aligned} \mathbf{S} &= -\mathbf{r}\nabla^2 P + \nabla(\nabla \cdot (\mathbf{r}P) - 2P) \\ &= -\mathbf{r}\nabla^2 P + \nabla(r\partial_r P + P). \end{aligned} \tag{104}$$

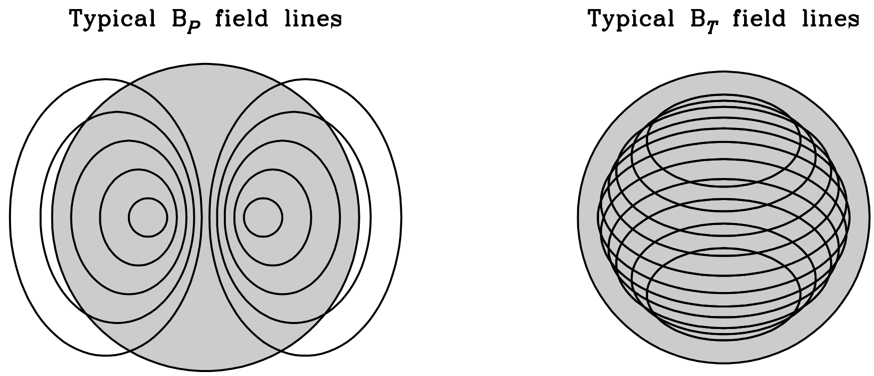


Figure 3.5.1

Finally, we take the curl of (104), noting the vanishing of the term involving a grad because $\nabla \times \nabla = 0$, so that

$$\nabla \times S = -\nabla \times (\mathbf{r}\nabla^2 P)$$

and we recover (100)

$$\mu_0 \mathbf{J} = -\nabla \times (\mathbf{r}\nabla^2 P) + \nabla \times \nabla \times (T\mathbf{r}) = \mu_0(\mathbf{J}_T + \mathbf{J}_P). \tag{105}$$

Thus the toroidal part of the current is derived from the toroidal scalar $-\nabla^2 P/\mu_0$.

The lengthy algebra may obscure what a toroidal or poloidal magnetic field might actually look like. The sketches in Figure 3.5.1 give an example of the simplest types of fields. You can easily see the toroidal system on the right as the currents generating the poloidal field on the left. But if you reverse the process, the field lines that stray outside the conductor (gray area) on the left are not allowed. Obviously not every toroidal or poloidal field has axial symmetry like these.

So we know that the geomagnetic field that we see is connected to a poloidal field in the core, which is generated by a toroidal current system J_T. The following nice result is also true (shown first by Gubbins 1975, and in more detail in *Foundations*, section 5.5, although there is an error of a factor 4π in the bounds given there):

$$\mu_0^2 \int_{r<c} \mathbf{J}_T \cdot \mathbf{J}_T d^3\mathbf{r} \geq c \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{(l+1)(2l+1)^2(2l+3)}{l} |b_{lm}|^2 \tag{106}$$

where the coefficients b_l^m are those obtained on the surface of the core, whose radius is c. The amount of heat generated by ordinary Ohmic losses (also called *Joule heating*) at a point is given by $\mathbf{J} \cdot \mathbf{E}$. Thus (106) allows us to compute the minimum Joule heating from toroidal currents:

$$Q = \int_{r<c} \frac{\mathbf{J}_T \cdot \mathbf{J}_T}{\sigma} d^3\mathbf{r}$$

$$\geq \frac{c}{\mu_0^2 \sigma} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{(l+1)(2l+1)^2(2l+3)}{l} |b_{lm}|^2. \qquad (107)$$

By using Q as a regularizing norm, we can find the minimum amount of heating in the core associated with the poloidal magnetic field, the only part we can see. Of course the actual ohmic heating could be much larger than this because of the poloidal current term which remains invisible to us and because the estimate from (96) is a lower bound on toroidal current power in any case. It is controversial in dynamo theory whether the toroidal component of **B** in the core is large compared with the poloidal part. Some models predict $|\mathbf{B}_T| \sim 50 \times |\mathbf{B}_P|$.

The Joule heating in the core is probably a small fraction of the heat budget (see the exercise), but it is believed that in certain neutron stars, called *magnetars*, the collapse of the magnetic field (with intensity 10^9 T!) is the major source of energy in the body. See S. Kalkarni, *Nature*, v 419, pp 121-2, 2002.

Exercise:

(a) Make an estimate of the minimum rate of Joule heating generated in core. Compare this with terrestrial heat flow at the surface.

(b) A toroidal magnetic field $\mathbf{B}_T$ fills a conducting sphere of radius a; its scalar is the function $T(\mathbf{r})$. Imagine creating a contour map of the values of T on the surface $S(b)$ with $b < a$. What connection does the contour map of the scalar T have with the magnetic field $\mathbf{B}_T$?

2. MAGNETOHYDRODYNAMICS IN THE CORE

Observations of the geomagnetic field show that the internal part changes on times scales ranging from about a year upwards. It is generally believed that secular variation at shorter time scales will be greatly attenuated by passage through the Earth's mantle which is only approximated by an insulator (perhaps more on this later). Observationally, the secular variation is usually distinguished from the more rapidly varying external field by its time spectrum (recall Figure 1.5). However, it should be noted that the external field generated by magnetic storms does not average to zero, and the number of these storms is greater at times of sunspot maximum than at sunspot minimum. Since the sunspot cycle has an approximate periodicity of 11 years (and there are also longer term contributions to solar variability, often associated with so-called grand minima or grand maxima of solar activity see for example, Usoskin et al, 2007, DOI: 10.1051/0004-6361:20077704) there is some overlap in the time spectra of internal and external field variations. Geomagnetic jerks, which are defined by very rapid changes in the second derivative of the field with time, are also believed to be of internal origin.

In the magnetic field models that we looked at in an earlier lecture the temporal variation is incorporated by allowing the Gauss coefficients to vary as a function of time; the temporal variation of the g_l^m, h_l^m can be constrained to be smooth through a parameterization in terms of cubic or higher order splines and the minimization of a penalty function like (68) involving the second temporal derivative of **B**, as well as some penalty on spatial structure, like (76)-(79). Spherical harmonic models now extend back to 100 ka, although the level of detail in paleofield models is much less than in those constructed from modern observations with good spatial coverage (see Treatise on Geophysics, volume 5, Chapter 9 for a review). Further back in time it is possible to reconstruct time variations in the dipole moment back to about 2 Ma, and intermittently at more remote times in the geological past. Because of lack of detailed age constraints in paleomagnetic work statistical representations of geomagnetic field variability are useful for earlier times. The occurrence times for geomagnetic reversals are well documented back to about 200 Ma via marine magnetic anomalies, and there have been numerous attempts to link changes in reversal frequency to long term changes in field strength.

Ultimately such observations and the models derived from them are used to try and make inferences about the state of Earth's core and the way the geodynamo operates. This requires that we go back to the physical cause of the magnetic field, and look at magnetohydrodynamics (MHD); these are the dynamics of fluids in which electromagnetic forces are important.

