SIOG 231
GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 10
Core field modeling and regularization.
2/8/2024

Today's Class

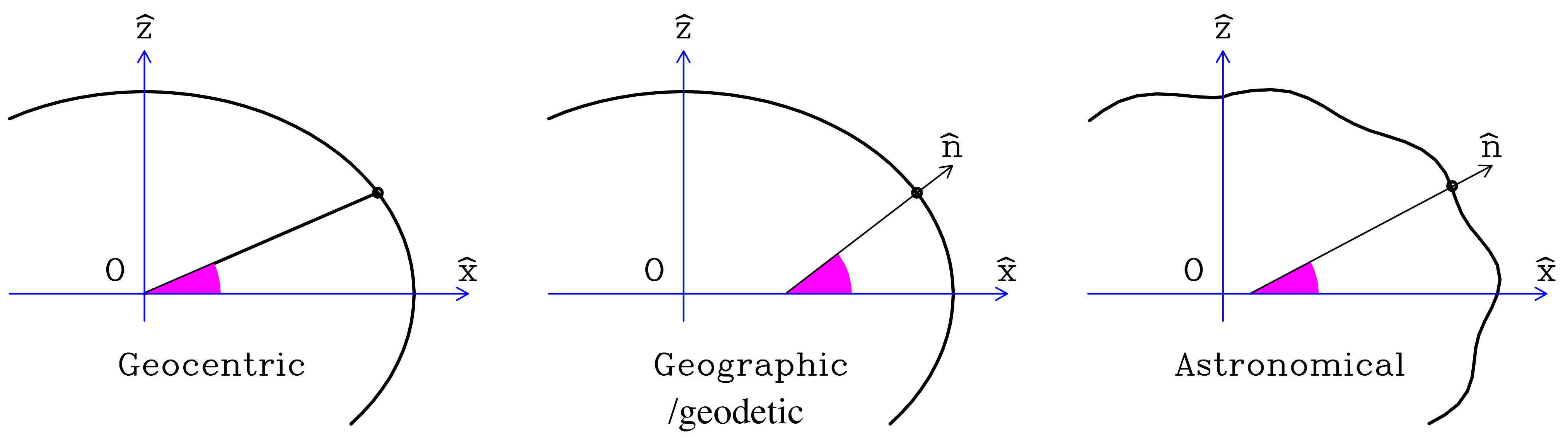
- What's a model?
- Kinds of data - Geomagnetic Elements
- A bit about Uniqueness
- Constructing Geomagnetic Field Models
    - least squares
    - regularization
    - some results - IGRF 2020 & regularized core field models

Geomagnetic Field Modeling

- Observations are measurements of the magnetic field at specific times and places.
- What about locations where we have no measurements?
- We can combine our observations to provide a **model,** a mathematical description in functional form of the Global Geomagnetic Field.
- Spherical Harmonic representations are the common language of magnetic field models.
- We measure various elements of $\mathbf{B}$, and use them to estimate the SH coefficients in the potential, Ψ.
- Then the gradient of Ψ can be used to give us the "best" estimate of the magnetic field at any desired altitude, latitude, and longitude. We need to decide what "best" means.

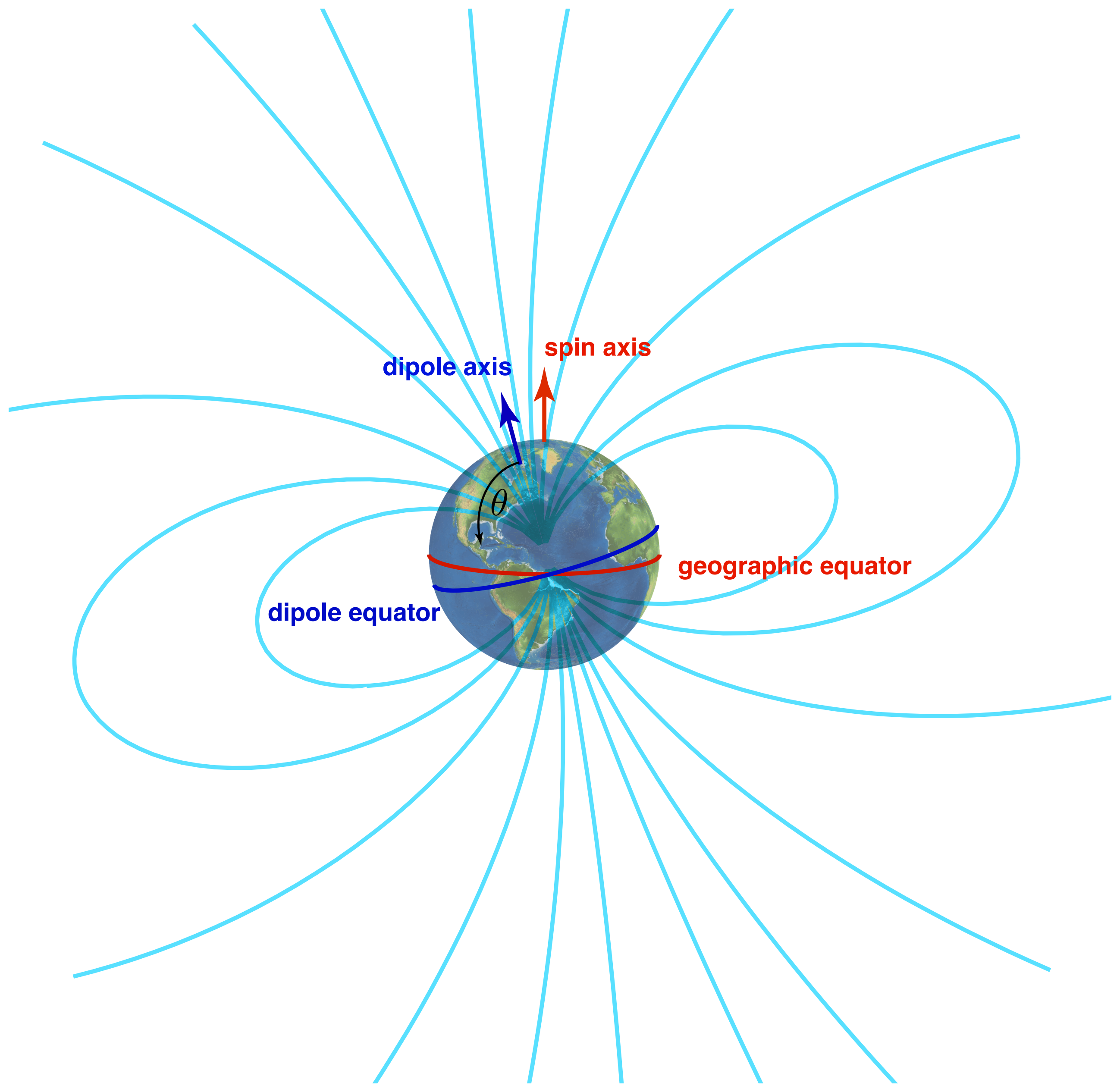
Geomagnetic elements depend on coordinate system

Geodetic vs Geocentric vs Geomagnetic coordinates



Latitude used in geographic coordinates is **geodetic** latitude. **Geodetic coordinates** are a type of curvilinear orthogonal coordinate system used in geodesy based on a *reference ellipsoid*. They include **geodetic latitude** (north/south) λ, *longitude* (east/west) φ, and **ellipsoidal height** h (also known as **geodetic height**[1]).

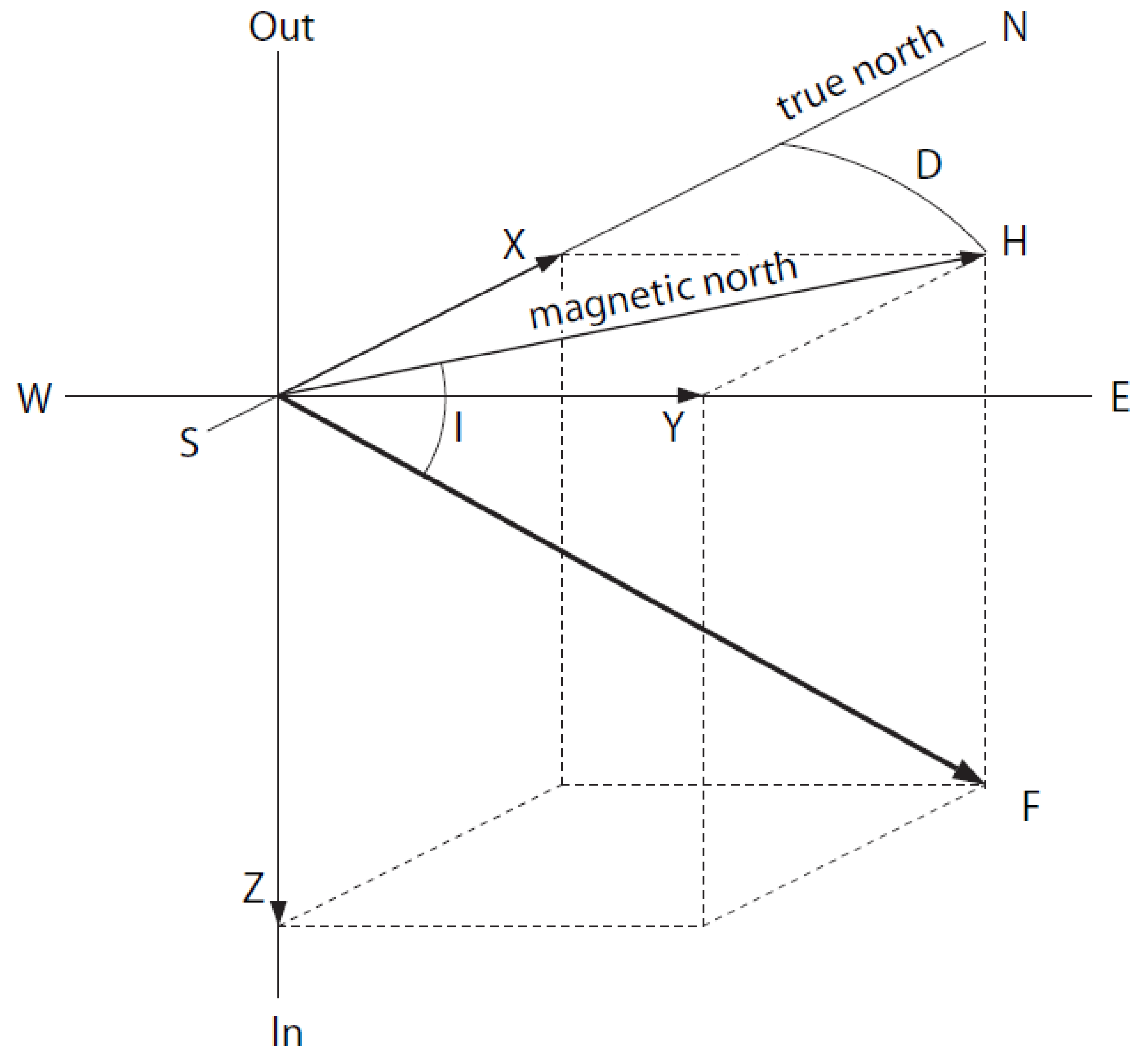
Similarly, geodetic altitude is defined as the height above the ellipsoid surface, normal to the ellipsoid; whereas geocentric altitude is defined as the distance to the reference ellipsoid along a radial line to the geocenter.



In geomagnetic coordinates, commonly used in external field studies, the **geomagnetic** colatitude is measured relative to the best fitting dipole axis, and geomagnetic latitude is measured relative to the dipole equator.

