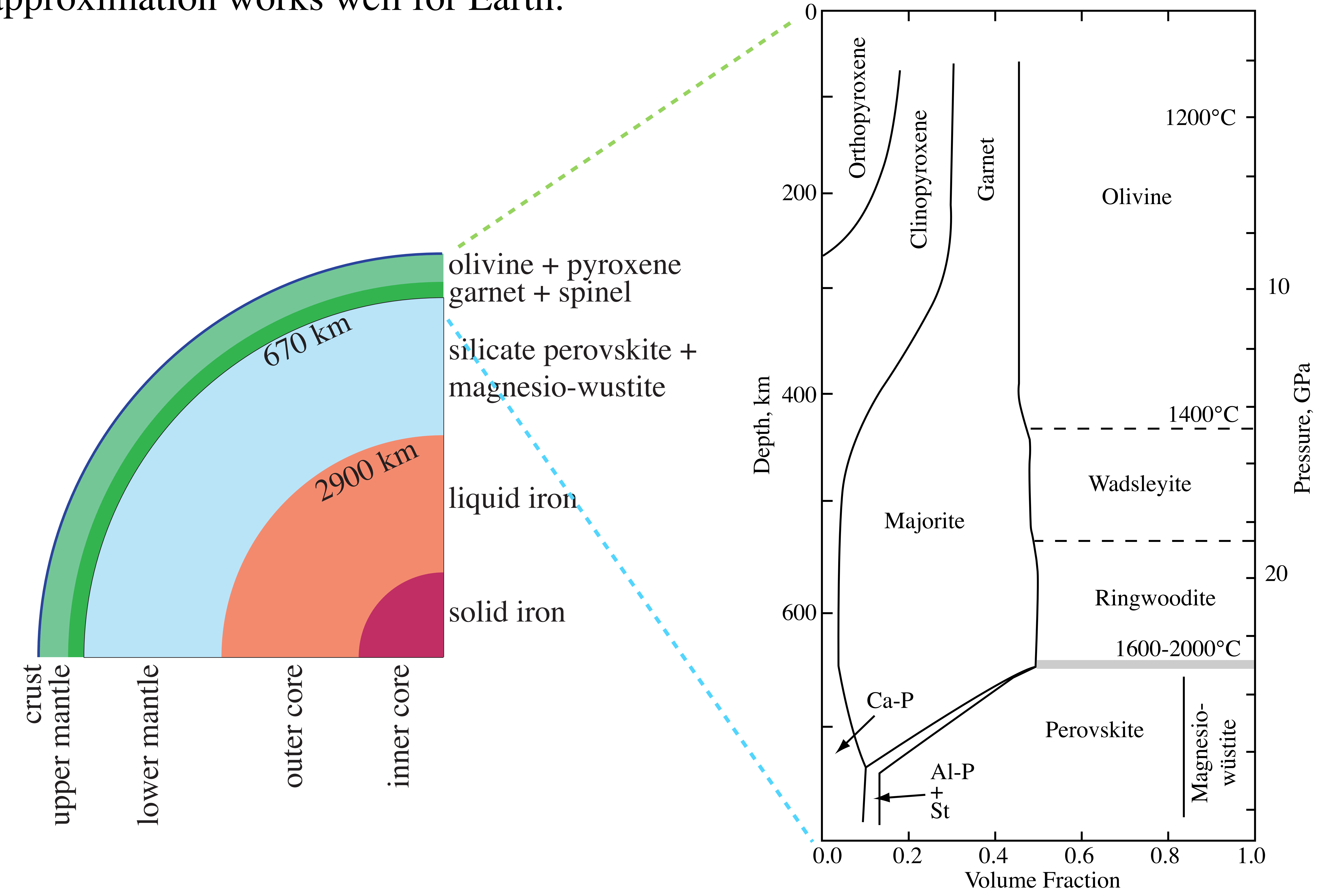
SIOG 231
GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 12