For our purposes this means looking at the time evolution of the magnetic field in the fluid part of Earth's core, where the fluid has non-zero conductivity and is permeated by a magnetic field. The temporal variation is described by the pre-Maxwell equations:

$$\nabla \cdot \mathbf{B} = 0 \tag{108a}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{108b}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{108c}$$

We have assumed that there is no permanent magnetization in the core so that in the last equation $\mathbf{H} = \mathbf{B}/\mu_0$. In static media we have Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ but in a fluid moving with respect to the (inertial) frame of reference with velocity **u**

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}') \tag{109}$$

where $\mathbf{E}' = \mathbf{u} \times \mathbf{B}$. You can see this might be correct from the force on a moving charge (3). For a derivation, consult *Foundations*. So the modified Ohm's law is

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{110}$$

Setting this result into (108c), we find

$$\nabla \times \mathbf{B} = \mu_0 \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{111}$$

Taking the curl of this equation and assuming that σ is uniform throughout the core, we get

$$\nabla \times \nabla \times \mathbf{B} = \mu_0 \sigma[\nabla \times \mathbf{E} + \nabla \times (\mathbf{u} \times \mathbf{B})]. \tag{112}$$

Recall the vector identity (91) which we apply to $\mathbf{B}$:

$$\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B} + \nabla(\nabla \cdot \mathbf{B}). \tag{113}$$

But $\mathbf{B}$ is solenoidal – equation (108a) – so the second term vanishes. We put (113) into (112). Next we appeal to Maxwell's equation (108b) to eliminate $\mathbf{E}$, and (112) becomes after some rearrangement:

$$\partial_t \mathbf{B} = \eta \nabla^2 \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{114}$$

This is called the *magnetic induction equation* for the geodynamo. Here $\eta = 1/\mu_0 \sigma$ is called the *magnetic diffusivity* – it has the usual units for a diffusion constant namely, m^2/s. η is the analog of the kinetic viscosity, ν in ordinary fluid flow. For the Earth's core recent revisions to the electrical conductivity (see Pozzo *et al.*, 2012, doi:10.1038/nature11031) yield a $\sigma \approx 1.1 \times 10^6$ S/m, and so $\eta \approx 0.7\mathrm{m}^2/\mathrm{s}$ under core conditions.

Jacobs' book *Geomagnetism, Vol 2*, Chapter 1, pp 1-23 gives a good description of the equations required to describe the basic state of the core. These include the magnetic induction equation (114) above and the nature of the magnetohydrodynamic approximation leading to it. In addition to this we need an equation describing the dynamics of the flow. This is the Boussinesq Navier-Stokes equation which assumes that the fluid is incompressible except for thermal expansion. The frame of reference is fixed in and rotating with the earth's mantle. Then the equation of motion for a volume element within the fluid is

$$\rho_0(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \times \mathbf{u}) = -\nabla p + \rho' \mathbf{g} + \mathbf{J} \times \mathbf{B} + \rho_0 \nu \nabla^2 \mathbf{u} \tag{115}$$

where ρ_0 is the hydrostatic density
ρ' is departure from hydrostatic density
Ω is the angular velocity vector of Earth's rotation
p is the non-hydrostatic pressure
$\mathbf{g}$ is gravitational acceleration
$\mathbf{J}$ is current density
$\mathbf{J} \times \mathbf{B}$ is called the Lorentz force.

You will recognize in (115) the terms on the left correspond to mass times acceleration, and those on the right to a sum of forces, in a statement of Newton's second law of motion for an element of the fluid. Equations (114) and (115) are the fundamental equations of magnetohydrodynamics. In solving the geodynamo problem one can identify three important aspects.

(1) Energy source: we need a source of energy for convection to occur in Earth's core. This source must be sufficient to overcome diffusion, not just at the present time, but to maintain the dynamo throughout Earth history. In most theories the driving term appears in additional equations that define how ρ', the density differential is governed; but in some the driving force enters through boundary conditions.

(2) Kinematic dynamo problem: this is basically the question of whether a particular fluid motion, that is **u**, is capable of generating a magnetic field. The problem can be studied mathematically using the magnetic induction equation (114), $\nabla \cdot \mathbf{B} = 0$ and appropriate boundary conditions for the geomagnetic field. A candidate fluid velocity **u** is tested for dynamo action by solving (114) for **B** and testing to see if the solution grows with time. The fluid velocity needs to be sufficiently large to overcome the effects of diffusion as well as having the right form to cause its regeneration. Two important types of kinematic dynamos are $\alpha - \omega$ models and α^2 models which we discuss later. In the first the ω effect is strong, core fluid motion is influenced by the Coriolis force, and is a combination of differential rotation and convective helical motion. In $\alpha - \omega$ type models the toroidal field is expected to be much stronger than the poloidal contribution, but when the fields are comparable the dynamo is an α^2 type.

(3) Dynamical dynamo problem: the feasibility of kinematic dynamo problems has now been amply demonstrated, and in dynamo theory the emphasis is now on MHD, that is studying the dynamics of the fluid flow. The full dynamo problem requires the solution of (114), (115) in three-dimensions and an appropriate equation of state for the core with suitable boundary conditions. This is a formidable problem, currently only approachable by computer modeling, and even then not with parameters accurately reflecting the conditions inside the core. For example the Ekman number, which measures the relative importance of viscosity and the Coriolis force, is usually larger than 10^{-5} in simulations: but the molecular value is $\approx 10^{-15}$ and even the turbulent value is $\approx 10^{-9}$. None-the-less the solutions are suggestive and resemble the real system in many ways. Well known models by Glatzmaier, G. A, and Roberts, P. H., *Phys. Earth Planet. Inter.,* 91, pp 63-75, 1995 were the first to spontaneously exhibit geomagnetic reversals among other Earth-like properties. For animations and other good stuff see these websites:

http://www.psc.edu/science/glatzmaier.html
http://es.ucsc.edu/˜glatz/geodynamo.html

Of the above list (2) and (3) each involve mathematical analysis of simplified models of Earth's core, while (1) is fundamentally a thermodynamic problem. Much of (1) depends on ideas about Earth's history and constitution which lie outside the scope of this course; despite these uncertainties (2) and (3) may be pursued essentially independently of the exact nature of the energy source.

2:1 Energy Sources for the Geodynamo

The amount of energy actually required to drive the geodynamo is at best poorly understood. The magnitude of ohmic dissipation depends on the electrical conductivity and the strength of the magnetic field in the core, and this must be made up by conversion from kinetic energy of the flow. The poloidal field at the base of the mantle is of the order of 500 μT, but this may not reflect the strength in the interior of the core. Surface observations place a lower bound on heating of 10^8 W, but strong toroidal fields may make 10^{12} W a more realistic estimate.

Two possible mechanisms can drive the flow, (1) internal buoyancy forces and (2) external forcing by boundary motion. Suggested energy sources have been:

(1) Thermal buoyancy from heat sources in the core: these are cooling of the core, release of latent heat during freezing of the outer core to form the inner core, and contributions from radioactive decay. The significance of the latter is disputed.

(2) Compositional buoyancy is expected because light constituents in the core are released at the inner core boundary, as the heavy fraction is preferentially incorporated into the solid part.

(3) In a Gravitationally powered dynamo the energy source is gravitational potential energy stored in the outer core, which is released as Earth cools and the inner core grows.

(4) Precessional driving of core flows due to gravitational torques from the Sun and Moon. Precession of the earth can produce fluid instability (and thus flow?), but this requires momentum transfer from mantle to core. Nevertheless this has not been completely discounted as a partial energy source.

Most numerical simulations rely on a cooling Earth model, with both compositional and thermal buoyancy effects incorporated into a so-called *co-density*. For a recent review of progress in understanding the origin of Earth's magnetism see Roberts & King (2013, Rep. Prog. Phys,, 76, doi:10.1088/0034-4885/76/9/096801) We will return briefly in 4.6 to the geodynamo after we have studied the behavior of a magnetic field in a conducting fluid.

2:2 Secular Variation

Secular variation is the name given to the changes in the geomagnetic field with timescales of the order of a few years to many centuries. As we have seen there are a number of regular features in the secular variation, such as the westward drift, that ought to be capable of theoretical explanation. Westward drift is illustrated in Figure 4.2 below, which shows the nondipole field at two times, exactly 10 years apart. The fields are contoured at 5 μT intervals. Notice there is good evidence for a general westward drift in the Atlantic ocean, but little sign over Siberia or the central Pacific oceans.