What are the Geomagnetic Elements?

In local geographic or geodetic coordinates



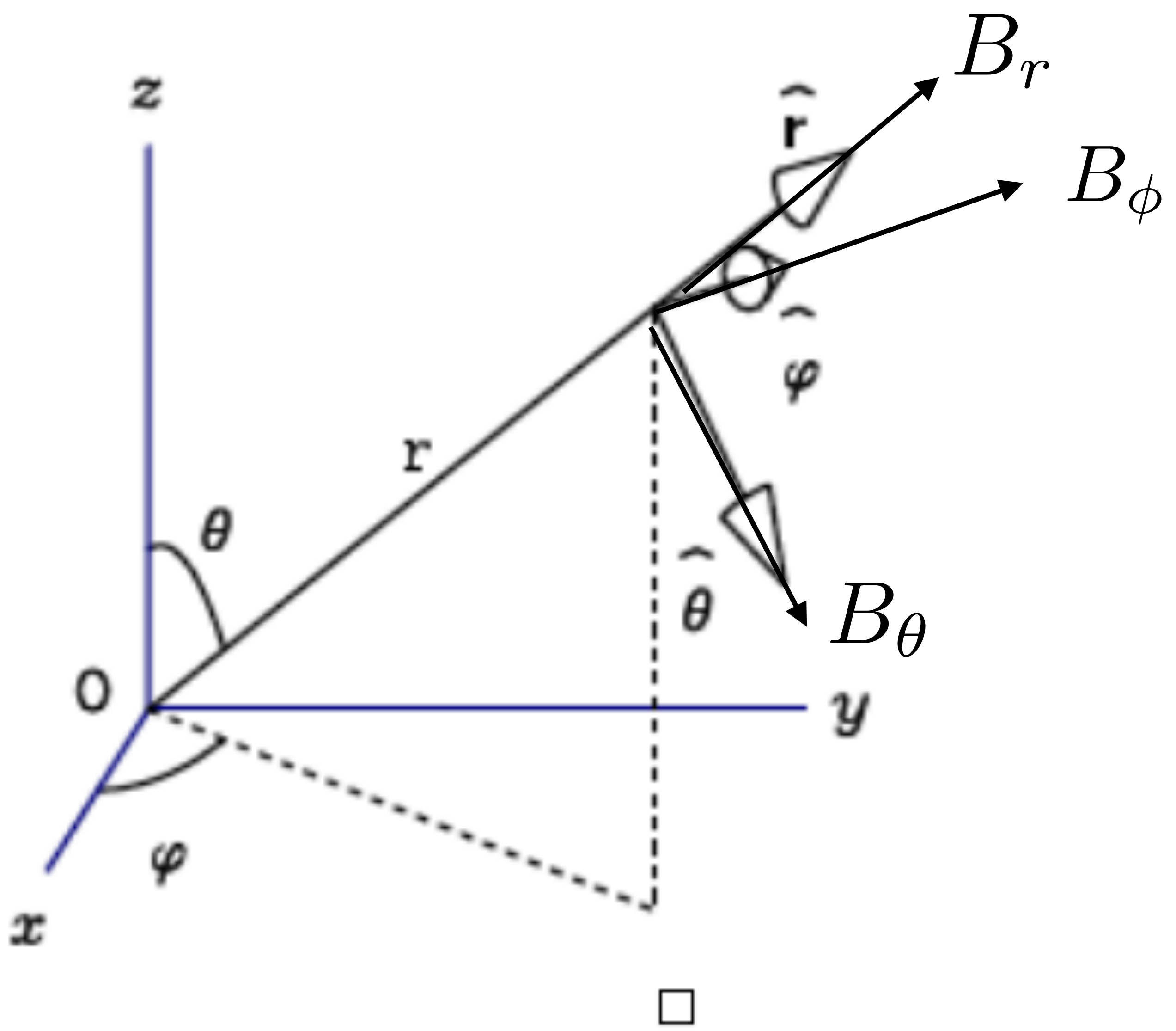
F (or B) - magnitude of total field **B**

X - north component
Y - east component
Z - vertical component, +ve down

D - declination, +ve east
H - horizontal component
I - inclination, +ve down

In geocentric coordinates

B_r, positive radially outward
B_θ, positive southward on reference sphere
B_ϕ, positive eastward on surface sphere



If we are prepared to assume Earth is a sphere, then for

X, Y, Z - orthogonal components of the geomagnetic field in local coordinate system
B_r, B_θ, B_ϕ – orthogonal components in geocentric reference frame

$$X = -B_\theta$$

$$Y = B_\phi$$

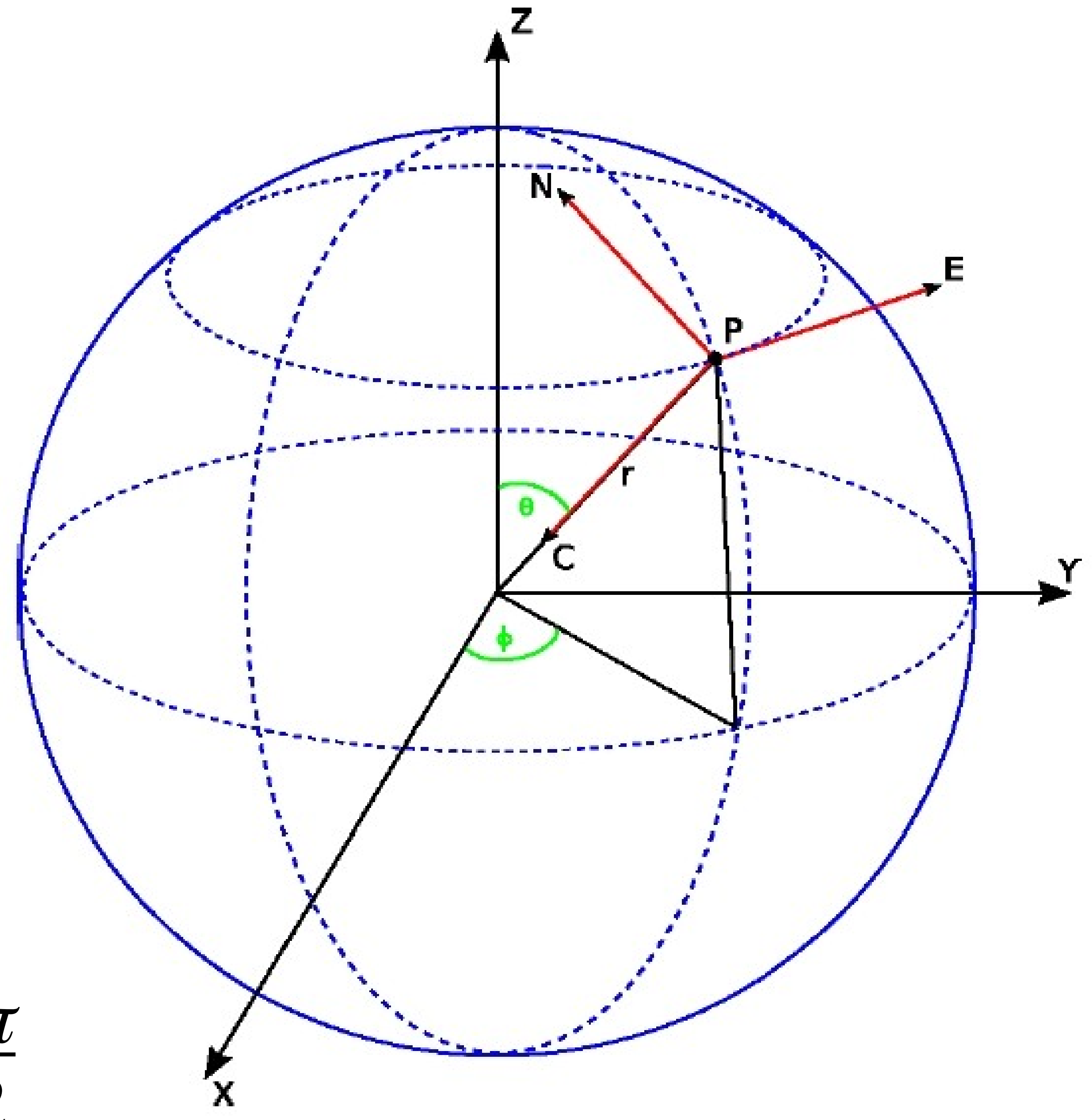
$$Z = -B_r$$

$$H = \left(B_\theta^2 + B_\phi^2\right)^{1/2} = (X^2 + Y^2)^{1/2}$$

$$F = \left(B_\theta^2 + B_\phi^2 + B_r^2\right)^{1/2} = (X^2 + Y^2 + Z^2)^{1/2}$$

$$I = \arctan\left[\frac{-B_r}{\left(B_\theta^2 + B_\phi^2\right)^{1/2}}\right] \quad \text{where} -\frac{\pi}{2} \leqslant I \leqslant \frac{\pi}{2}$$

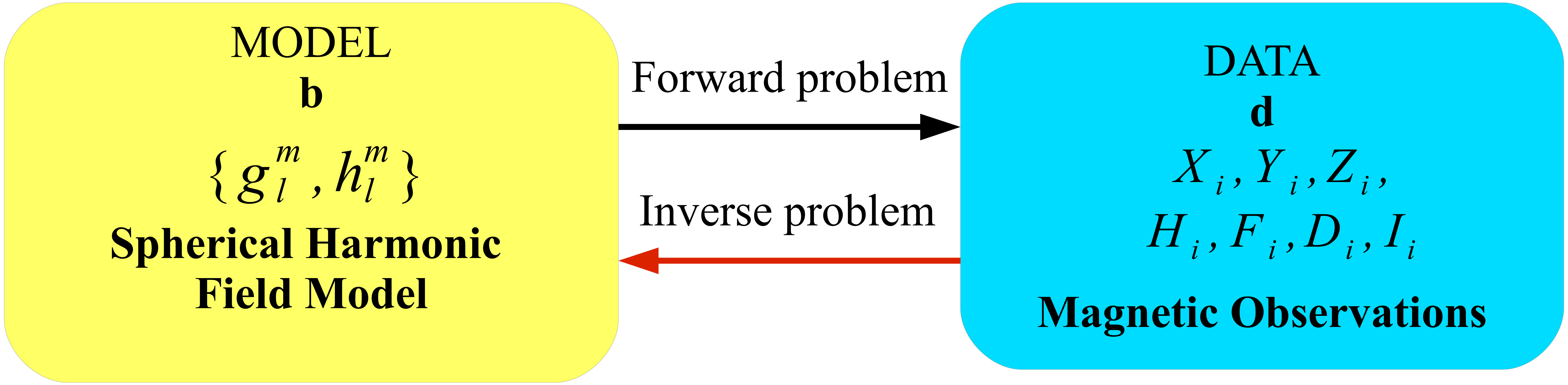
$$D = \arctan\left[\frac{B_\phi}{-B_\theta}\right] \quad \text{where} -\pi \leqslant D \leqslant \pi$$



Note, the spherical approximation does lead to detectable errors in accurate field models.

Construction of Field Models

- Spherical harmonics can be used to define a global geomagnetic field model.
- From this model we can determine any magnetic component at any location of interest (outside source regions).
- But, how do we determine the Gauss coefficients defining the spherical harmonic model best fitting observations?



Can we determine the field uniquely?

Suppose we specialize to the case of internal field only

- If we know B_r exactly everywhere on a spherical surface, we can find a unique representation of the field everywhere Laplace's equation holds (exterior Dirichlet Boundary Value Problem or BVP).