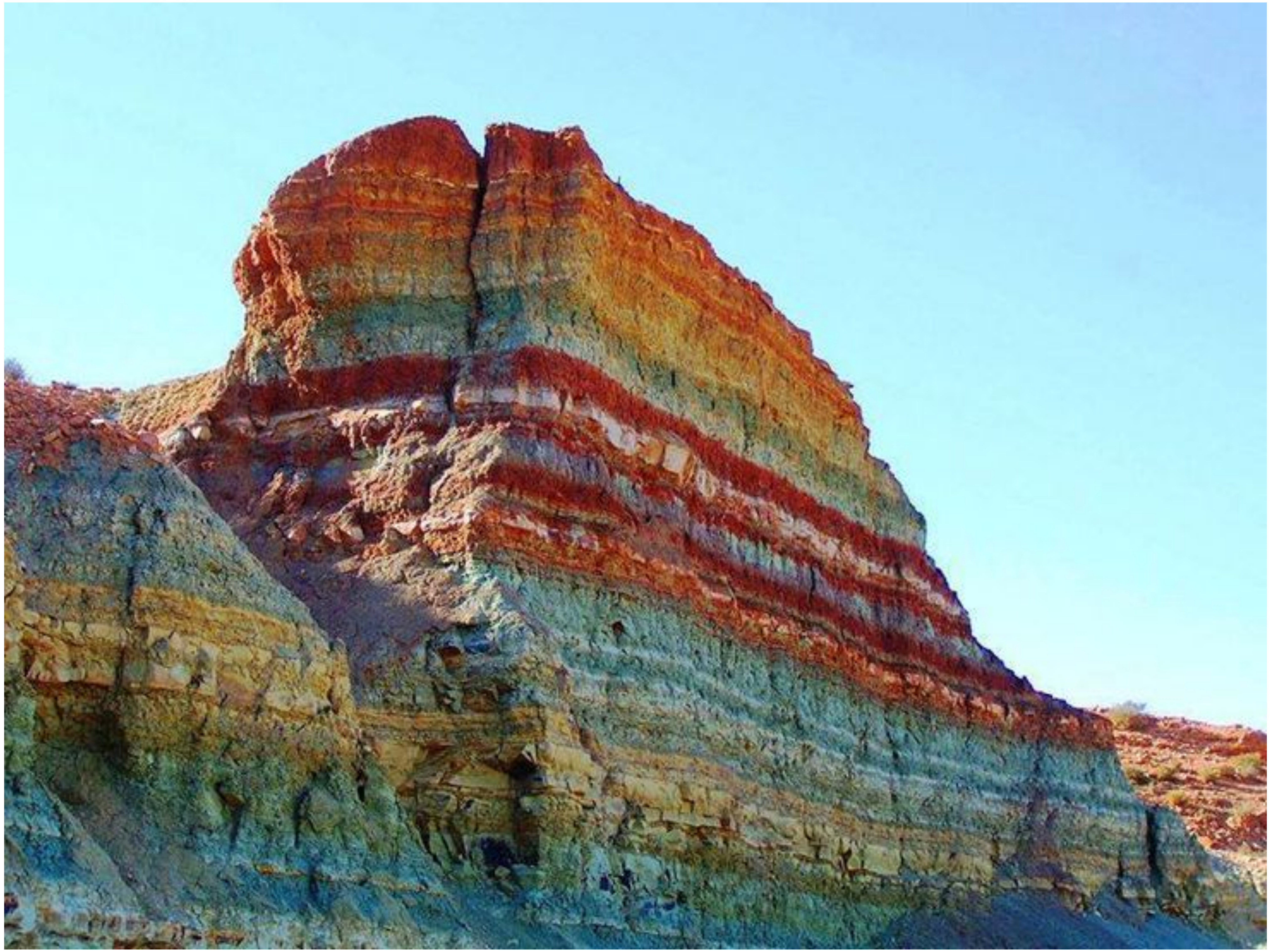
The One Dimensional Earth

2/15/2024

The 1D approximation works well for Earth:



As well as some geologies:



Rainbow Sediment Stratum, Oued Metlili Ghardaia, Algeria

The forward problem - given conductivity as a function of depth, solve for c (GDS) or apparent resistivity (MT).

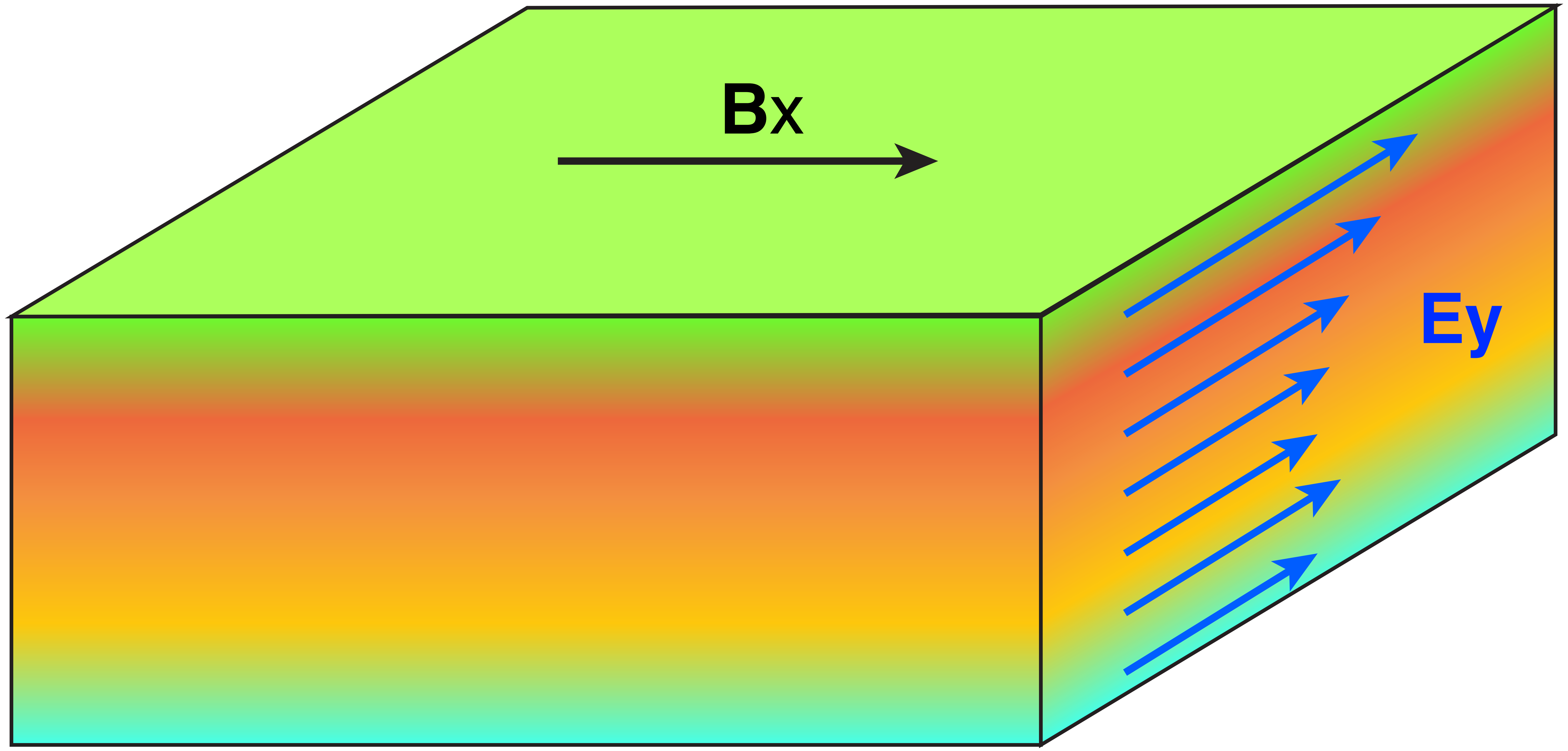
Can do this by solving for electric field inside the model for a given magnetic field at the surface (FD, FE, analytical solutions)

Or can solve for the response directly (stack of layers)