We gain some insight into the motions of the fluid outer core by considering various approximations to the magnetic induction equation in which each term in turn is taken to be negligibly small. The idea here is to forget about the forces and just look at how a magnetic field is affected by motions in the fluid core. Here is the magnetic induction equation again:

$$\partial_t \mathbf{B} = \eta \nabla^2 \mathbf{B} \ + \ \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{116}$$

The left side is the local rate of change of the geomagnetic field. Setting it to zero corresponds to a steady state condition in which the field is unchanging; this is of no interest in describing secular variation. If we neglect the second term on the right (which contains effects of advection) then (116) becomes a vector diffusion equation, corresponding to no fluid motion in the outer core. Alternatively, if the core were a perfect conductor $\sigma \to \infty$, we would have $\eta = 0$, and the ∇^2 term could be neglected. What determines the relative importance of advection and diffusion? When L and U are typical length and velocity scales then the ratio of the two terms right is roughly

$$\begin{aligned} \frac{|\nabla \times (\mathbf{u} \times \mathbf{B})|}{|\eta \nabla^2 \mathbf{B}|} &= \frac{UB/L}{\eta B/L^2} \\ &= \frac{UL}{\eta} = \mu_0 \sigma U L = R_m. \end{aligned} \tag{117}$$

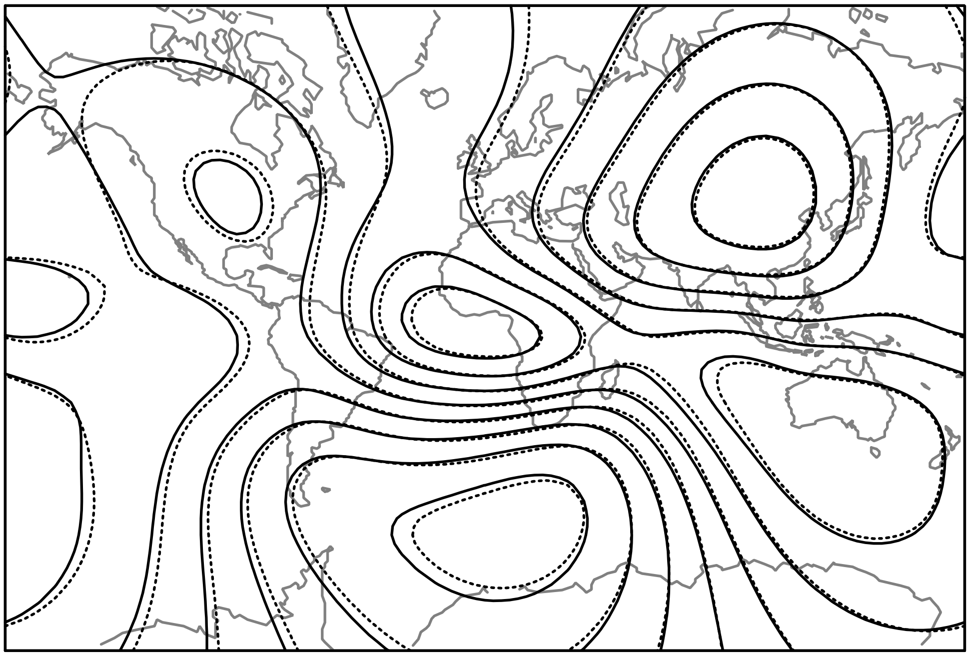


Figure 4.2

The dimensionless number R_m is called the *magnetic Reynold's number* by analogy with the Reynold's number, Re, used in fluid mechanics. Large magnetic Reynold number corresponds to the dominance of advection over diffusion, while small values indicate a diffusive situation with negligible advection. We will study each of these next. Note that if we use the core diameter as L, and $U \approx$ 0.4 mm/s, we find a rather large value: $R_m \approx 4 \times 10^{-4} \times 7 \times 10^6 / 0.7$ =5,000.

2:3 Diffusion of the Magnetic Field

Suppose that R_m is small. Then we neglect the advection term in (116) and we study

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}. \tag{118}$$

You will recall the diffusion equation, as the one that governs the behavior of heat for example, or salt dissolving in water. Equation (118) is a vector equation and the various components will not in general diffuse separately because they get mixed together in the boundary conditions. It is not hard to solve (118) exactly for a uniform conducting sphere: one uses the decomposition into spheroidal and poloidal parts, and there quickly emerges a diffusion equation for each of the two scalars T and P; the only trick is in fitting proper boundary conditions: see *Foundations* if you are interested.

We can get an idea of what is going on by assuming that the conductor is infinite (thus getting rid of the nuisance of boundary conditions), and looking at the evolution of a magnetic field that initially is in the form of an infinite sine wave with wavelength λ: we simply assume a behavior of the form $\mathbf{B}(x, t) = \mathbf{B}_0 e^{2\pi i x/\lambda} f(t)$. Substitute into (118):

$$\partial_t f = -\frac{4\pi^2 \eta}{\lambda^2} f. \tag{119}$$

The solution to this differential equation is easily seen to be

$$f(t) = f(0)\exp(\frac{-t}{t_0})$$

where $t_0 = \lambda^2/4\pi^2\eta$, the characteristic time of exponential decay. Thus

$$\mathbf{B}(x,t) = \mathbf{B}_0 \mathrm{e}^{2\pi \mathrm{i} x} \mathrm{e}^{-t/t_0}. \tag{120}$$

Thus we expect all magnetic fields to decay away in time, at a rate inversely proportional to their length scales. The energy stored in the field is dying away as the electric currents transform the field energy into heat through Joule heating $\mathbf{J} \cdot \mathbf{J}/\sigma$.

A more precise argument is the following. Suppose the core is a sphere radius c, in an insulator. Dot equation (118) with **B** and integrate over all space:

$$\int_{|R^3} d^3\mathbf{s}\ \mathbf{B} \cdot \partial_t \mathbf{B} = \int_{|R^3} d^3\mathbf{s}\ \eta \mathbf{B} \cdot \nabla^2 \mathbf{B} \tag{121}$$

On the left you will easily verify that $\mathbf{B} \cdot \partial_t \mathbf{B} = \frac{1}{2}\partial_t(\mathbf{B} \cdot \mathbf{B})$ and so the left side of (121) is proportional to the rate of increase of the total field energy. On the right introduce the summation convention and then apply vector identity 4:

$$\int_{|R^3} d^3\mathbf{s}\ \frac{1}{2}\partial_t |\mathbf{B}|^2 = \int_{|R^3} d^3\mathbf{s}\ \eta B_j \nabla^2 B_j = \int_{|R^3} d^3\mathbf{s}\ \eta[\nabla \cdot (B_j \nabla B_j)\ -\ \nabla B_j \cdot \nabla B_j] \tag{122}$$

When Gauss's theorem is applied to the first term on the right and we integrate over a very large sphere, it is clear this term vanishes, leaving us with

$$\int_{|R^3} d^3\mathbf{s}\ \frac{1}{2}\partial_t |\mathbf{B}|^2 = -\int_{|R^3} d^3\mathbf{s}\ \eta \sum_j |\nabla B_j|^2 \tag{123}$$

Since the right side is always negative, this equation shows the total magnetic energy must decrease for all times. Because the field vanishes at infinity, B_j cannot be constant, and thus the field energy declines inexorably to zero. In fact it can be shown (*Foundations,* p 271) that

$$\int_{|R^3} d^3\mathbf{s}\ |\nabla B_j|^2\ \geq\ (\pi/c)^2 \int_{|R^3} d^3\mathbf{s}\ B_j^2 \tag{124}$$

from which we can conclude the magnetic field energy dies away at least as fast as $\exp(-\pi^2\eta/c^2)t$.

Recall in the earth's core we have $\sigma \approx 1.1 \times 10^6$ S/m or $\eta \approx 0.7\mathrm{m}^2/\mathrm{s}$; the core radius is $c = 3.5 \times 10^6$ m; then the longest characteristic decay time, $c^2/\pi^2\eta = 7.7 \times 10^{11}$ s or 55,000 years. In fact, the same answer is obtained from (120) if we fit a half wavelength across the diameter of the core, and if we determine the longest-lived field in a spherical conductor (see *Foundations* , Chap 5). This calculation shows that any magnetic field for which $\mathbf{u} = 0$ cannot persist very long without motions in the fluid, yet we know there has been a field for over 3 billion years from paleomagnetism. Thus we need velocities **u** to prevent the death of the main field through Joule heating.