- But |B| is not enough - unlike the gravity case ambiguity about the sign is enough to cause problems (George Backus, in 1968 published an example. This led to launch of MAGSAT, the first vector magnetic satellite).

- Nor does knowing $\hat{\mathbf{B}}$, the direction of **B,** everywhere determine **B** to within a scalar multiple.

- In any case we do not know **B** either everywhere or exactly - this produces fundamental non-uniqueness in the results of any modeling activity.

A slightly more realistic view of data gathering and impact on uniqueness

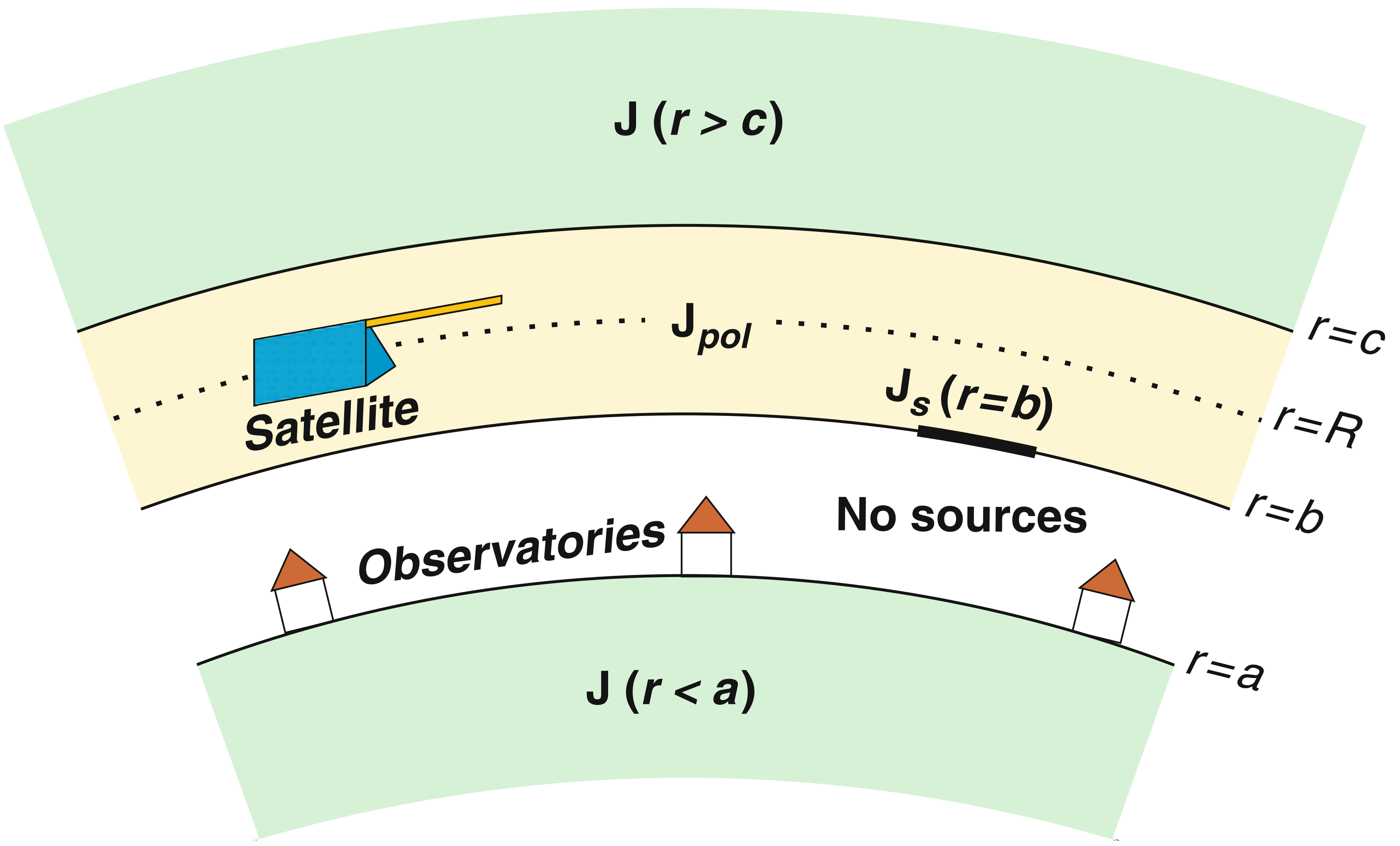
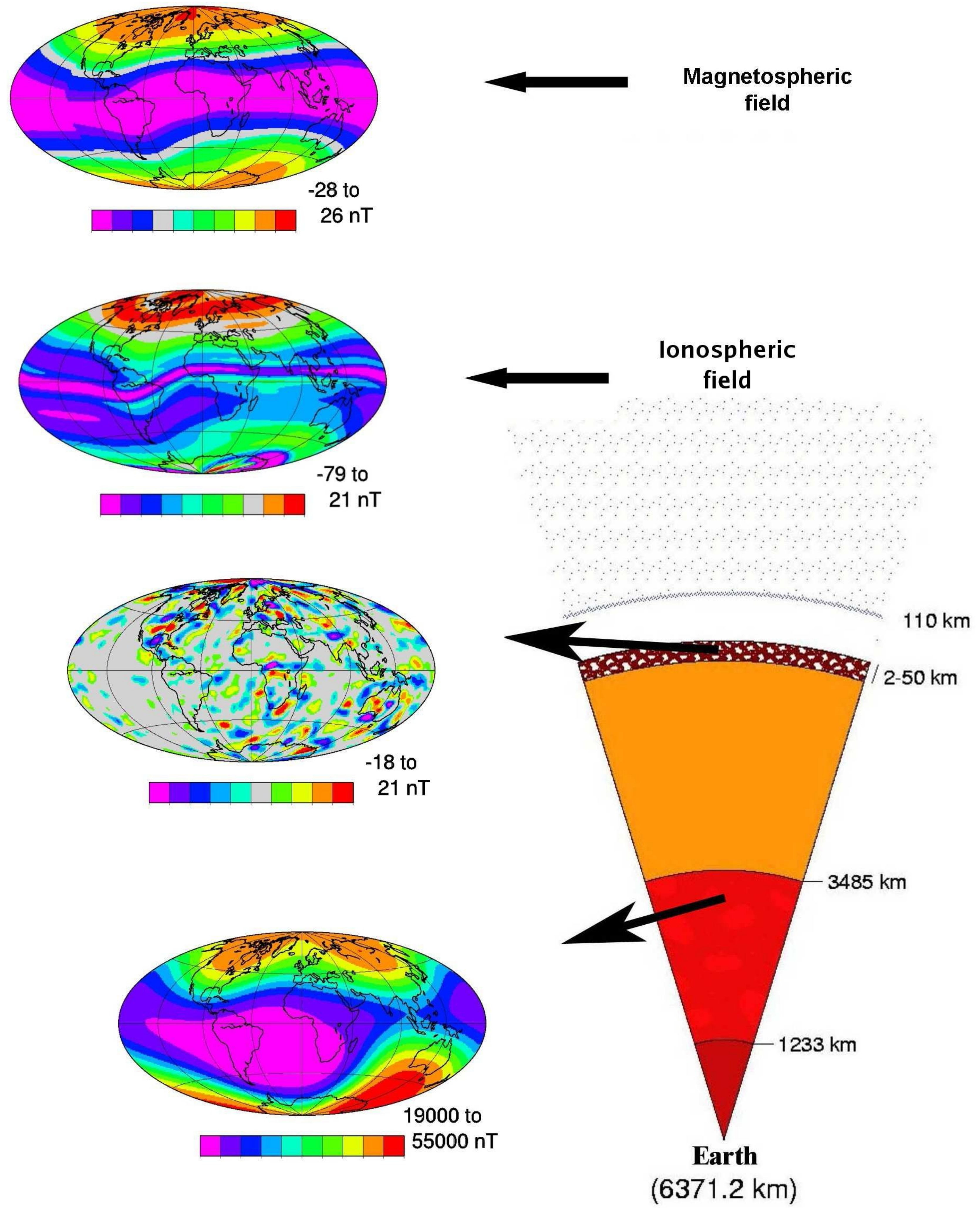


Fig. 4

Uniqueness of a magnetic field recovered from partial information within a current-carrying shell. In this special case relevant to geomagnetism, it is assumed that any source can lie below $r = a$ (*internal* $J(r < a)$ sources), and above $r = c$ (*external* $J(r > c)$ sources), no sources can lie within the lower subshell ($a < r < b$, the neutral atmosphere), a spherical sheet current can lie at $r = b$ (the *E*-region $J_s(r = b)$ sources), and only poloidal sources can lie within the upper subshell ($b < r < c$, the *F*-region ionosphere). The knowledge of B on a sphere $r = R$ in the upper subshell (as provided by, e.g., a satellite) and of enough components of B on the sphere $r = a$ (as provided by, e.g., observatories at the Earth's surface), is then enough to recover the field produced by most sources in many places (see text for details)



Construction of Field Models

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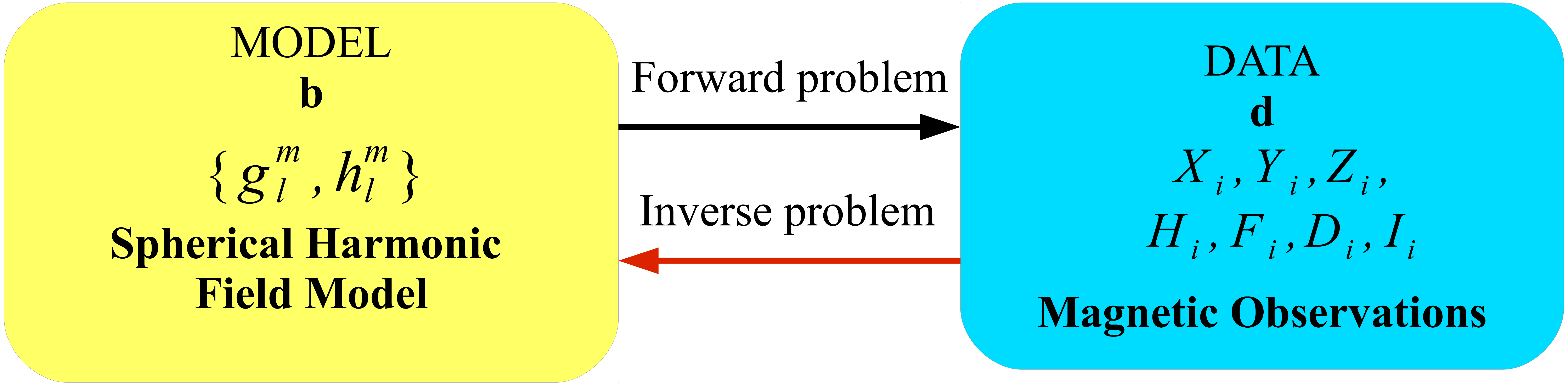


Last time we introduced Least Squares Estimation

Data vector $\mathbf{d} = (d_1, d_2, d_3, ..., d_M)^T$ Uncertainty estimates $\sigma = (\sigma_1, \sigma_2, ..., \sigma_M)^T$

Prediction of data based on forward function ***f***, model vector **m**, and observation position **x**,

$$\hat{\mathbf{d}} = f(\mathbf{x}, \mathbf{m})$$

$$\mathbf{x} = (x_1, x_2, x_3, ..., x_M) \qquad \mathbf{m} = (m_1, m_2, ..., m_N)^T$$

Least Squares Estimation finds m that minimizes the sum of the squared residuals

$$\chi^2 = \sum_{i=1}^{M} \frac{1}{\sigma_i^2}\left[d_i - f(x_i, \mathbf{m})\right]^2$$

in matrix notation

$$\chi^2 = ||\mathbf{W}(\mathbf{d} - \hat{\mathbf{d}})||^2 = ||\mathbf{Wd} - \mathbf{W}f(\mathbf{m})||^2 = ||\mathbf{Wd} - \mathbf{WFm}||^2$$

$\mathbf{W} = \mathrm{diag}(1/\sigma_1, 1/\sigma_2, ...1/\sigma_M)$ and when ***f*** is linear we can write it as the design matrix, $\mathbf{F}$.