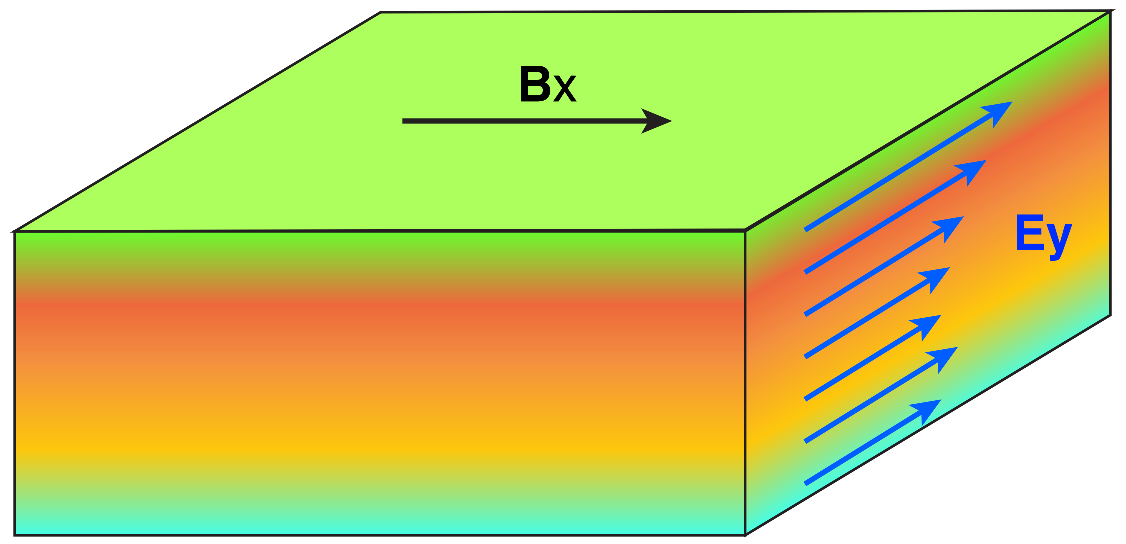
MT in 1D - Analytical methods:

A harmonic, *x*-directed magnetic field at Earth's surface $B_x = B_o e^{i\omega t}$

generates a *y*-directed electric field that varies with depth $E_y = E(z) e^{i\omega t}$



Analytical methods:



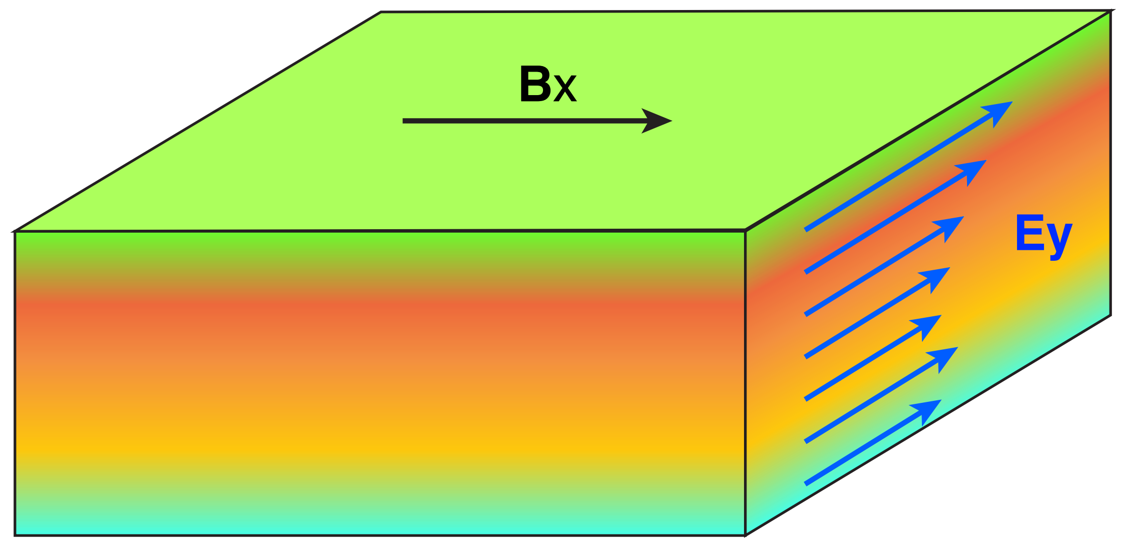
A harmonic, *x*-directed magnetic field at Earth's surface $$B_x = B_o e^{i\omega t}$$

generates a *y*-directed electric field that varies with depth $$E_y = E(z)e^{i\omega t}$$

Combine Ampere's and Ohm's Laws $$\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E}$$

and take the divergence $$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_o \nabla \cdot \sigma(z)\mathbf{E}$$

Analytical methods:



A harmonic, *x*-directed magnetic field at Earth's surface $$B_x = B_o e^{i\omega t}$$

generates a *y*-directed electric field that varies with depth $$E_y = E(z)e^{i\omega t}$$

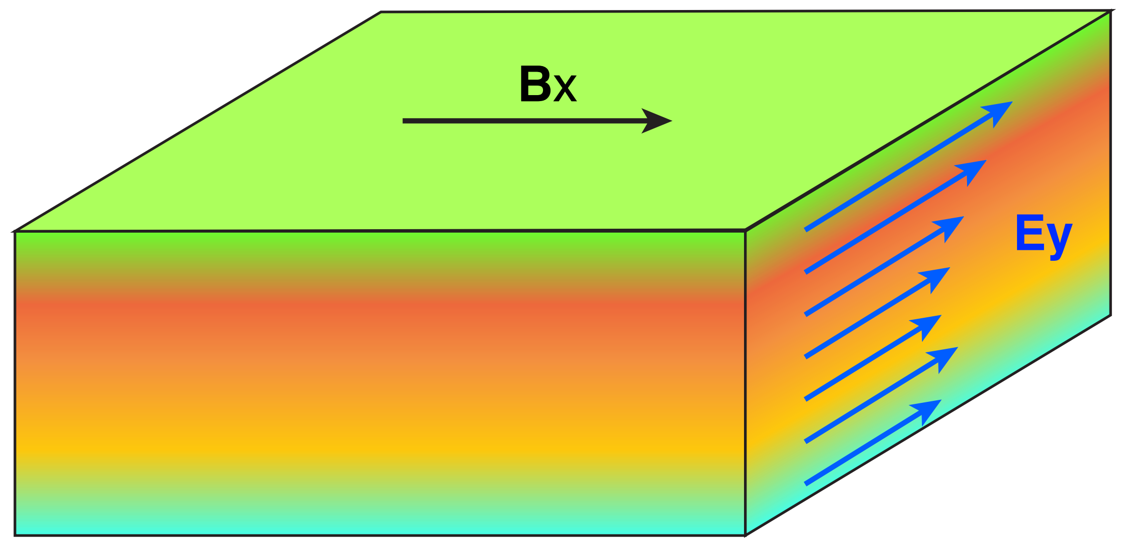
Combine Ampere's and Ohm's Laws $$\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E}$$

and take the divergence $$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_o \nabla \cdot \sigma(z)\mathbf{E}$$

use a couple of vector identities $$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s\nabla \cdot \mathbf{A}$$

and we have $$0 = \sigma(z)\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \sigma(z)$$