2:4 Frozen Flux and Alfvén's Theorem

In this section we investigate the much more interesting, and geophysically more appropriate, case in which the magnetic Reynolds number is so large that we can neglect the diffusion term in the magnetic induction equation. Then the fluid effectively becomes infinitely conducting and is called a *perfectly conducting* fluid. Superconductors conduct perfectly too, but they don't allow magnetic fields inside them, so a superconductor is different from a perfect conductor.

We look at the way the magnetic field changes according to:

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{125}$$

which is (116) with $\eta = 0$. If we use vector identity number 8 to write

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{B} \cdot \nabla\mathbf{u} \; - \; \mathbf{u} \cdot \nabla\mathbf{B} \; + \; \mathbf{u}\nabla \cdot \mathbf{B} \; - \; \mathbf{B}\nabla \cdot \mathbf{u} \tag{126}$$

then since $\nabla \cdot \mathbf{B} = 0$, and dropping the $\eta\nabla^2\mathbf{B}$, we transform (125) into

$$\partial_t \mathbf{B} \; + \; (\mathbf{u} \cdot \nabla)\mathbf{B} \; + \; \mathbf{B}(\nabla \cdot \mathbf{u}) \; - \; (\mathbf{B} \cdot \nabla)\mathbf{u} = 0 \tag{127}$$

or, equivalently

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B} \cdot \nabla)\mathbf{u} \; - \; \mathbf{B}(\nabla \cdot \mathbf{u}) \tag{128}$$

where $D\mathbf{B}/Dt$ is called the *material derivative* of $\mathbf{B}$ and is the rate of change of the field as experienced by a particle moving with the fluid: in general the material derivative is given by

$$\frac{D}{Dt} = \partial_t \; + \; \mathbf{u} \cdot \nabla \tag{129}$$

where the first term on the right accounts for the time derivative at a fixed point, and second gives the apparent change as the particle moves along the spatial gradient.

Although it is unnecessary, let us simplify our discussion to incompressible fluids, which means, as you can easily verify from conservation of matter, that $\nabla \cdot \mathbf{u} = 0$. Now (128) reduces to

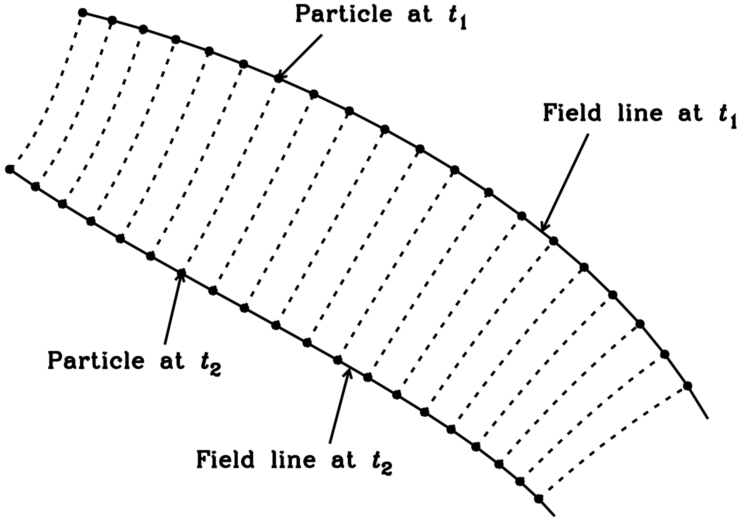
$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B} \cdot \nabla)\mathbf{u}. \tag{130}$$

We wish to show that the same equation is satisfied by a line element joining two neighboring points in the fluid. At time t we mark a particle at the point $\mathbf{x}$ and a nearby particle is at point $\mathbf{y}$. Time flows on by a small amount dt; now the first particle has moved to $\mathbf{x} + dt\mathbf{u}(\mathbf{x})$ plus terms in dt^2, and the second has traveled to $\mathbf{y} + dt\mathbf{u}(\mathbf{y})$. Since $\mathbf{y}$ is near to $\mathbf{x}$ we can write $\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + (\mathbf{y} - \mathbf{x}) \cdot \nabla\mathbf{u}$ plus small terms in the square of $|\mathbf{y} - \mathbf{x}|$. Thus the new vector between the two particles is

$$\begin{aligned} \mathbf{y}(t + dt) \; - \; \mathbf{x}(t + dt) &= (\mathbf{y} + dt[\mathbf{u}(\mathbf{x}) + (\mathbf{y} - \mathbf{x}) \cdot \nabla\mathbf{u}]) \; - \; (\mathbf{x} + dt\mathbf{u}(\mathbf{x})) \; + \; O(dt^2) \\ &= (\mathbf{y} - \mathbf{x}) \; + \; dt[(\mathbf{y} - \mathbf{x}) \cdot \nabla\mathbf{u}] \; + \; O(dt^2). \end{aligned} \tag{131}$$

Hence the material derivative describing a short line element embedded in the fluid is

$$\frac{D(\mathbf{y} - \mathbf{x})}{Dt} = (\mathbf{y} - \mathbf{x}) \cdot \nabla\mathbf{u}. \tag{132}$$



As promised this is the same differential equation as (130). Suppose the two particles lie on the same magnetic line of force at time t. The two equations (130) and (132) show that at a later time they will remain on the same line of force because the vector $\mathbf{B}$ and the vector $\mathbf{y} - \mathbf{x}$ evolve identically and must remain parallel. This famous result is known as *Alfvén's Theorem* after Hannes Alfvén a Swedish plasma physicist who in his later years was a faculty member at UCSD. The theorem tells us that in perfectly conducting material the magnetic lines of force are permanently attached to the material and move with it.

Equations (130) and (132) also show that the amplitude of the field is proportional to the density multiplied by the length of the fluid element, so that elongation of the material line will increase the field strength.

Next consider a simple surface P with boundary points attached to the material of the fluid and enclosing a number of lines of force; the number of lines of force is proportional to the magnetic flux Φ threading P:

$$\Phi = \int_P \mathbf{B} \cdot \hat{\mathbf{n}} d^2\mathbf{s} \,. \tag{133}$$

At a later time P has a different shape since the particles have moved, but none of the field lines that started inside P has crossed the perimeter, because none of the interior particles has either – we assume the fluid is moving continuously without tearing. Thus the number field lines crossing P is unchanged and therefore Φ is unchanged also. We call this the *frozen flux* condition for perfect conductors. You may not have found this argument very convincing (I know that I don't).

Here is a more conventional proof which I have found in several places though usually with mistakes. You may skip it if the line-of-force argument satisfies you. *Foundations* has a very different and more difficult proof. We consider a simple patch P in the fluid bounded by the line ∂P which is defined by a set of points attached to material particles. We wish to show that the material derivative of magnetic flux through the patch vanishes:

$$\frac{D\Phi}{Dt} = \frac{D}{Dt} \int_P \mathbf{B} \cdot \hat{\mathbf{n}} \, d^2\mathbf{s} = 0. \tag{134}$$

In time dt the change of the flux through a surface that moves with the fluid is the contribution of two terms: one from the time derivative with the patch fixed in space, the second from the constant magnetic field when the patch is moved from its old position to the new one, the same idea as in (129):

$$dt\frac{D\Phi}{Dt} = dt\int_P \partial_t \mathbf{B}\cdot\hat{\mathbf{n}}\, d^2\mathbf{s} \;+\; [\int_{P(t+dt)} \mathbf{B}\cdot\hat{\mathbf{n}}\, d^2\mathbf{s} \;-\; \int_{P(t)} \mathbf{B}\cdot\hat{\mathbf{n}}\, d^2\mathbf{s}]$$

$$= dt\int_P \partial_t \mathbf{B}\cdot\hat{\mathbf{n}}\, d^2\mathbf{s} \;+\; dF. \tag{135}$$