Construction of Field Models via Least Squares fit to Spherical Harmonic expansion (SHE)

Back to the fully normalized SHE for the internal part of the field
We now truncate the expansion at some degree L - presumed large enough to accommodate all relevant field structure.

$$\Psi(r, \theta, \phi) = a \sum_{l=1}^{L} \sum_{m=-l}^{l} b_l^m \left(\frac{a}{r}\right)^{l\,+\,1} Y_l^m(\theta, \phi) \tag{60}$$

and we measure

$$\mathbf{B} = -\nabla\Psi. \tag{61}$$

and we want to find the b_l^m by least squares estimation.

Construction of Field Models via Least Squares fit to Spherical Harmonic expansion (SHE)

Suppose we measure all three orthogonal components of the field B_r, B_θ, B_ϕ at P locations, a set of $M = 3P$ observations of magnetic elements at sites $\mathbf{r}_p = (\hat{r}_p, \hat{\theta}_p, \hat{\phi}_p)$, the vector field is written $\mathbf{B}(\mathbf{r}_p), p = 1, \ldots, P$.
let $\hat{\mathbf{s}}_{p_i}$, for $i = 1, \ldots, 3$ be the unit vector along one of the orthogonal (r, θ, ϕ) directions at location $\mathbf{r}_j$,

$$\begin{aligned} d_j &= \hat{\mathbf{s}}_{p_i} \cdot \mathbf{B}(\mathbf{r}_j), \quad j = 1, ..., M \\ &= \sum_{l=1}^{L} \sum_{m=-l}^{l} b_l^m a^{l\,+\,2} \hat{\mathbf{s}}_{p_i} \cdot \nabla \left[\frac{Y_l^m(\hat{\mathbf{r}}_j)}{r_j^{l\,+\,1}} \right] \,+\, \epsilon_j. \end{aligned} \tag{1}$$

Note that we have allowed for uncertainty in each observation through ϵ_j.
We can write a prediction for our observations d_j as a matrix equation.

$$\mathbf{d} = \mathbf{G}\mathbf{b} \,+\, \mathbf{e} \tag{63}$$

We need an indexing scheme for:

$$\mathbf{d} = \mathbf{G}\mathbf{b} + \mathbf{e} \tag{63}$$

$\mathbf{d}$ and $\mathbf{e} \in R^M$,

$\mathbf{b} \in R^K$ is a vector containing an ordered list of the spherical harmonic coefficients b_l^m. K, is given by the truncation level: $K = L(L + 2)$

$\mathbf{G}$ is an $M \times K$ matrix that tell us how to predict the observations based on their positions $\mathbf{r}_j$ and the various $Y_l^m(\theta, \phi)$.

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_M \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1^{-1} \\ b_1^0 \\ b_1^1 \\ \vdots \\ b_L^L \end{bmatrix} \quad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_M \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots & g_{1K} \\ g_{21} & g_{22} & g_{23} & \cdots & g_{2K} \\ g_{31} & g_{32} & g_{33} & \cdots & g_{3K} \\ \vdots & & & & \\ g_{M1} & g_{M2} & g_{M3} & \cdots & g_{MK} \end{bmatrix}$$

Least squares estimation involves finding the values for $\mathbf{b}$ that minimize $||\mathbf{e}||^2 = ||\mathbf{d} - \mathbf{G}\mathbf{b}||^2$

the LS solution vector $\tilde{\mathbf{b}}$ can be written in terms of the solution to the normal equations:

$$\tilde{\mathbf{b}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{d}. \tag{65}$$

The least squares solution is the best linear unbiased estimate (BLUE) available - the resulting coefficients have the smallest variance amongst such estimates. Also

$$\mathcal{E}[||\mathbf{e}||^2] = (M - K)\sigma^2 \tag{66}$$

This is an overdetermined LS problem with $M > K$, and uncertainty in the resulting parameter estimates will be proportional to $\sqrt{M - K}$

Least squares produces results like the IGRF which since 1965 extends to SH degree n=13

g/h	Deg	Ord	DGRF	DGRF	DGRF	DGRF	DGRF	DGRF	DGRF	DGRF	DGRF	DGRF	DGRF	IGRF	SV
	n	m	1965.0	1970.0	1975.0	1980.0	1985.0	1990.0	1995.0	2000.0	2005.0	2010.0	2015.0	2020.0	20-25
g	1	0	−30334	−30220	−30100	−29992	−29873	−29775	−29692	−29619.4	−29554.63	−29496.57	−29441.46	−29404.8	5.7
g	1	1	−2119	−2068	−2013	−1956	−1905	−1848	−1784	−1728.2	−1669.05	−1586.42	−1501.77	−1450.9	7.4
h	1	1	5776	5737	5675	5604	5500	5406	5306	5186.1	5077.99	4944.26	4795.99	4652.5	−25.9
g	2	0	−1662	−1781	−1902	−1997	−2072	−2131	−2200	−2267.7	−2337.24	−2396.06	−2445.88	−2499.6	−11.0
g	2	1	2997	3000	3010	3027	3044	3059	3070	3068.4	3047.69	3026.34	3012.20	2982.0	−7.0
h	2	1	−2016	−2047	−2067	−2129	−2197	−2279	−2366	−2481.6	−2594.50	−2708.54	−2845.41	−2991.6	−30.2
g	2	2	1594	1611	1632	1663	1687	1686	1681	1670.9	1657.76	1668.17	1676.35	1677.0	−2.1
h	2	2	114	25	−68	−200	−306	−373	−413	−458.0	−515.43	−575.73	−642.17	−734.6	−22.4
g	3	0	1297	1287	1276	1281	1296	1314	1335	1339.6	1336.30	1339.85	1350.33	1363.2	2.2
g	3	1	−2038	−2091	−2144	−2180	−2208	−2239	−2267	−2288.0	−2305.83	−2326.54	−2352.26	−2381.2	−5.9
h	3	1	−404	−366	−333	−336	−310	−284	−262	−227.6	−198.86	−160.40	−115.29	−82.1	6.0
g	3	2	1292	1278	1260	1251	1247	1248	1249	1252.1	1246.39	1232.10	1225.85	1236.2	3.1
h	3	2	240	251	262	271	284	293	302	293.4	269.72	251.75	245.04	241.9	−1.1
g	3	3	856	838	830	833	829	802	759	714.5	672.51	633.73	581.69	525.7	−12.0
h	3	3	−165	−196	−223	−252	−297	−352	−427	−491.1	−524.72	−537.03	−538.70	−543.4	0.5
g	4	0	957	952	946	938	936	939	940	932.3	920.55	912.66	907.42	903.0	−1.2
g	4	1	804	800	791	782	780	780	780	786.8	797.96	808.97	813.68	809.5	−1.6
h	4	1	148	167	191	212	232	247	262	272.6	282.07	286.48	283.54	281.9	−0.1
g	4	2	479	461	438	398	361	325	290	250.0	210.65	166.58	120.49	86.3	−5.9
h	4	2	−269	−266	−265	−257	−249	−240	−236	−231.9	−225.23	−211.03	−188.43	−158.4	6.5
g	4	3	−390	−395	−405	−419	−424	−423	−418	−403.0	−379.86	−356.83	−334.85	−309.4	5.2
h	4	3	13	26	39	53	69	84	97	119.8	145.15	164.46	180.95	199.7	3.6
g	4	4	252	234	216	199	170	141	122	111.3	100.00	89.40	70.38	48.0	−5.1
h	4	4	−269	−279	−288	−297	−297	−299	−306	−303.8	−305.36	−309.72	−329.23	−349.7	−5.0
g	5	0	−219	−216	−218	−218	−214	−214	−214	−218.8	−227.00	−230.87	−232.91	−234.3	−0.3
g	5	1	358	359	356	357	355	353	352	351.4	354.41	357.29	360.14	363.2	0.5
h	5	1	19	26	31	46	47	46	46	43.8	42.72	44.58	46.98	47.7	0.0
g	5	2	254	262	264	261	253	245	235	222.3	208.95	200.26	192.35	187.8	−0.6
h	5	2	128	139	148	150	150	154	165	171.9	180.25	189.01	196.98	208.3	2.5
g	5	3	−31	−42	−59	−74	−93	−109	−118	−130.4	−136.54	−141.05	−140.94	−140.7	0.2
h	5	3	−126	−139	−152	−151	−154	−153	−143	−133.1	−123.45	−118.06	−119.14	−121.2	−0.6
g	5	4	−157	−160	−159	−162	−164	−165	−166	−168.6	−168.05	−163.17	−157.40	−151.2	1.3
h	5	4	−97	−91	−83	−78	−75	−69	−55	−39.3	−19.57	−0.01	15.98	32.3	3.0
g	5	5	−62	−56	−49	−48	−46	−36	−17	−12.9	−13.55	−8.03	4.30	13.5	0.9
h	5	5	81	83	88	92	95	97	107	106.3	103.85	101.04	100.12	98.9	0.3
g	6	0	45	43	45	48	53	61	68	72.3	73.60	72.78	69.55	66.0	−0.5
g	6	1	61	64	66	66	65	65	67	68.2	69.56	68.69	67.57	65.5	−0.3
h	6	1	−11	−12	−13	−15	−16	−16	−17	−17.4	−20.33	−20.90	−20.61	−19.1	0.0
g	6	2	8	15	28	42	51	59	68	74.2	76.74	75.92	72.79	72.9	0.4
h	6	2	100	100	99	93	88	82	72	63.7	54.75	44.18	33.30	25.1	−1.6
g	6	3	−228	−212	−198	−192	−185	−178	−170	−160.9	−151.34	−141.40	−129.85	−121.5	1.3
h	6	3	68	72	75	71	69	69	67	65.1	63.63	61.54	58.74	52.8	−1.3
g	6	4	4	2	1	4	4	3	−1	−5.9	−14.58	−22.83	−28.93	−36.2	−1.4
h	6	4	−32	−37	−41	−43	−48	−52	−58	−61.2	−63.53	−66.26	−66.64	−64.5	0.8
g	6	5	1	3	6	14	16	18	19	16.9	14.58	13.10	13.14	13.5	0.0
h	6	5	−8	−6	−4	−2	−1	1	1	0.7	0.24	3.02	7.35	8.9	0.0
g	6	6	−111	−112	−111	−108	−102	−96	−93	−90.4	−86.36	−78.09	−70.85	−64.7	0.9
h	6	6	−7	1	11	17	21	24	36	43.8	50.94	55.40	62.41	68.1	1.0