Analytical methods:



A harmonic, *x*-directed magnetic field at Earth's surface $B_x = B_o e^{i\omega t}$

generates a *y*-directed electric field that varies with depth $E_y = E(z)e^{i\omega t}$

Combine Ampere's and Ohm's Laws $\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E}$

and take the divergence $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_o \nabla \cdot \sigma(z)\mathbf{E}$

use a couple of vector identities $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ $\nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s \nabla \cdot \mathbf{A}$

and we have $0 = \sigma(z)\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \sigma(z)$

= 0 because E is horizontal and gradient of conductivity is vertical

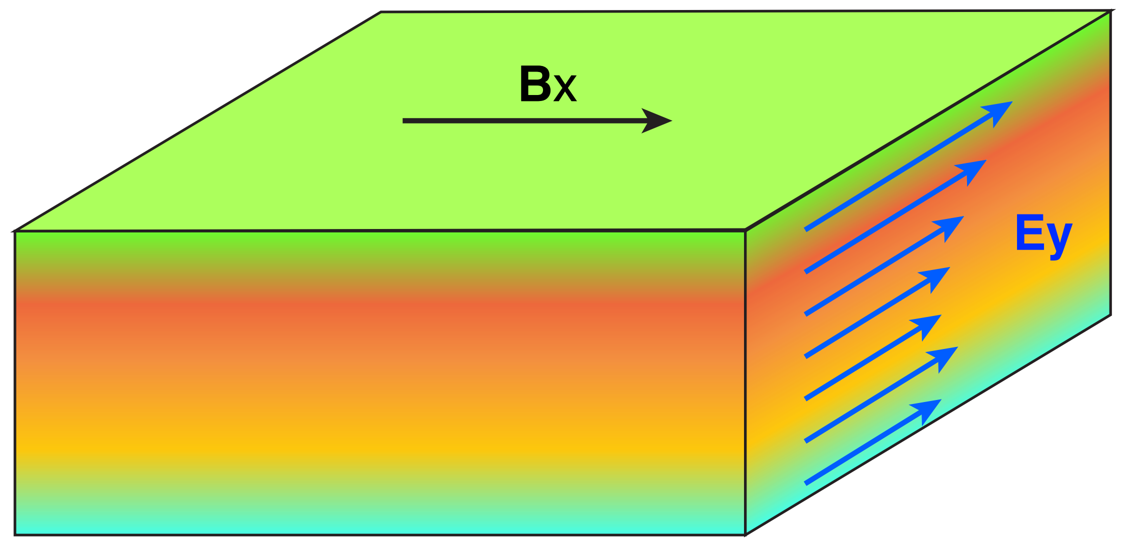
We thus show that $\nabla \cdot \mathbf{E} = 0$

and our diffusion equation still holds

$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

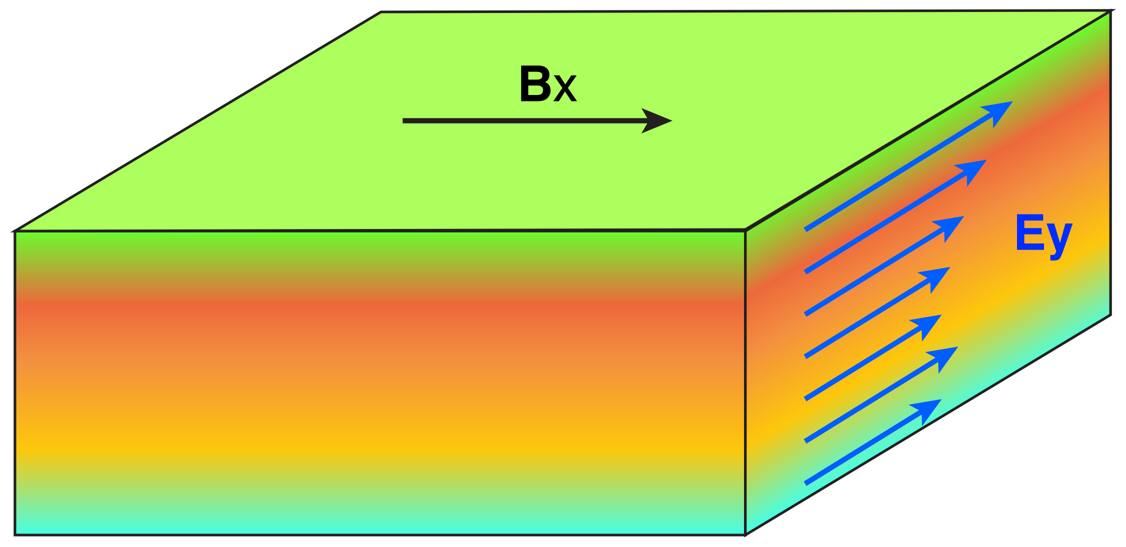
(last time we needed to invoke constant σ)

$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$ becomes $$\frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z)$$



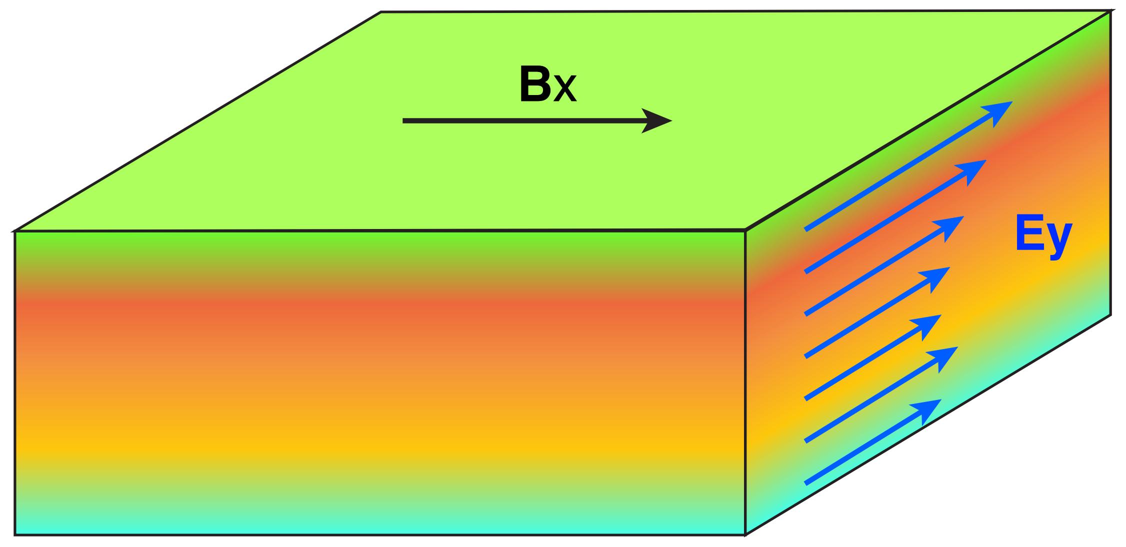
Given the electrical conductivity $\sigma(z)$, find E throughout z