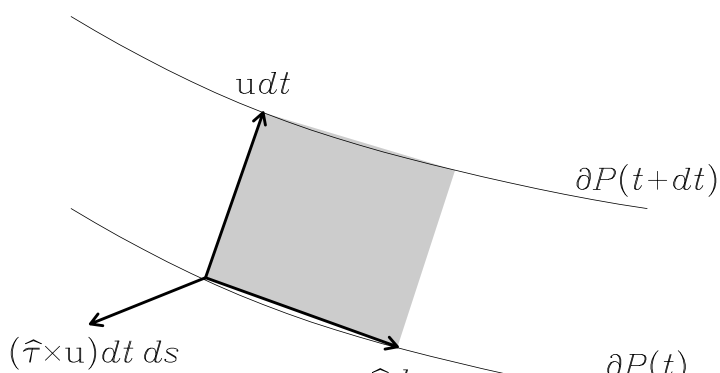
Consider the volume of space V swept out by the particles as they move from $P(t)$ to $P(t + dt)$; the flux of **B** out of this region is zero by the divergence theorem:

$$\int_{\partial V} \mathbf{B}\cdot\hat{\mathbf{s}} d^2\mathbf{s} = \int_V \nabla\cdot\mathbf{B} d^3\mathbf{s} = 0 \tag{136}$$

where $\hat{\mathbf{s}}$ denotes the outward normal to the surface ∂V. Noting that the normal $\hat{\mathbf{n}}$ on $\partial P(t)$ points inward we see that the flux integral in (136) is also given by

$$\int_{\partial V} \mathbf{B}\cdot\hat{\mathbf{s}} d^2\mathbf{s} = \int_{P(t+dt)} \mathbf{B}\cdot\hat{\mathbf{n}}\, d^2\mathbf{s} \;-\; \int_{P(t)} \mathbf{B}\cdot\hat{\mathbf{n}}\, d^2\mathbf{s} \;+\; \int_{dS} \mathbf{B}\cdot\hat{\mathbf{s}} d^2\mathbf{s}$$

$$= dF \;+\; \int_{dS} \mathbf{B}\cdot\hat{\mathbf{s}} d^2\mathbf{s} \tag{137}$$



where dS is the surface strip joining $P(t)$ and $P(t + dt)$. Since (136) shows that the left side of (137) vanishes, we have:

$$\int_{dS} \mathbf{B}\cdot\hat{\mathbf{s}} d^2\mathbf{s} = -dF. \tag{138}$$

Now we calculate dF in a different way. From Figure 4.4.2 $(\hat{\tau} ds) \times (\mathbf{u}dt) = \hat{\tau} \times \mathbf{u}dtds$ is the elementary area swept out by ∂P as it moves from $\partial P(t)$ to $\partial P(t + dt)$. The cross product points in the direction of $\hat{\mathbf{s}}$ normal to this surface and so the flux integral can also be written

$$\int_{dS} \mathbf{B}\cdot\hat{\mathbf{s}} d^2\mathbf{s} = \int_{\partial P} \mathbf{B}\cdot(\hat{\tau}\times\mathbf{u}dtds)$$

$$= dt\int_{\partial P} (\mathbf{u}\times\mathbf{B})\cdot\hat{\tau}\; ds = dt\int_P \nabla\times(\mathbf{u}\times\mathbf{B})\cdot\hat{\mathbf{n}} d^2\mathbf{s} \tag{139}$$

where we have applied Stokes' integral theorem to turn the line integral to a surface integral. We can substitute this expression into dF in (138) and then into (135); then divide by dt:

$$\frac{D\Phi}{Dt} = \int_P \partial_t \mathbf{B} \cdot \hat{\mathbf{n}}\, d^2\mathbf{s} \;-\; \int_P \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot \hat{\mathbf{n}} d^2\mathbf{s}$$

$$= \int_P [\partial_t \mathbf{B} \;-\; \nabla \times (\mathbf{u} \times \mathbf{B})] \cdot \hat{\mathbf{n}} d^2\mathbf{s}. \tag{140}$$

Finally we simply appeal to (125) which shows that the right side must vanish, and therefore $D\Phi/Dt = 0$, which is the frozen flux condition.

Roberts and Scott (*J. Geomag. Geoelec.* 17, p 137, 1965) noted that the time scale for diffusion, t_0, can be expected to be much longer than that for advection, given the high electrical conductivity in the core: in effect they are saying that for fields with the scales we can detect from the Earth's surface, R_m is large – the same calculation we did at the end of section 4.2, but now with $L \approx 1{,}000$ km (then $R_m = 180$). They called this idea the *frozen-flux hypothesis*, and declared that under this approximation we are allowed to use (118) in studying the secular variation. If fluid particles and field lines are so closely linked, it seems likely that we can learn something rather directly about **u** in the core by observing **B** and its evolution in time. This is indeed the case, as we shall see in the next section.

> *Exercise:*
> The proof in the notes of the frozen-flux condition is too long. Here is a shorter one. Consider three closely neighboring, non-colinear particles in the surface P at **x**, **y** and **z** at time t. An element of area is given by $\mathbf{s} = (\mathbf{y} - \mathbf{x}) \times (\mathbf{z} - \mathbf{x})$, and an elementary contribution to the flux integral over P is given by $\mathbf{s} \cdot \mathbf{B}$ where **B** is measured at **x**. Consider a fourth point also close to **x** at $\mathbf{p} = \mathbf{x} + \kappa \mathbf{B}$. The product $dV = (\mathbf{p} - \mathbf{x}) \cdot ((\mathbf{y} - \mathbf{x}) \times (\mathbf{z} - \mathbf{x}))$ is the volume of a small parallelepiped of fluid. As time goes on it deforms with the motion. If the fluid is incompressible, dV cannot change; hence κdV, which is an element of flux, is invariant in time. Then the sum of all such contributions, which gives the total flux, is also invariant in time.
> Show from $\nabla \cdot \mathbf{u} = 0$ that the elementary volume doesn't change as the fluid unit moves. Does the long proof in the notes require the fluid to be incompressible as this one does?

2:5 Applications of the Frozen Flux Hypothesis

The beauty of the frozen flux hypothesis is that it allows one to visualize a very direct connection between the core fluid motions and the magnetic field. Indeed it is the case that we can calculate the magnetic field at all future times if we know it at one instant, and we are given the velocity at all times. We won't do exactly this calculation (again, see *Foundations*), but we will look at a simplified version for the surface of the core. Is the frozen-flux hypothesis really valid for the core? It is so useful for other calculations and deductions we would like to check it from observations. We will discuss how that is done.

2:5.1 Observational Consequences of the FFH

Let us continue to assume the FFH (frozen-flux hypothesis) applies to the core. We are going to look at the magnetic fields right at the surface of the core, a place where we hope downward continuation from the surface makes them observable. We start with the evolution equation (125), and apply standard vector identity 8:

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$= \mathbf{B} \cdot \nabla \mathbf{u} \; - \; \mathbf{u} \cdot \nabla \mathbf{B} \; + \; \mathbf{u} \nabla \cdot \mathbf{B} \; - \; \mathbf{B} \nabla \cdot \mathbf{u}. \tag{141}$$

Now dot this equation with $\hat{\mathbf{r}}$ which is a fixed vector and so it can travel through the time derivative:

$$\partial_t B_r = \mathbf{B} \cdot \nabla u_r \; - \; \mathbf{u} \cdot \nabla B_r \; + \; u_r \nabla \cdot \mathbf{B} \; - \; B_r \nabla \cdot \mathbf{u}$$

$$= \nabla \cdot (\mathbf{B} u_r) \; - \; \nabla \cdot (\mathbf{u} B_r). \tag{142}$$

Remember $\mathbf{B} \cdot \nabla \mathbf{u}$ is just the dot product of $\mathbf{B}$ with the gradients of each of the components of $\mathbf{u}$; so when we dot with $\hat{\mathbf{r}}$ this simply picks out the r component under the gradient. And the second line is just two applications of that most useful identity number 4. At the CMB the core fluid cannot flow radially because it is confined by the solid mantle; hence $u_r = 0$. Similarly radial components of $\mathbf{u}$ are irrelevant in the first term on the right and thus (142) at the CMB reduces to:

$$\partial_t B_r = -\nabla_s \cdot (\mathbf{u}_s B_r). \tag{143}$$

This equation shows us that if we know B_r at one time, and we are given the surface velocities on the top of the core, we can calculate B_r at all future times. Of course given B_r we can also find $\mathbf{B}$ outside the core uniquely too. This is a remarkably simple solution to what would appear to be a complicated system.