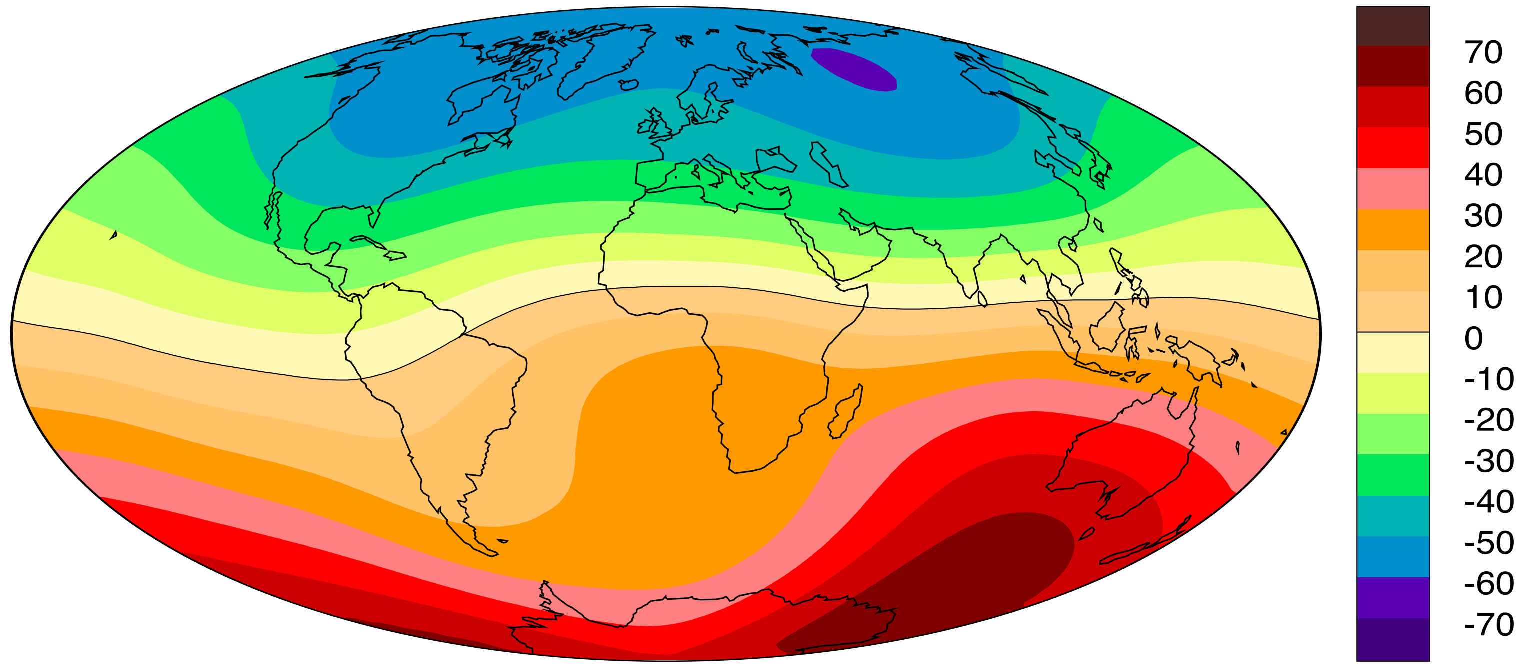
The time dependence of these parameters is modeled as piecewise linear, and is given by

$$g_n^m(t) = g_n^m(T_t) + (t - T_t)\dot{g}_n^m(T_t),$$

Alken et al. Earth, Planets and Space (2021) 73:49
https://doi.org/10.1186/s40623-020-01288-x