$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E} \quad \text{becomes} \quad \frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z)$$



Boundary conditions: $\mathbf{E} \to 0 \quad \text{as} \quad z \to \infty$

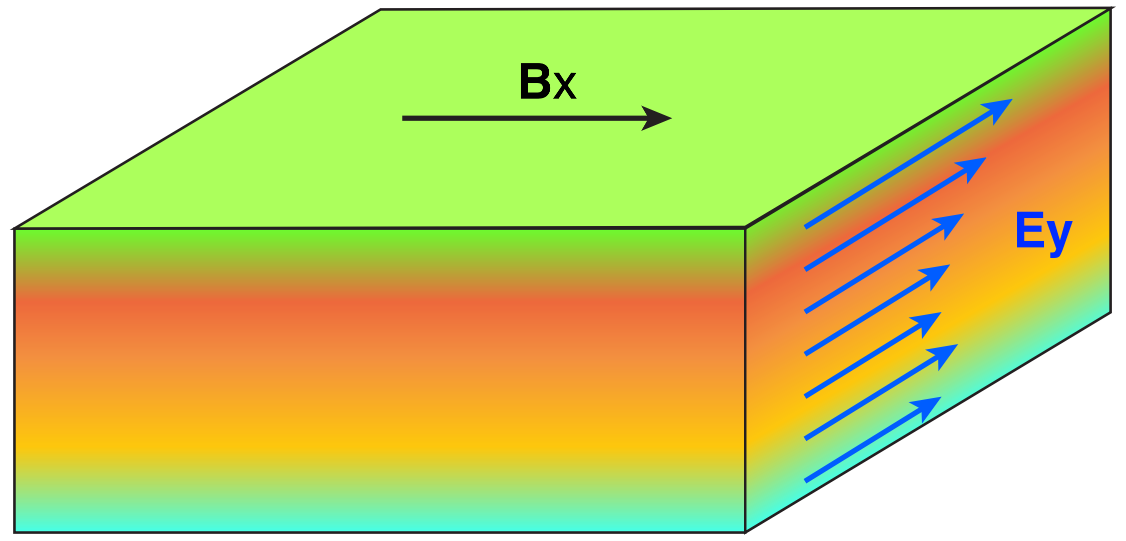
$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$ becomes $$\frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z)$$



Boundary conditions: $\mathbf{E} \to 0$ as $z \to \infty$

For harmonic **B**: $\frac{\partial \mathbf{B}}{\partial t} = i\omega\mathbf{B}$ Faraday's Law $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ becomes $\nabla \times \mathbf{E} = -i\omega\mathbf{B}$

so **B** is given by the curl of **E** $$\mathbf{B} = -\frac{1}{i\omega}\nabla \times \mathbf{E}$$



$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E} \quad \text{becomes} \quad \frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z)$$

Boundary conditions: $\mathbf{E} \to 0 \quad \text{as} \quad z \to \infty$

For harmonic **B**: $\dfrac{\partial \mathbf{B}}{\partial t} = i\omega\mathbf{B}$ Faraday's Law $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ becomes $\nabla \times \mathbf{E} = -i\omega\mathbf{B}$

so **B** is given by the curl of **E** $\mathbf{B} = -\dfrac{1}{i\omega}\nabla \times \mathbf{E}$

only non-zero term

no *Ex* term because *E* only in *y*

$$\nabla \times \mathbf{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

in 1D no variations in *x* or *y*

So $B_x = \dfrac{1}{i\omega}\dfrac{dE_y}{dz}$ and at the surface ($z = 0$) $E'(0) = i\omega B_o$

$E'(0) = i\omega B_o$ Our admittance, c, is now $c(\omega) = -\frac{E_y}{i\omega B_x} = -\frac{E(0)}{E'(0)}$ $\left(= -\frac{Z}{i\omega}\right)$

B has gone away - good. Let's choose B_0 so that $E'(0) = -1$ and $E(0) = c(\omega)$

$E'(0) = i\omega B_o$ Our admittance, c, is now $c(\omega) = -\frac{E_y}{i\omega B_x} = -\frac{E(0)}{E'(0)} \quad \left(= -\frac{Z}{i\omega}\right)$

B has gone away - good. Let's choose B_0 so that $E'(0) = -1$ and $E(0) = c(\omega)$

Given $\sigma(z)$ solve $\frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z)$ for $E(\infty) = 0$ and $E'(0) = -1$

Weidelt (1972) showed that if you know c exactly for all frequencies, you can recover $\sigma(z)$ uniquely.

Lahiri and Price (1939) used an analytical solution for a polynomial to fit GDS data.

Parker and Whaler (1981) created a solution that was invertible.

and across the insulating interval $z_k < z < z_{k+1}$ we find

$$E_{k+1} = E_k + (z_{k+1} - z_k) D_k^+ \tag{52}$$

$$= E_k + (z_{k+1} - z_k) D_{k+1}^- \tag{53}$$

Define the admittance just above the k-th conductor in the usual way

$$C_k = -E_k / D_k^- . \tag{54}$$

Then by means of equations (50), (51) and (53) we can eliminate the E_k and $D_k^\pm$ as we did for uniform layers (although C_k is not continuous):

$$C_k = \frac{E_k}{-D_k^-} = \frac{E_k}{i\omega\mu_0\tau_k E_k - D_k^+} = \frac{1}{i\omega\mu_0\tau_k - D_k^+/E_k} \tag{55}$$

$$= \frac{1}{i\omega\mu_0\tau_k - D_{k+1}^-/E_k} = \frac{1}{i\omega\mu_0\tau_k - \dfrac{D_{k+1}^-}{E_{k+1} - (z_{k+1} - z_k) D_{k+1}^-}} . \tag{56}$$