In reality we have the other side of the coin: we know B_r moderately accurately for 100 years – can we deduce $\mathbf{u}_s$ from (143)? The answer is no. We can see this crudely by observing that $\mathbf{u}_s$ has two unknown components at every point, while (143) supplies only a single constraint, not enough to pin down the velocity without further information. As we will see in a moment there are places where we can gain partial information about $\mathbf{u}_s$.

We now derive an observable consequence of (142). At the surface of the earth there is a line running very roughly around the equator where $B_r = 0$ called, appropriately enough, the *magnetic equator*. When we downward continue the field we find there are at least seven such lines (see Figure 4.5.2.1) that George Backus named *null-flux curves*. These mini-equators are the site of some interesting properties.

First we show the material particles on a null flux curve move with the curve. We name ∂P to be a particular (closed) curve on the core on which $B_r = 0$; it is the boundary of a patch P on the core. Consider a particle on ∂P and a neighboring particle at the position $\epsilon \mathbf{B}$; since $\mathbf{B} \cdot \hat{\mathbf{r}} = 0$ the neighboring particle is in ∂P as well. As time moves on the two particles can only move tangentially because they are at the CMB, confined to the boundary, thus the line connecting them remains tangential. Since by Alfvén's theorem that line remains a magnetic field line, magnetic field remains tangential too, and therefore the radial component B_r continues to be zero for all time.

This already tells us something interesting. By following ∂P in time we can see the fluid velocity normal to ∂P; the particles on ∂P cannot escape from it, but they can move along ∂P so we have no way of finding the component tangential to ∂P; see Figure 4.5.3.1.

But there is more. Since ∂P is a line of material particles, the frozen-flux condition applies to the flux coming through the area P:

$$\int_P B_r d^2\mathbf{s} = \text{constant}. \tag{144}$$

This is an observable consequence of the FFH. We can model the magnetic field on the core over a period of time, and calculate the fluxes through the various null flux curves as a function of time. If the FFH is correct the values within each patch should not change, even though the shapes and field values will evolve through secular variation.

2:5.2 Testing the FFH

Does the set of magnetic observations pass this test? The answer turns out to be disappointing and various authors (e.g. Bloxham and Gubbins, *Nature* 317, pp 777-81, 1985) claimed in fact that the geomagnetic field at the core's surface did *not* satisfy the FFH because of disagreement between fluxes calculated at various times. But their conclusions are suspect because it is quite difficult to come up with a truly believable estimate for the uncertainty in the geomagnetic models. Whether or not the test succeeds or fails depends vitally on how accurate you think the calculated fluxes really are.

The problem is that the magnetic field at the surface of the core requires infinitely many numbers to describe it completely, but we have only finitely many observations. Therefore while it is possible to say that we know the first L SH coefficients to a certain accuracy, it is unreasonable to expect us to know *every* b_l^m to some precision. Another equivalent way of looking at the question, which derives from inverse theory, is to say that we know the field at the core up to a certain resolution, meaning that features below a certain length scale are unresolved, and therefore inaccessible to us in the present state of knowledge.

Constable, Parker, and Stark (*Geophys. J. Royal Astron. Soc.* 113, pp 419-33, 1993) argued that the issue could be studied another way: start with magnetic data at two epochs and see if it is possible to actually construct a core field that has satisfies the integral constraints from the FFH. At first this would seem to settle the accuracy problem because now we are asking questions about the accuracy of the data, something we have an idea about. If we can find models obeying the constraint, then the constraint is a physically plausible one; if we cannot, then we must reject the FFH.

But, as these authors showed the question is more subtle than that: if one is allowed to manipulate features on an arbitrarily fine scale, *it is always possible to find pairs of models satisfying the FFH*. In other words, you cannot actually decide the issue with models of a given resolution, or from a finite data set! The best you can say is that the models that fit the observations are implausible in some way, for example, in having to much short-wavelength energy.

In that paper and a later one (O'Brien, Constable, and Parker, *Geophys. J. Int.* 128, pp 434-50, 1997) reasonable-looking geomagnetic models were constructed satisfying the FFH for the epochs 1915, 1945, 1980, and the idea that the FFH is generally agreed to have passed the test. For the past decade continuous satellite observations have improved recent field models (see for example Chulliat & Olsen, J. Geophys. Res.,115, 2010, doi:10.1029/2009JB006994) and the fit to the observation is beginning to look less plausible. It is also true that when we look back on historical timescales to 1590AD we see violation of the FFH as features like the South Atlantic Anomaly appear, growing a reverse flux patch at the CMB.

2:5.3 Calculation of Fluid Flow Models at the Surface of the Core

If we agree to accept the FFH we can hope to say something about the velocity field $\mathbf{u}_s$ on $S(c)$. As discussed earlier it turns out not to be possible to get the velocity from B_r and its time derivatives alone, as we crudely indicated by a counting argument. A more sophisticated treatment is given in *Foundations*. But as we already indicated, we can find unambiguously the component of $\mathbf{u}_s$ normal to the null-flux curves. This is easily done – just calculate the radial field at two close times t_1 and t_2, then draw the null-flux curves at each time. The normal distance moved between the two is just $\mathbf{u}_s \cdot \hat{\nu}(t_2 - t_1)$. Figure 4.5.3.1 shows one such calculation based on the IGRF-1980 model and its time derivative.

The map is drawn in Mercator projection, so it doesn't preserve area or scales, but locally angles between features are correct. The sizes of the normal velocity vectors are drawn to scale, except for the imploding patch under the western Pacific, where the lines have been reduced by a factor of four. Recall the velocity component of $\mathbf{u}_s$ tangent to the null-flux curves is unknown. The map of the world is there for orientation purposes only.