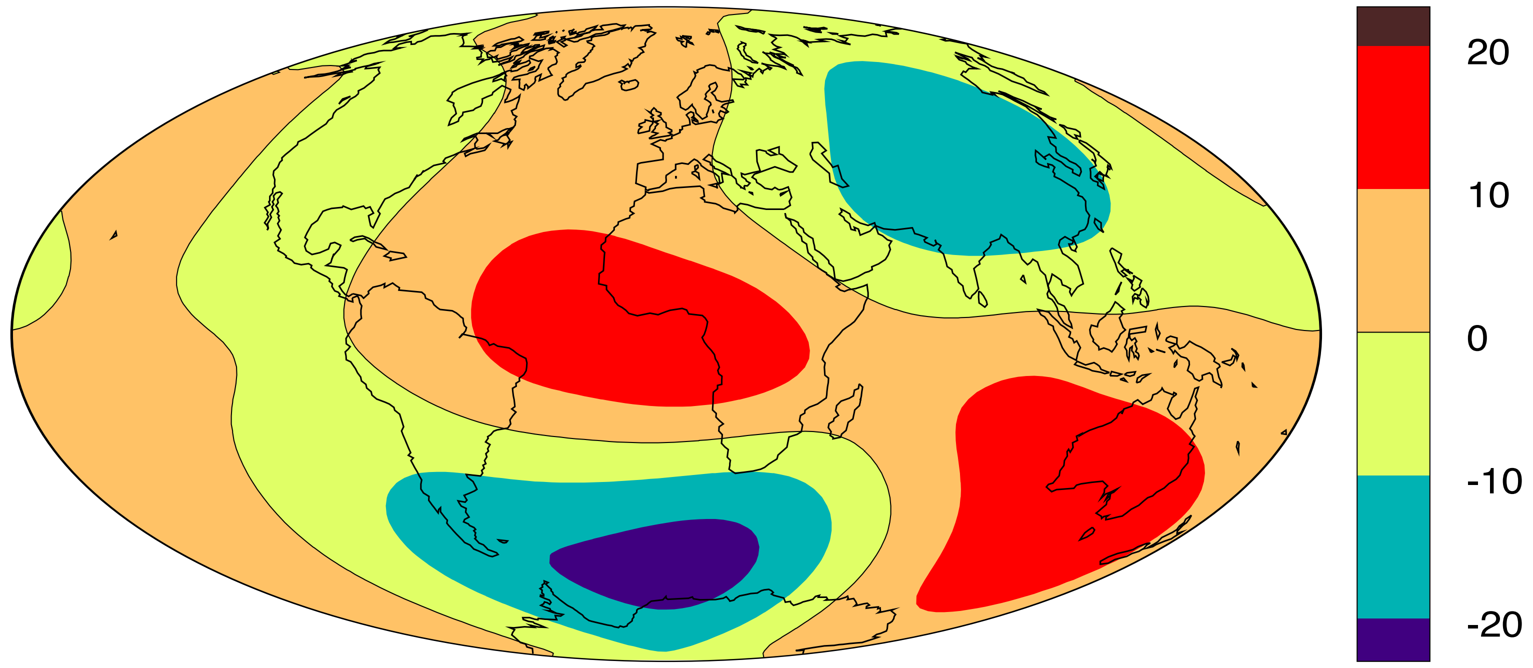
B_r for the IGRF in 2020

Radial field at r=a



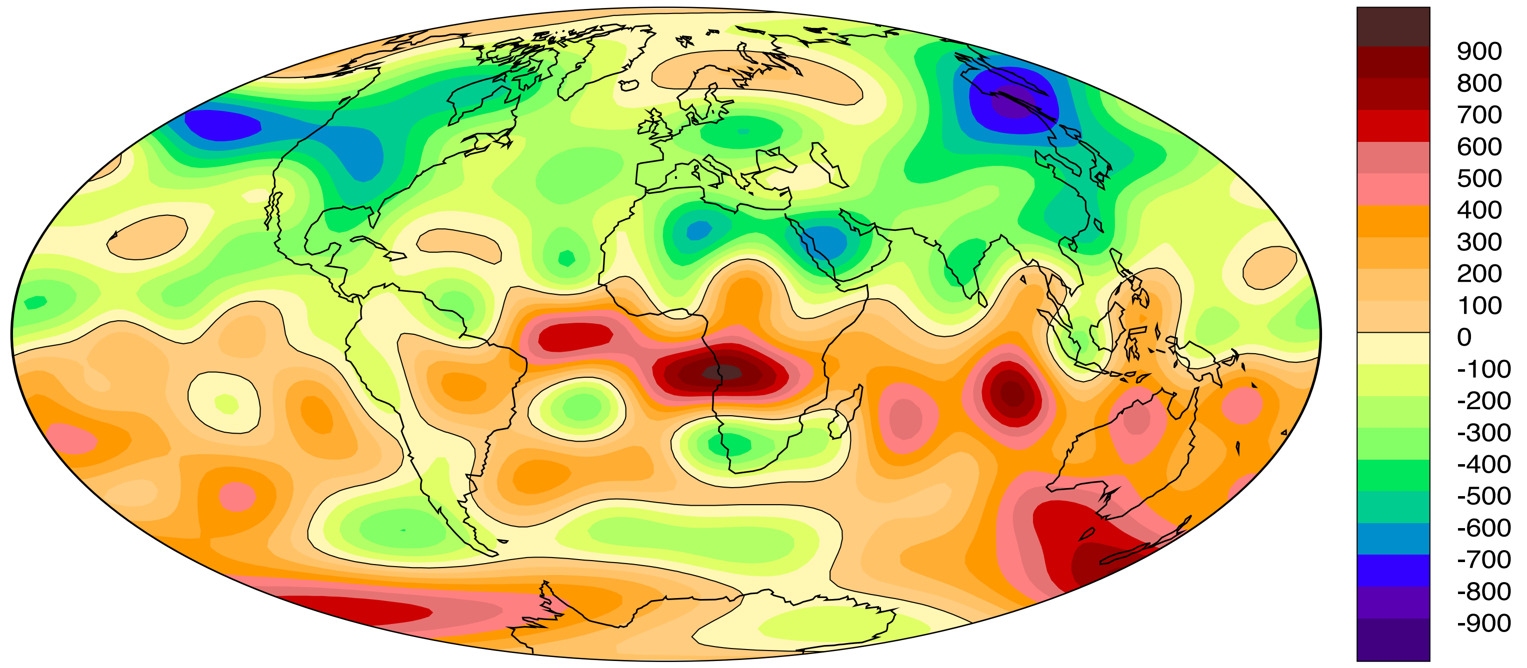
IGRF 2020 Radial field Br at r=a

Non-dipole radial field at r=a



IGRF 2020 Non-Dipole Radial field Br at r=a

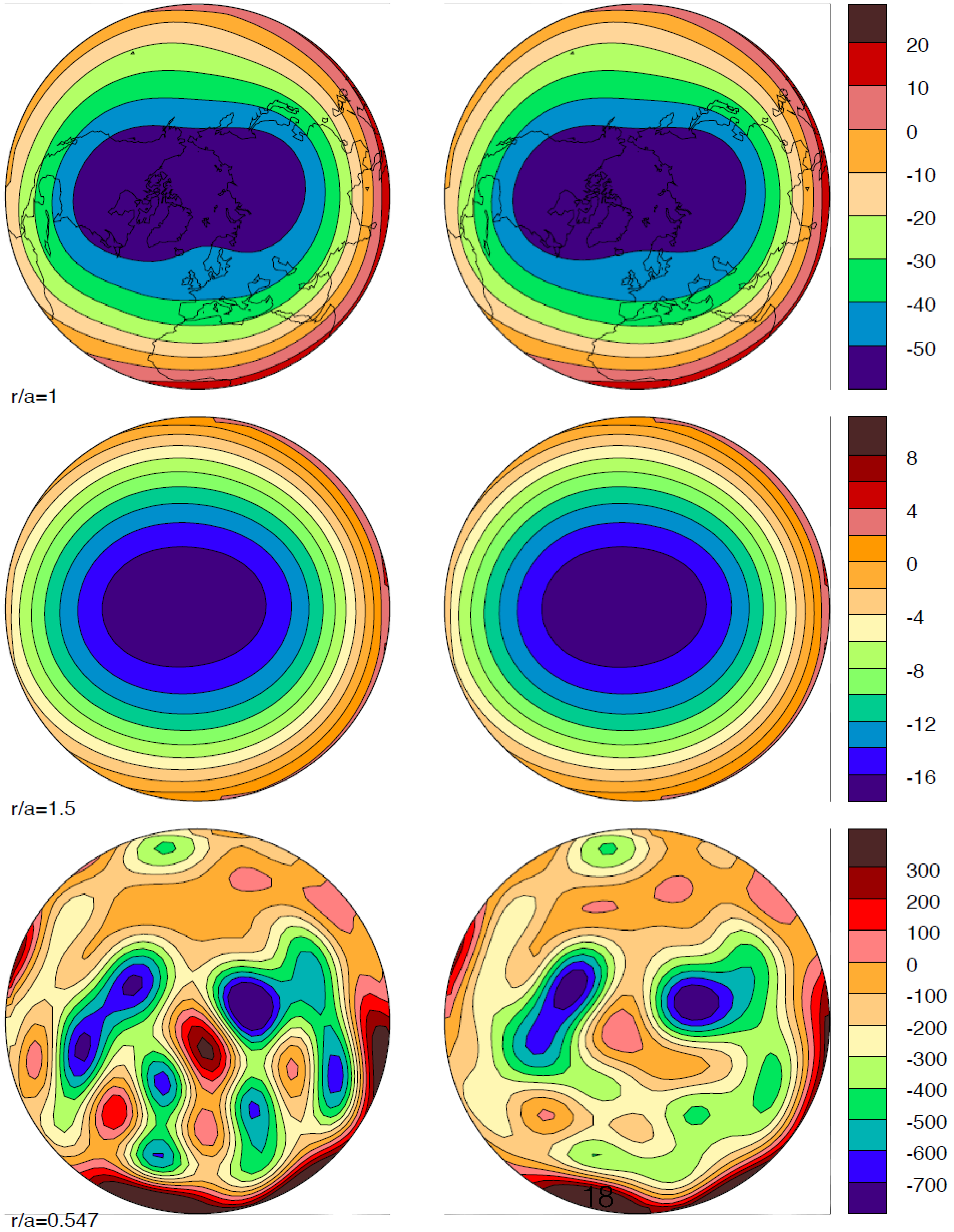
Downward continued radial field at r=c



IGRF 2020 Radial field Br at r=c

Radial component of the IGRF1980 magnetic field

with added noise without noise



Earth Surface, r=a

r=1.5a

CMB, r=0.547a

Regularization – an Alternative to Least Squares

• In many geophysical inverse problems:

(i) Models can be very complex, especially those arising from the discretization of continuous physical fields, and may possess large ‘null spaces’ not constrained by data.

(ii) Data are contaminated by large errors or parts of the model are very sensitive to data noise.

Then model non-uniqueness and instability are problems:

i.e. Many models can fit the data within the error estimates and minimizing the least squares criteria alone will not necessarily yield the most plausible model.

Recall the discussion of interpolation from Lecture 7

- We used cubic splines to “regularize” time variations by minimizing complexity as measured by the integrated second derivative in time of interpolating function.

- Making a static model of the geomagnetic field is similar to interpolation or fitting a function to data in time, but now we’re interested in filling in the spatial domain.

- For a snapshot of the field in time we can instead minimize spatial complexity in the field represented by our SH model.

Splines are piecewise degree *j* polynomials in time used to make a continuous function and up to degree (*j-1*) derivatives at knot points where they join

Example: cubic spline interpolation

Minimize the objective functional

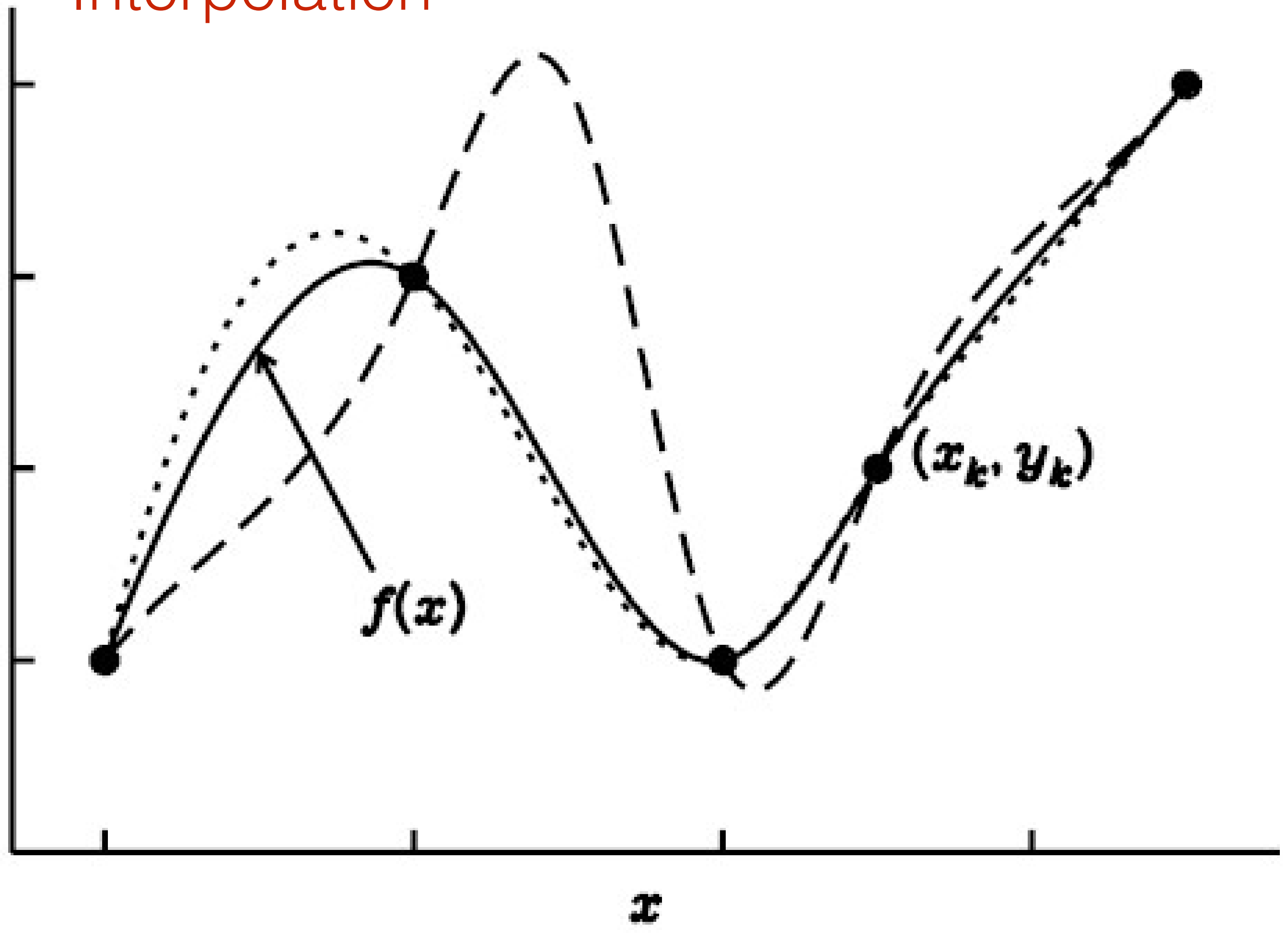
$$U = \sum_{j=1}^{N} \frac{\left(f(x_j) - y_j\right)^2}{\sigma_j^2} + \lambda \int_{x_1}^{x_N} \left[\frac{d^2 f}{dx^2}\right]^2 dx$$

smoothest curve connecting the points (RMS second derivative)

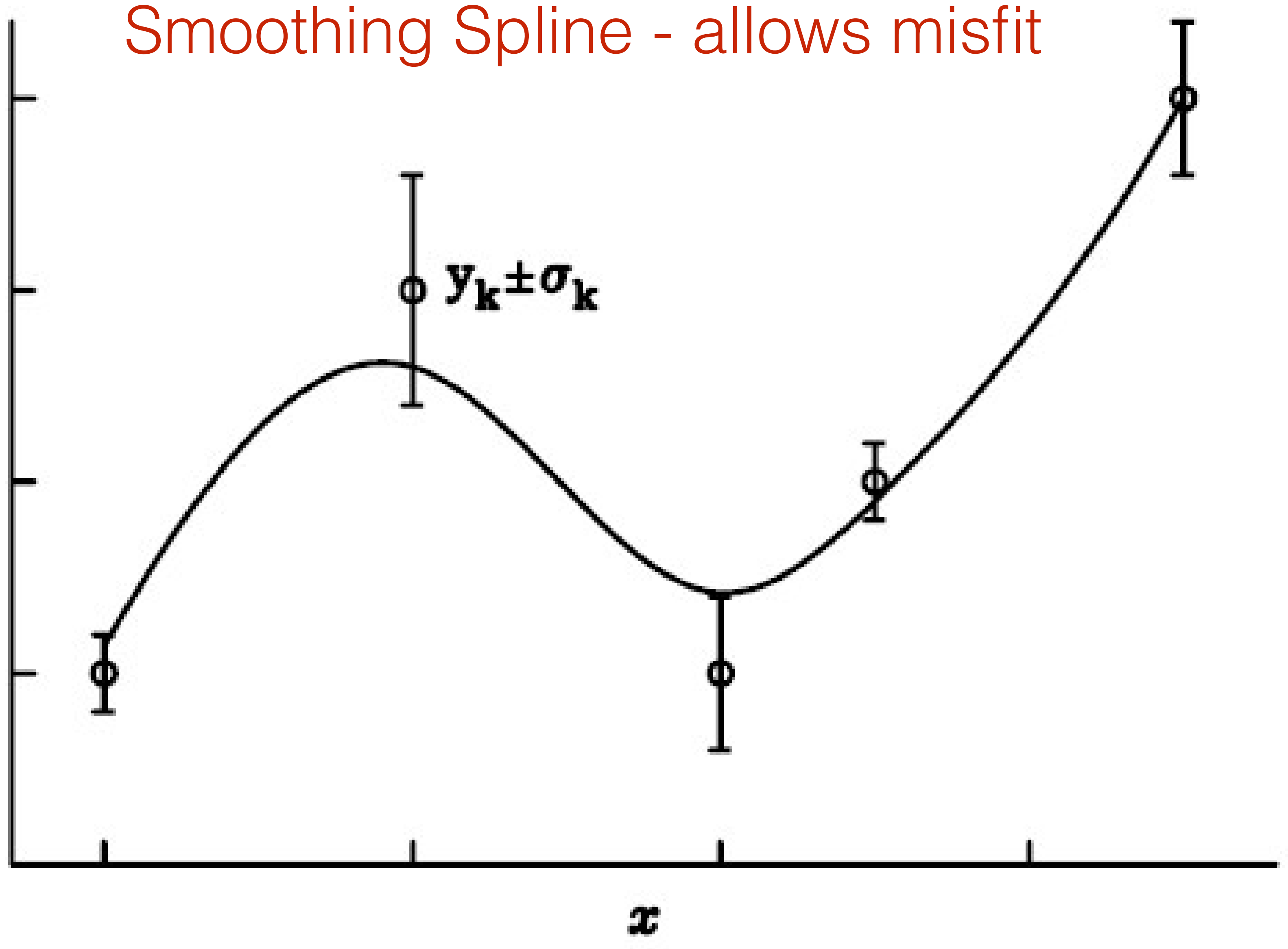
subject to $$\sum_{j=1}^{N} \frac{\left(f(x_j) - y_j\right)^2}{\sigma_j^2} = T$$

λ **is a Lagrange multiplier chosen to make this true**

Interpolation



Smoothing Spline - allows misfit



Limitation of Least Squares

- We don't know how we should choose L. Truncation may exclude parts of the model that are needed.
- Misfit can be specified by a target value for chi-squared by choosing K - the normalized sum of squares of residuals- but it includes both measurement error and model error
- What model property would it make most sense to minimize? e.g., total energy stored in the field for $r>a$

Regularizing Magnetic Field Models

- Regularization provides a tradeoff between misfit to the observations and some property of the field model.
- Misfit can be specified by a target value for chi-squared - the normalized sum of squares of residuals
- What model property would it make most sense to minimize? e.g., total energy stored in the field for $r>a$

An Objective Functional for Magnetic field Modeling

- In the spline case we trade off misfit against minimizing the norm given by the second derivative of the function $f(t)$ in time

- Misfit can be specified by a target value for chi-squared - the normalized sum of squares of residuals

- What model property would it make most sense to minimize? e.g., total energy stored in the field for $r>a$

Many interesting field spatial complexity properties can be written in the form of sums of SHE squared coefficients

Regularization – an Alternative to Least Squares

$$\|\boldsymbol{B}\|_w^2 = \sum_{l=1}^{\infty} w_l \sum_{m=-l}^{l} |b_l^m|^2, \qquad w_l > 0$$

Penalty functions:

$$\int_{r>a} |\boldsymbol{B}|^2 d^3\hat{\boldsymbol{r}} \qquad w_l = (l+1)$$

total energy in field outside of Earth radius

$$\int_{S(a)} \boldsymbol{B}\cdot\boldsymbol{B}\, d^2\hat{\boldsymbol{r}} \qquad w_l = (21+1)(l+1)$$

spatial power spectrum on Earth Surface

These are examples of norms, specific measures of the size of the geomagnetic field model b.
Where did these expression come from? See equation 70-79 in ch10.pdf for examples using information from the Table of SH Lore.