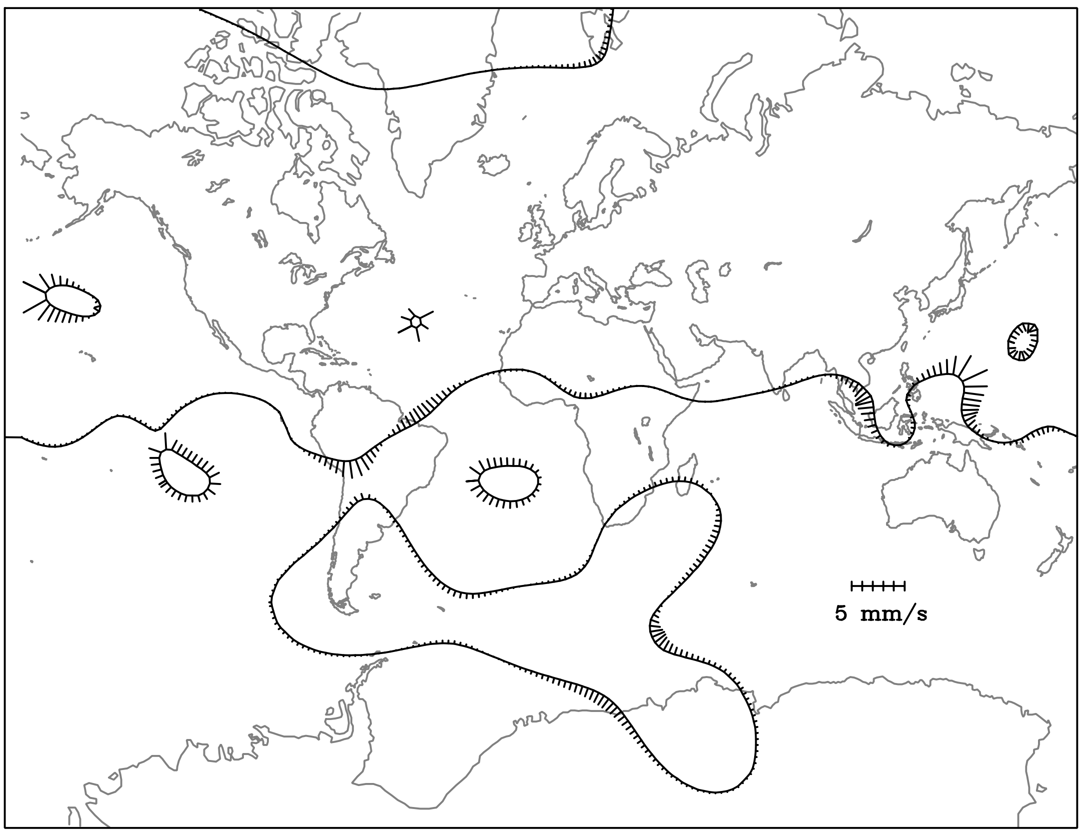


Figure 4.5.3.1

There are several other additional assumptions one can make about the field, which make it possible to find surface fluid flows. One is the *geostrophy*. In the atmosphere a geostrophic wind would arise from exact balance between Coriolis force and the pressure gradient force. In the core this means that the pressure balance is between Coriolis force and the Lorentz force $\mathbf{J} \times \mathbf{B}$. As Backus and Le Mouël showed (*Geophys. J. Internat.* 85, pp 617-29, 1986) however, this constraint does not make the flow unique – there are large patches where we have no information still!

This is a currently fast-developing field. For a thorough, if now somewhat dated review see Bloxham and Jackson, Fluid flow near the surface of Earth's outer core, *Rev. Geophys.* 29, pp 97-120, 1991. A more up to date discussion can be found in the Space Science Reviews article by Hulot et al (2010, DOI 10.1007/s11214-010-9644-0).

2:5.4 Determination of the Magnetic Core Radius of the Earth and Planets

The *pole strength* of the earth is given by the magnetic flux coming out of the core, without regard to the sign:

$$N = \int_{S(c)} |\mathbf{B} \cdot \hat{\mathbf{s}}| \, d^2\mathbf{s} = \int |B_r| \, d^2\mathbf{s}. \tag{145}$$

We can obviously calculate this number for any radius spherical surface:

$$N(r) = \int_{S(r)} |B_r(r, \theta, \phi)| \, d^2\mathbf{s}. \tag{146}$$

Recall that the flux coming through each null-flux curve is invariant in time under the FFH. This means that (145) is a constant of the magnetic field. But there is no reason to suppose the integral (146) at any other radius is constant in time, so this singles out the core radius as a special one. Suppose we compute this number from field models as a function of time and radius. The radius that gives the most nearly constant value for $N(r)$ is an estimate of the magnetic core radius. The result claimed is $c_m = 3,484 \pm 48$km, which is in surprisingly good agreement with the value derived from seismic observations $c = 3,485$ km. We have already seen another way to get the core's magnetic radius, by the slope of the spectrum, assuming the spectrum is white at the core.

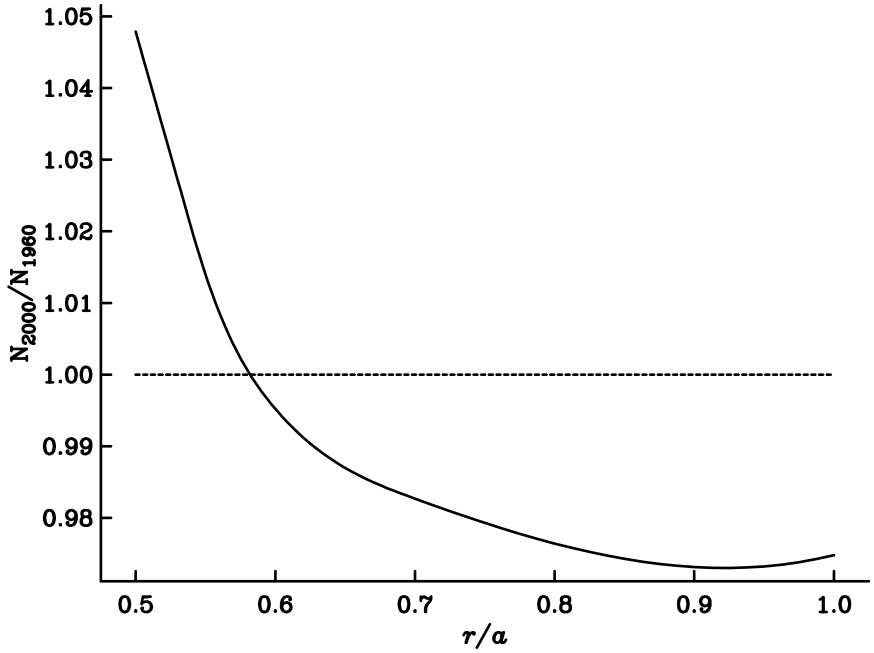
The method was invented by Raymond Hide (*Nature* 271, pp 640-1, 1978) who also proposed it to determine the structure of other magnetic planets, such as Jupiter, Saturn and Mercury, if we could obtain sufficiently good measurements of their magnetic fields and secular variations.

I have repeated the calculations with the IGRF models and the results are on the next page. I have plotted values of N through time, normalized by the 1960 values, and at the various radii shown; one needs to normalize because the variations in $N(r)$ are large as r varies. You will see I don't get such great agreement as other workers claim, I find approximate constancy for $r/a = 0.58$, or $r = 3,695$ km.

2:6 A Very Little Dynamo Theory

We have seen in 4.3 that if the conductor does not move the magnetic field must die away; in 4.4 we saw at the other extreme, when advection dominates, the field is trapped in the conductor, but there is no increase in magnetic energy. On this evidence it seems unlikely that with an intermediate conductivity fluid motions could maintain or even amplify the magnetic field. Indeed, in 1934 Thomas Cowling published a proof that many believed demonstrated dynamo action to be impossible; however, the proof showed only that purely axially symmetric fields could not be maintained. Intuition is wrong: we will discuss results proving that the proper kinds of **u** can amplify **B** for intermediate magnetic Reynolds numbers.

Section 4.5 was devoted to the approximation that $\eta = 0$ in (114), the induction equation. While this is likely to be good approximation over centuries, we saw it surely fails over timescales of 10^4 years or more. Diffusion and advection each play a part in the real dynamo. But what we have learned from Alfvén's