For the Magnetic Field

Least squares estimation involves finding the values for $\mathbf{b}$ that minimize

$$||\mathbf{e}||^2 = ||\mathbf{d} - \mathbf{G}\mathbf{b}||^2$$

As with the splines **regularized field models** trade off misfit against a property that minimizes complexity

Note we've gone back to $l = \infty$!
It will need to be large to allow for any necessary complexity.

$$U(\mathbf{b}) = ||\mathbf{d} - \mathbf{G}\mathbf{b}||^2 + \lambda \sum_{l=1}^{\infty} w_l |b_l^m|^2, \qquad w_l > 0$$

or more compactly

$$U(\mathbf{b}) = ||\mathbf{d} - \mathbf{G}\mathbf{b}||^2 + \lambda ||\mathbf{b}||_{w_l}^2$$

subject to

$$||\mathbf{d} - \mathbf{G}\mathbf{b}||^2 = T$$

Regularization - An Alternative to Least Squares (2)

Some more examples of norms for the field model of the form

$$||\mathbf{B}||_w^2 = \sum_{l=1}^{\infty} w_l \sum_{m=-l}^{l} |b_l^m|^2, \qquad w_l > 0.$$

the w_l functions penalize short wavelength structure in the field model with successively heavier weight for large l:

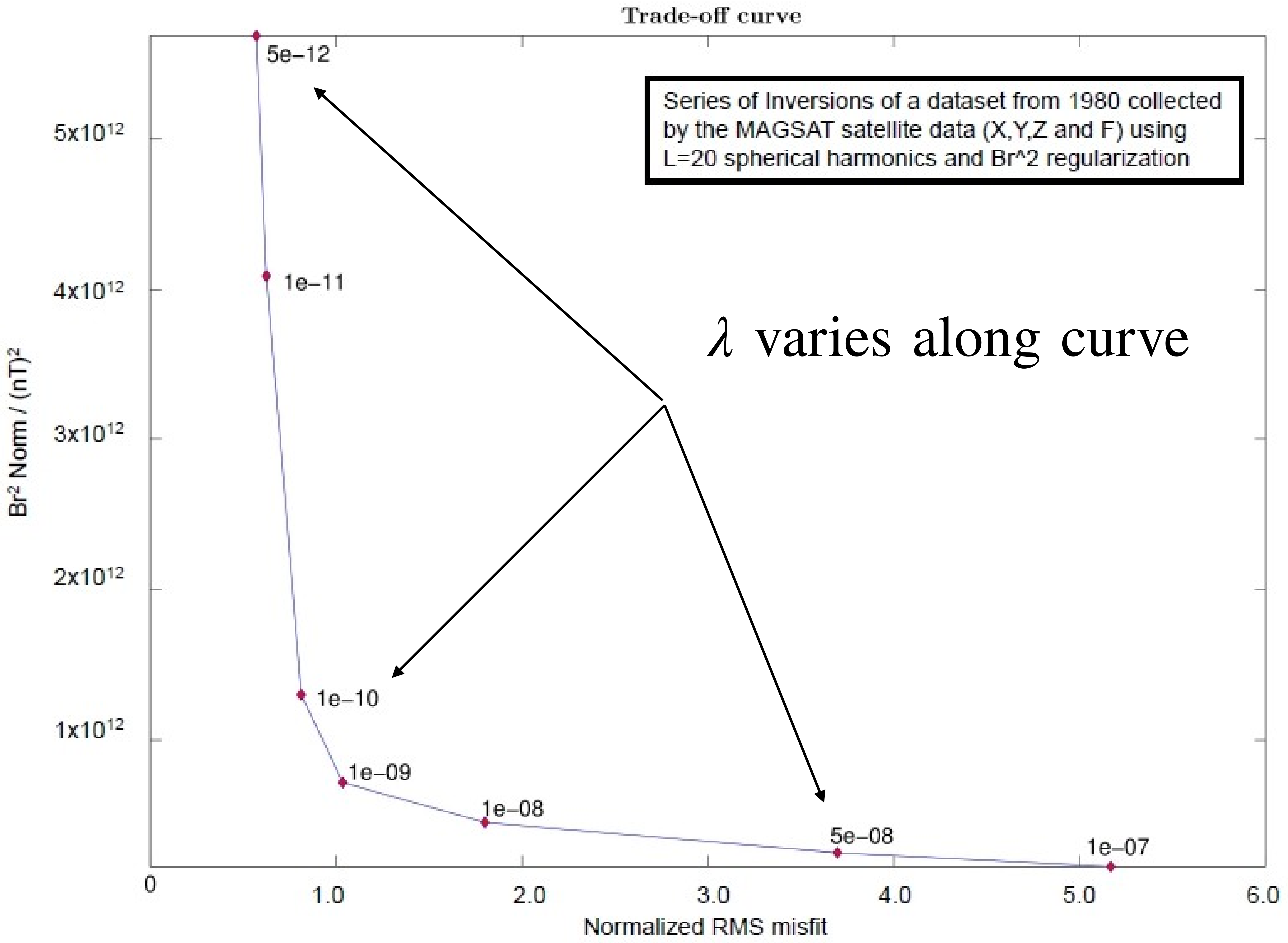
$$\int_{S(a)} [\nabla_1(\hat{\mathbf{r}} \cdot \mathbf{B})]^2 d^2\hat{\mathbf{r}} \qquad w_l = l(l\ +\ 1)^2(l\ +\ \frac{1}{2}) \qquad \text{surface gradient of } B_r$$

$$\int_{S(a)} [\nabla_1^2(\hat{\mathbf{r}} \cdot \mathbf{B})]^2 d^2\hat{\mathbf{r}} \qquad w_l = l^2(l\ +\ 1)^4 \qquad \text{surface Laplacian of } B_r$$

$$\int_{r<a} \mathbf{J}_T \cdot \mathbf{J}_T d^3\mathbf{r} \qquad w_l \approx (l\ +\ 1)(2l\ +\ 1)^2(2l\ +\ 3) \qquad \text{minimum toroidal current in Earth's outer core}$$

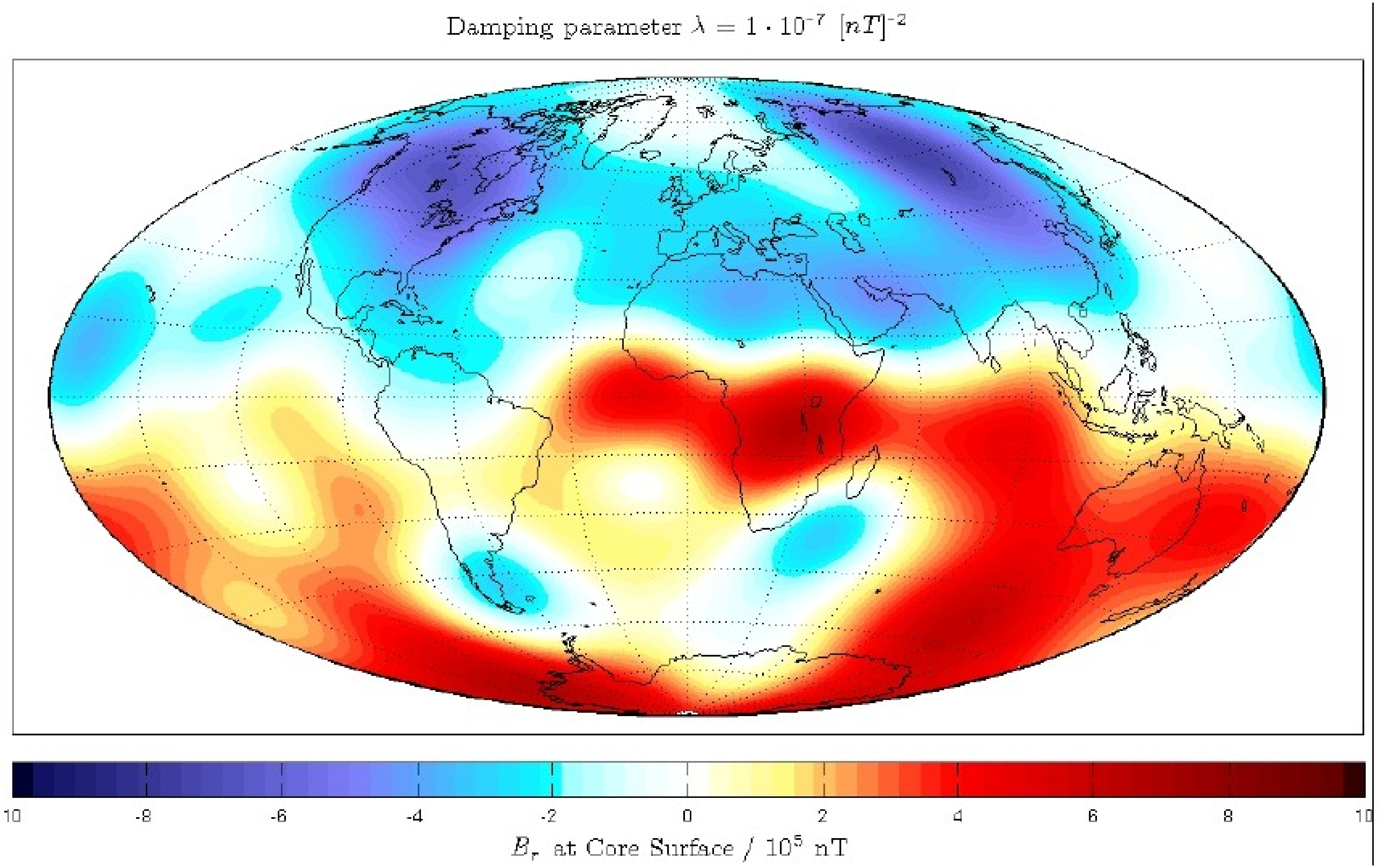
Regularization

Example of core surface field trade-off curve



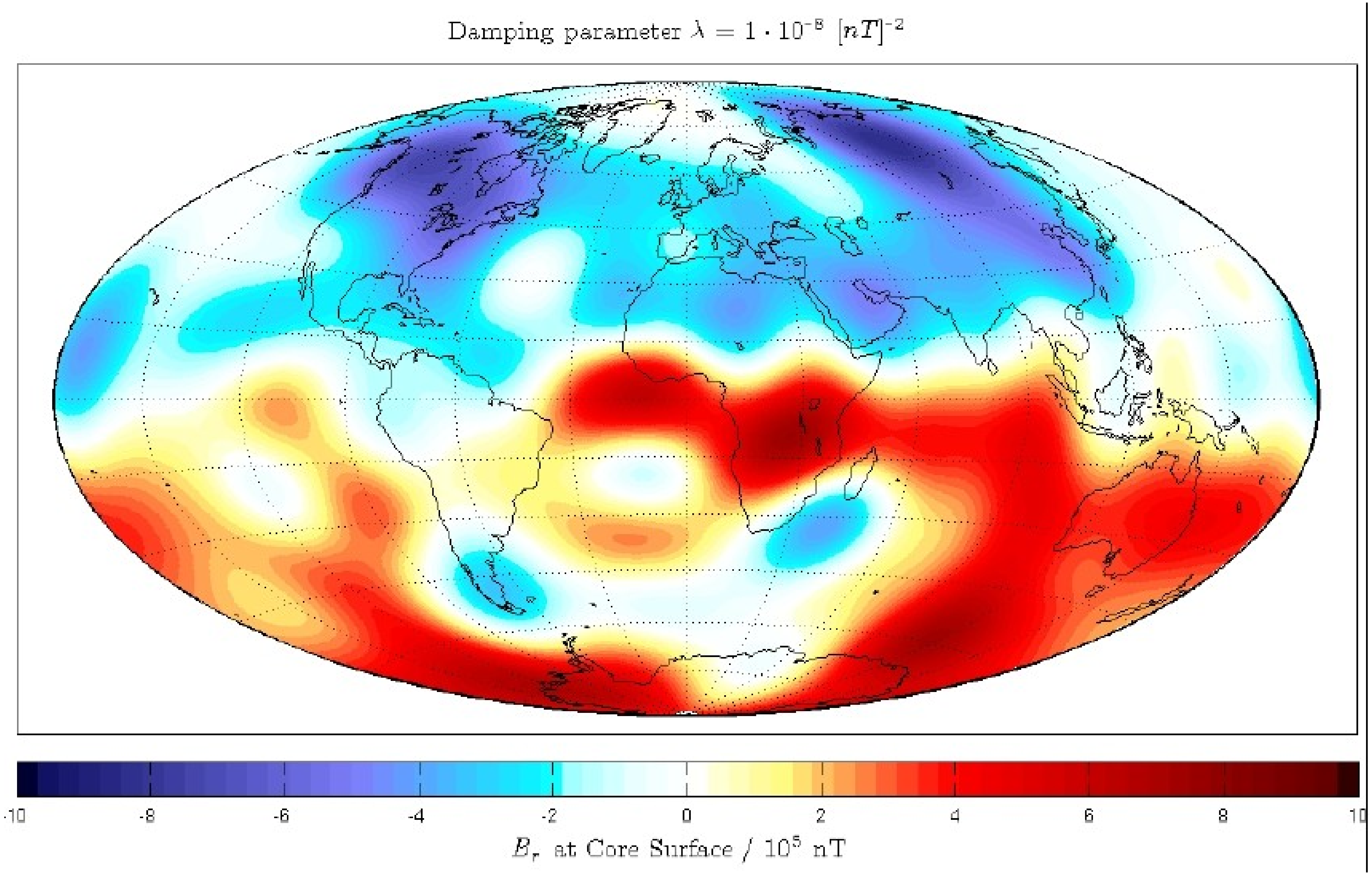
Regularization

Solution on the trade-off curve



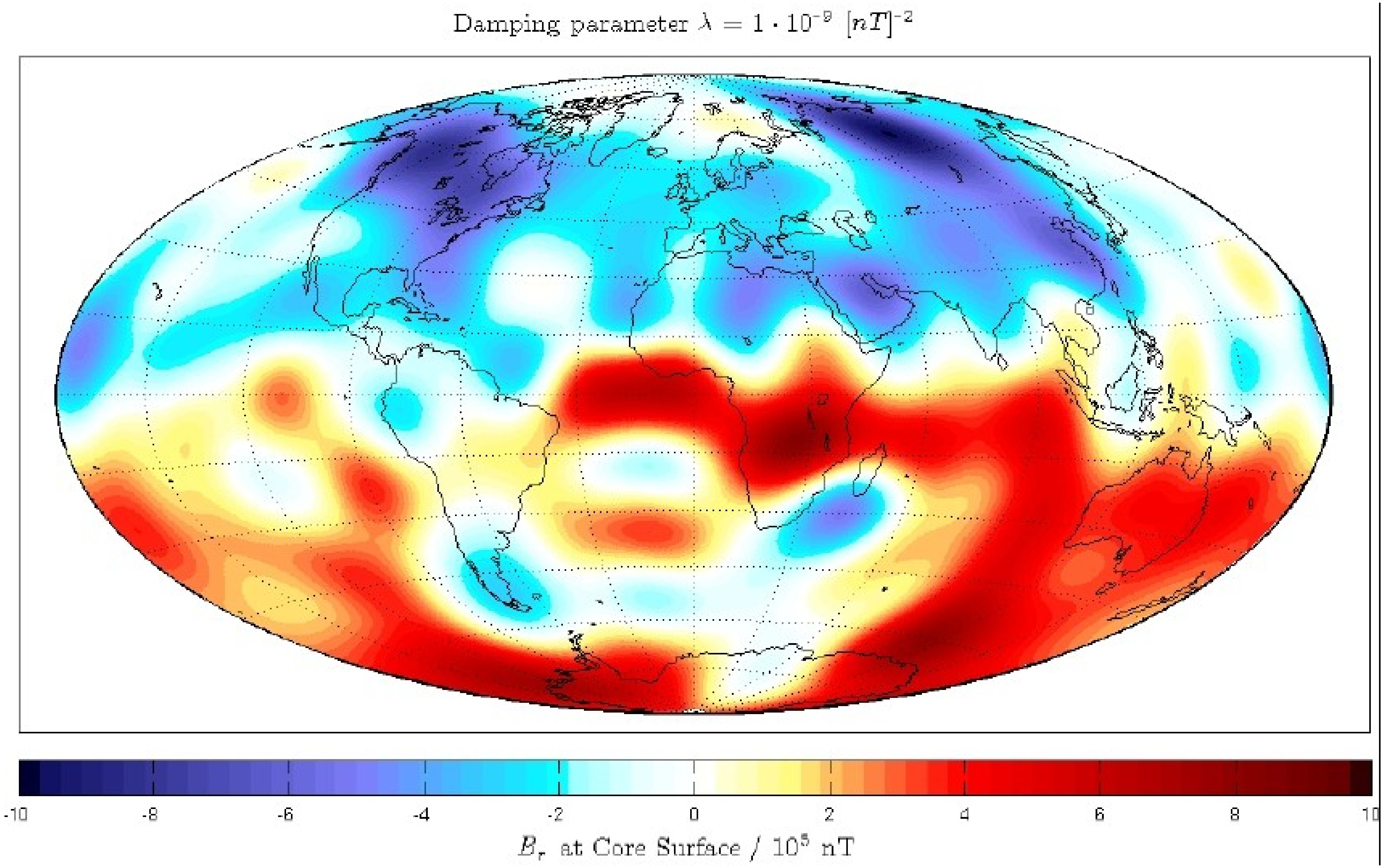
Regularization

Solution on the trade-off curve



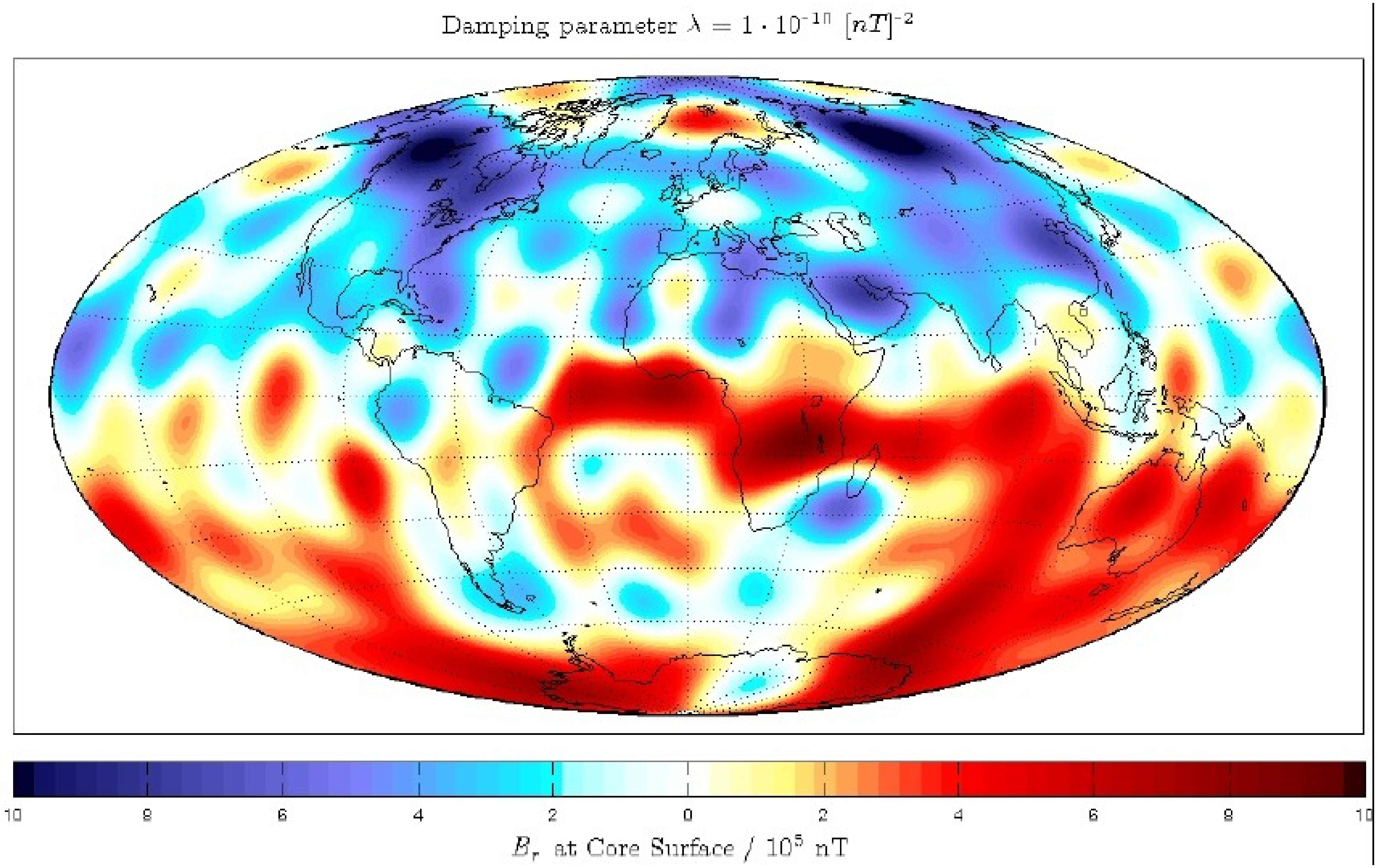
Regularization

Solution on the trade-off curve



Regularization

Solution on the trade-off curve



Regularization

olution on the trade-off curve