1.05
1.04
1.03
1.02
1.01
1.00
0.99
0.98
N2000/N1960
0.5
0.6
0.7
0.8
0.9
1.0
r/a

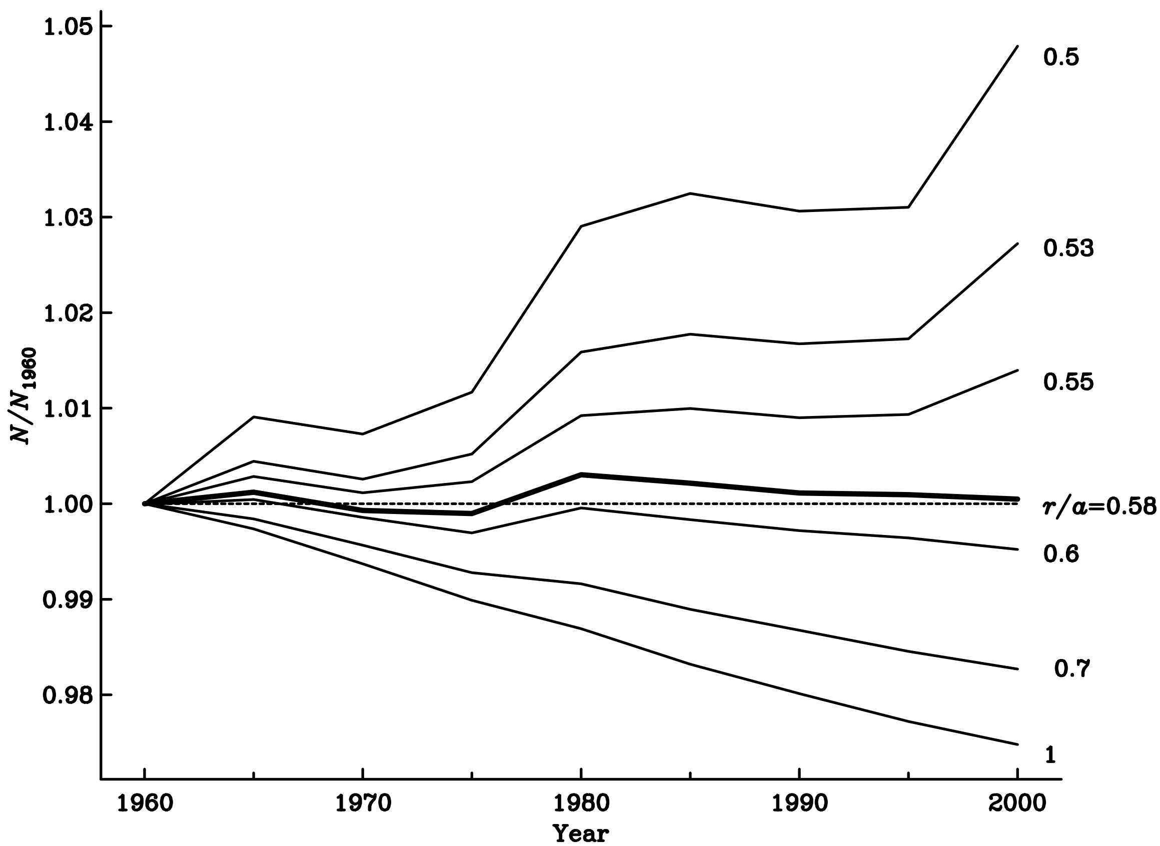

1.05
1.04
1.03
1.02
1.01
1.00
0.99
0.98
N/N1960
0.5
0.53
0.55
r/a=0.58
0.6
0.7
1
1960
1970
1980
1990
2000
Year

Figure 4.5.4.1

theorem is a handy way of think about how the conducting fluid interacts with the magnetic field. When the magnetic Reynold's number UL/η is very high, the field is "dragged" around by the fluid motion, or more precisely, by the component of motion normal to the field lines. When diffusion is important, the fluid is not 100 percent effective in dragging the field lines, and they tend to slip. It can be shown from energy considerations that the field lines act dynamically like stretched strings in these circumstances and exert a force resisting the dragging motion. These qualitative ideas allow us to do an arm-waving kind of analysis of some aspects of kinematic dynamo theory. In particular we will briefly discuss what is commonly believed to be a major mechanism for sustaining the magnetic field: the $\alpha - \omega$ effect.

Recall that the effectiveness of diffusion is increased for small scale fields (the decay time is like L^2). The difficulty in getting a dynamo to work (which means not losing all its energy through diffusion to Joule heating) is to find a way so that short wavelength fields, which are easily created by stirring the conducting fluid (and dragging the field lines into small eddies) can be combined together to create long-wavelength fields.

First the α effect. Recall the form of Ohm's law we needed in a moving medium (109) which we will write as $\mathbf{J} = \sigma \mathbf{E}_1$, where

$$\mathbf{E}_1 = \mathbf{E} \ + \ \mathbf{u} \times \mathbf{B}. \tag{147}$$

The effective electromotive force (emf) for driving currents has the unfortunate tendency from this equation to run perpendicularly to already existing magnetic fields, thus making it hard for advection to reinforce fields already present. Steenbeck, Krause and Rädler (see Roberts, P. H., and M. Stix, The turbulent dynamo: a translation of a series of papers by F. Krause, et al., *Tech. Note* 60, NCAR, Boulder, CO, 1971) in an astrophysical study of magnetic fields in the galaxy, postulated that there might be small-scale **u** superimposed by turbulence on the large-scale flows; writing $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$. Let us write averaging over a large length scale as $< \cdot >$ so that $< \mathbf{u}' >= 0$. Next we introduce a quantity called *helicity*:

$$h =< \mathbf{u} \cdot \nabla \times \mathbf{u} > . \tag{148}$$

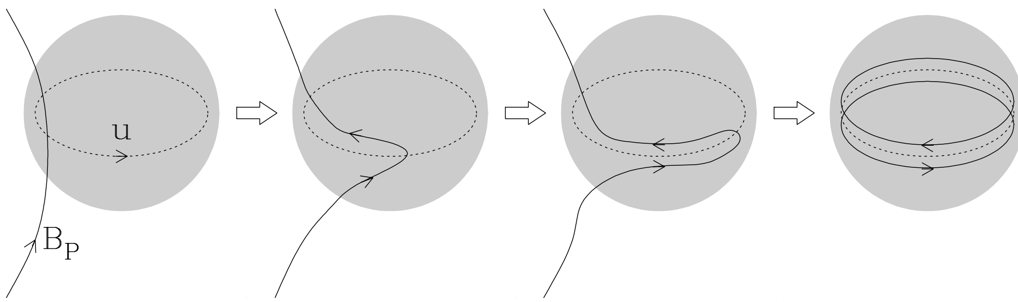
Helicity gives the average amount of "screw-like" motion, because we are dotting the velocity **u** at a point with its vorticity, which is just (twice) the local angular vector velocity. It is known that in a rapidly rotating convecting system, the turbulent motion has a net average helicity, as given by (148). Steenbeck and Krause showed by averaging over a velocity field with a nonzero average helicity (147) becomes, through interaction of the small-scale terms:

$$< \mathbf{E}_1 > \ = \ < \mathbf{E} > \ + \alpha < \mathbf{B} > \tag{149}$$

where α is a constant (or a tensor) depending on the details of the motion. But the important fact is that the helical motions on a fine scale produce emfs that are *parallel* to the large scale **B**; this is the (unimaginatively named) α effect.

With the α effect it is trivial to maintain a dynamo. Even if the average fluid flow is zero, we can get growing fields. This is shown in *Foundations*, Chapter 6.

Armed with the α effect many dynamo theorists look only at large-scale **u** and **B** and assume that the small-scale motions are harmless and just produce the useful **B**-parallel emf, through (149). This is given the grand name of *mean-field electrodynamics*. Here is a qualitative description of a popular dynamo, the α-ω model. We begin by assuming there is a large-scale poloidal field in a rotating spherical conductor, aligned with the spin axis. The fluid near the equator is assumed to rotate a bit faster than that in higher latitudes (observed in rotating fluids, like the atmosphere) and over time this drags the poloidal field into a wound up spiral through the attachment of the fluid to the field. But diffusion causes the spiral to condense into a pair of opposite-signed toroidal fields, one in each hemisphere. This process is called the ω effect, and though we have described it as a discrete process, it will go on continuously, transforming the large poloidal $\mathbf{B}_P$ into a quadrupole $\mathbf{B}_T$. Now the α effect operates, but you will perhaps believe it would possesses opposite signs in the two hemispheres, because the helicity derives from the overall rotation. So the toroidal $\mathbf{B}_T$ generates an effective electric field, with the same direction in the upper and lower hemispheres, in turn causing a toroidal $\mathbf{J}$ to be created, which is the source for more poloidal $\mathbf{B}_P$. The loop is closed. The figure illustrates the fate of a single field line in this scenario. I don't know how satisfactory this kind of "explanation" is without the algebra, but the equations do back it up. There are many other mechanisms.



For an advanced but very satisfactory review of dynamo theory read the two articles in *Geomagnetism, Vol 2* by Gubbins and Roberts.