Finally, dividing by D_{k+1}^- in the bottom tier we find the connection between the admittance at one level to the one above:

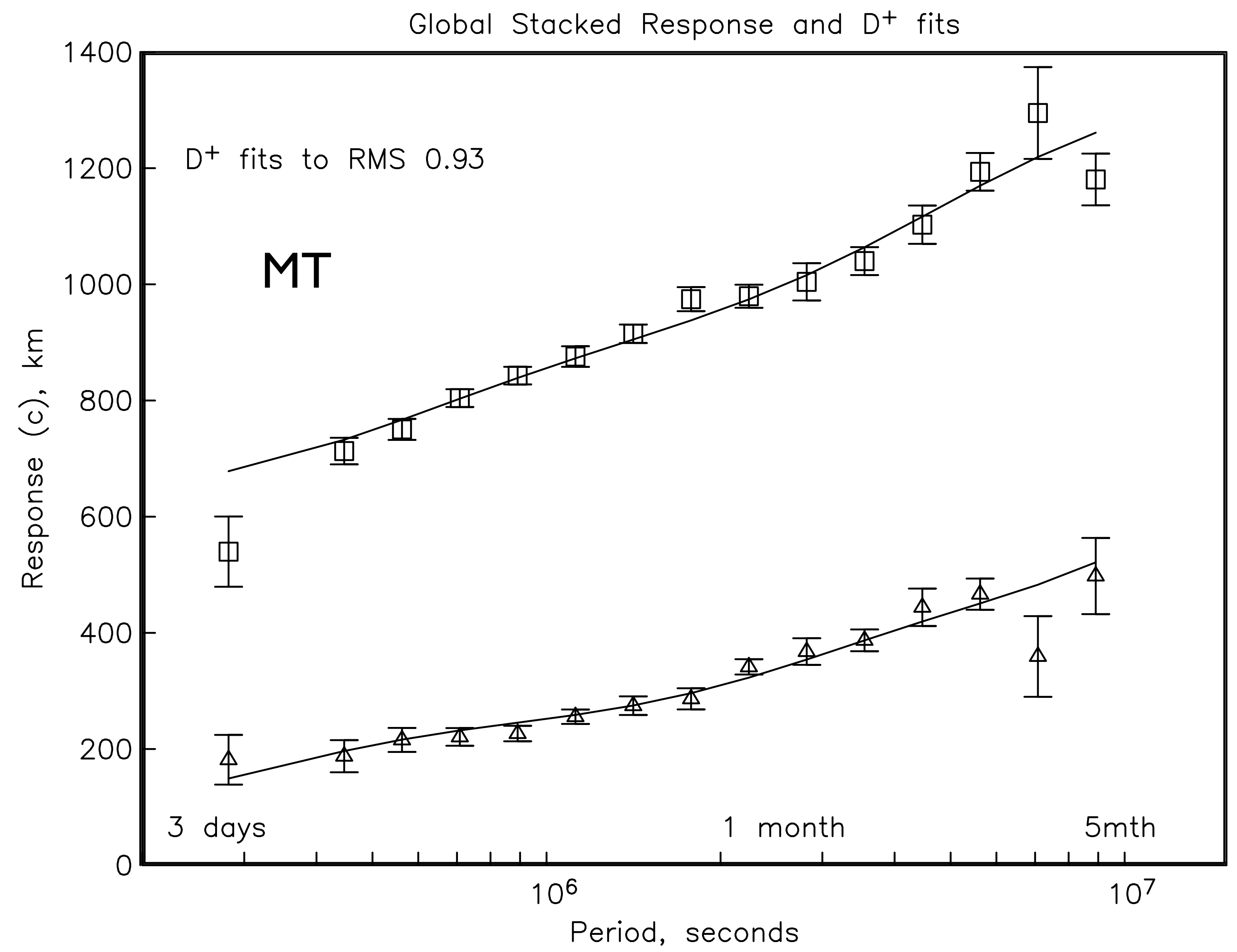
$$C_k = \frac{1}{i\omega\mu_0\tau_k + \dfrac{1}{z_{k+1} - z_k + C_{k+1}}} . \tag{57}$$

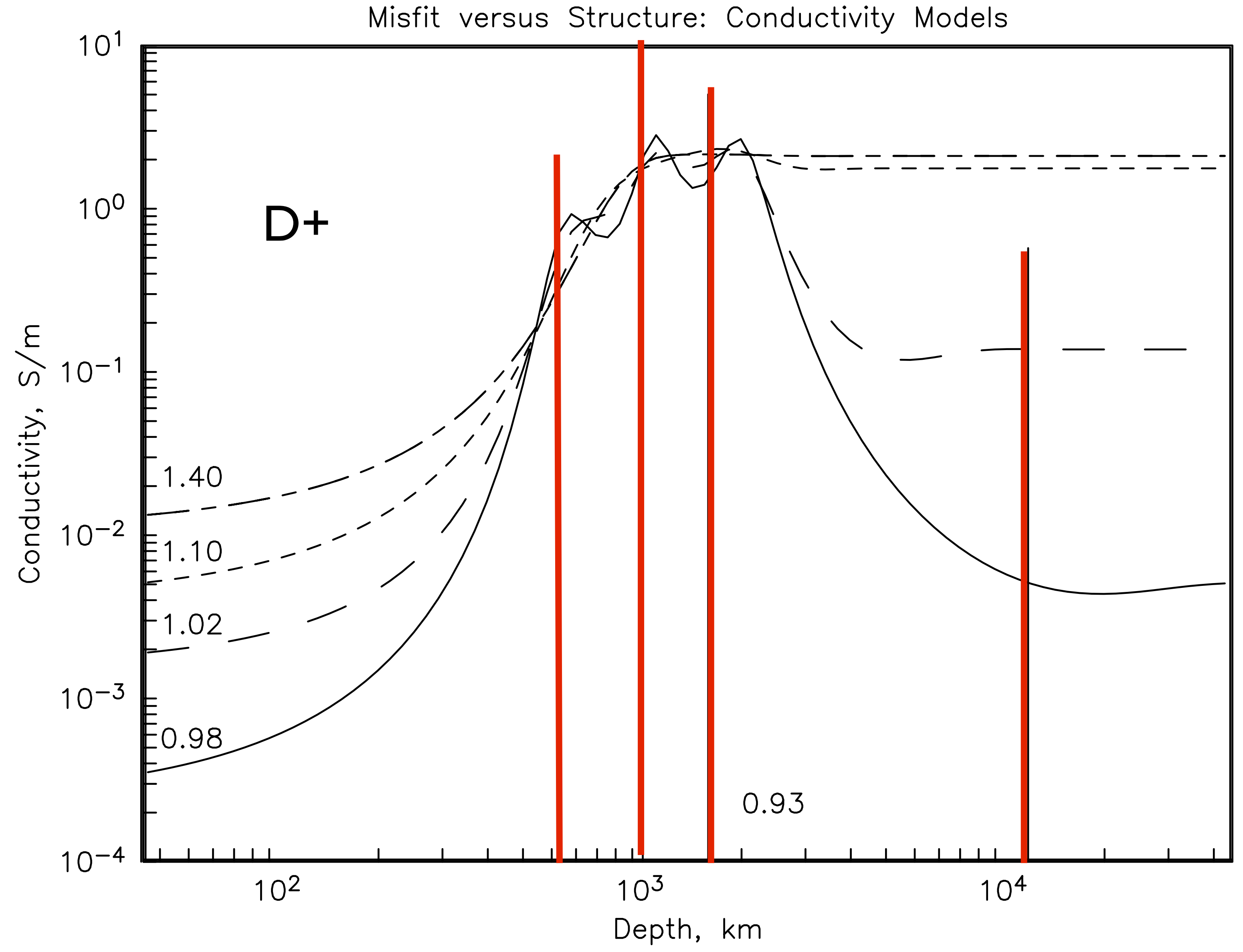
We could solve (48) by recurring upwards in the familiar way, starting with $E(H) = 0 = C_{K+1}$, to get the value of $E(0)$ and hence of $C_1 = c(\omega)$. But now we do something different: we substitute repeatedly from the top, and we get a magnificent **continued fraction** for the admittance:

$$c(\omega) = z_1 + \cfrac{1}{i\omega\mu_0\tau_1 + \cfrac{1}{z_2 - z_1 + \cfrac{1}{i\omega\mu_0\tau_2 + \cfrac{1}{z_3 - z_2 + \cfrac{1}{i\omega\mu_0\tau_3 + \cdots \cfrac{1}{H - z_K}}}}}} . \tag{58}$$

The initial z_1 allows us to put an insulator at $z = 0$, rather than a conducting sheet at the surface. While not exactly the same as the continued fractions described in the introduction, (58) can be rearranged by similar elementary algebra to be a *finite* partial fraction expansion:

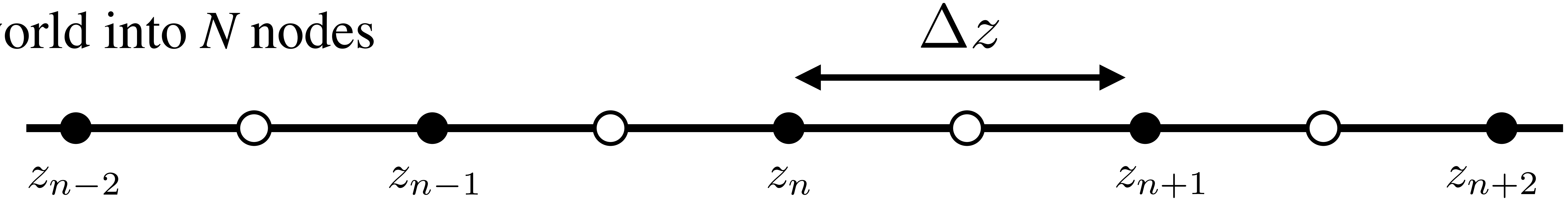
$$c(\omega) = z_1 + \sum_{k=1}^{K} \frac{\alpha_k}{\lambda_k + i\omega} . \tag{59}$$





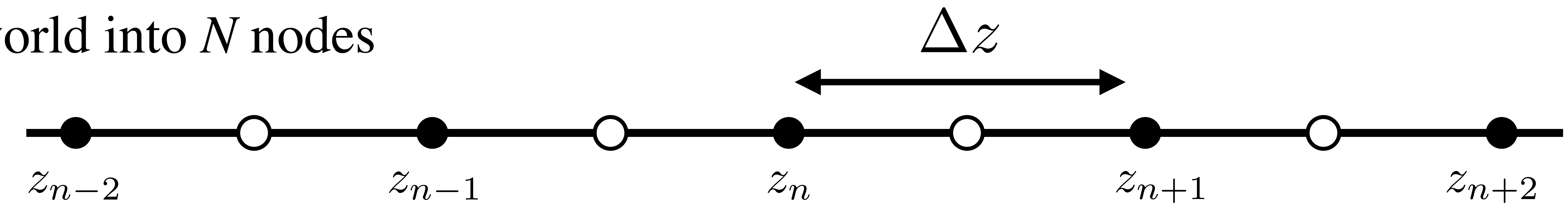
Finite differences:

Divide the world into N nodes

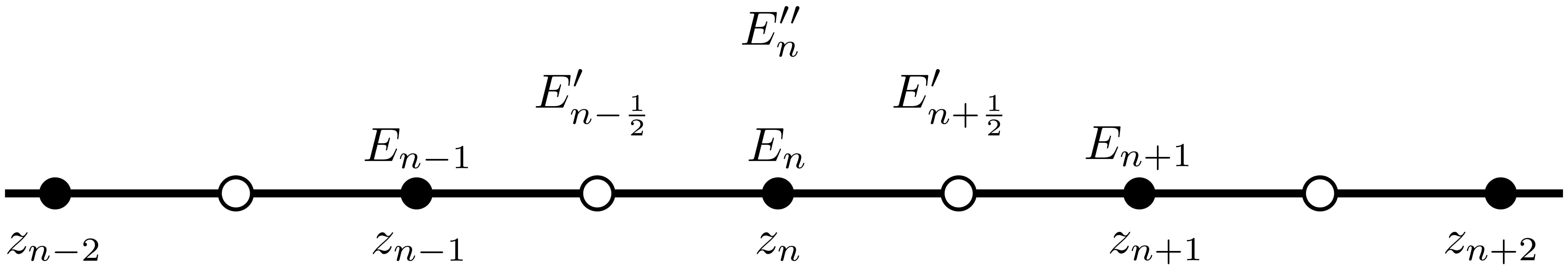


Finite differences:

Divide the world into N nodes

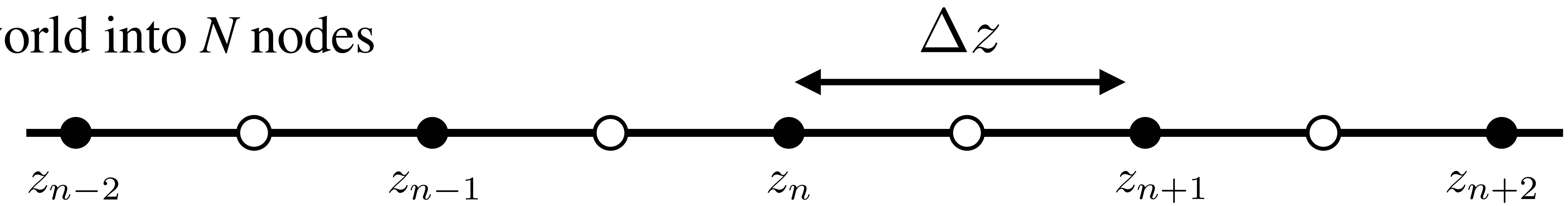


$$E'(z_n) = \frac{E_{n+1} - E_n}{\Delta z} + O(\Delta z) \qquad E''(z_n) = \frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} + O(\Delta z^2)$$

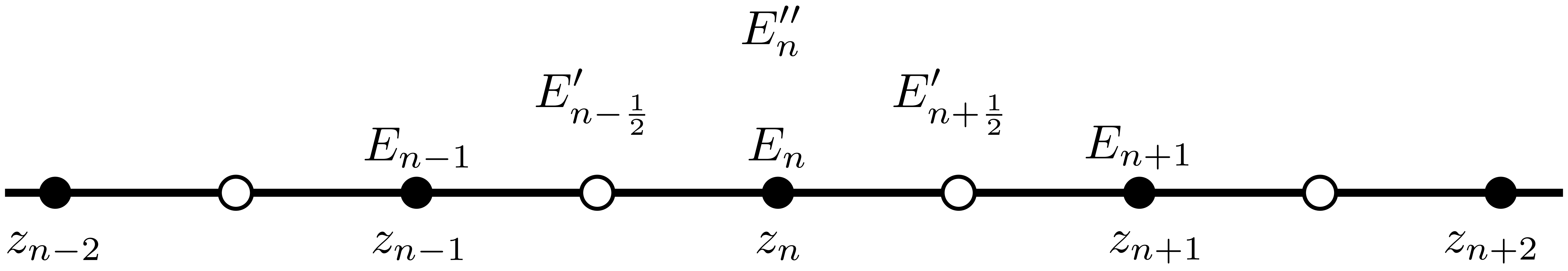


Finite differences:

Divide the world into N nodes



$$E'(z_n) = \frac{E_{n+1} - E_n}{\Delta z} + O(\Delta z) \qquad E''(z_n) = \frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} + O(\Delta z^2)$$

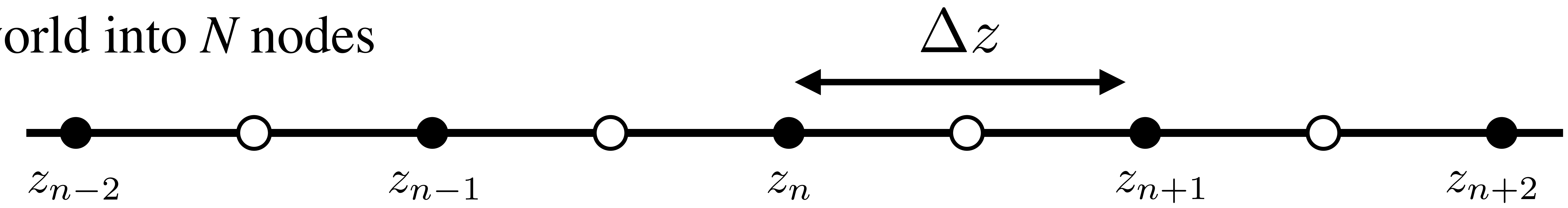


$$\frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z) \quad \text{becomes} \quad \frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} - i\omega\mu_0\sigma_n E_n = 0$$

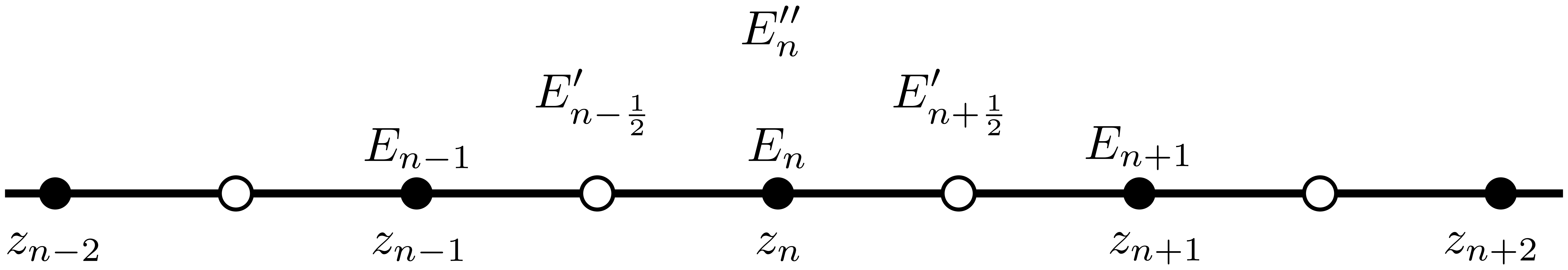
$$n = 2, 3, ..., N-1$$

Finite differences:

Divide the world into N nodes



$$E'(z_n) = \frac{E_{n+1} - E_n}{\Delta z} + O(\Delta z) \qquad E''(z_n) = \frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} + O(\Delta z^2)$$



$$\frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z)$$ becomes $$\frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} - i\omega\mu_0\sigma_n E_n = 0$$

$$n = 2, 3, ..., N-1$$

$$\frac{E_2 - E_1}{\Delta z} = -1 \qquad E_N = 0$$

A linear system

$$\frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} - i\omega\mu_0\sigma_n E_n = 0 \qquad n = 2, 3, ..., N-1 \qquad \frac{E_2 - E_1}{\Delta z} = -1 \qquad E_N = 0$$

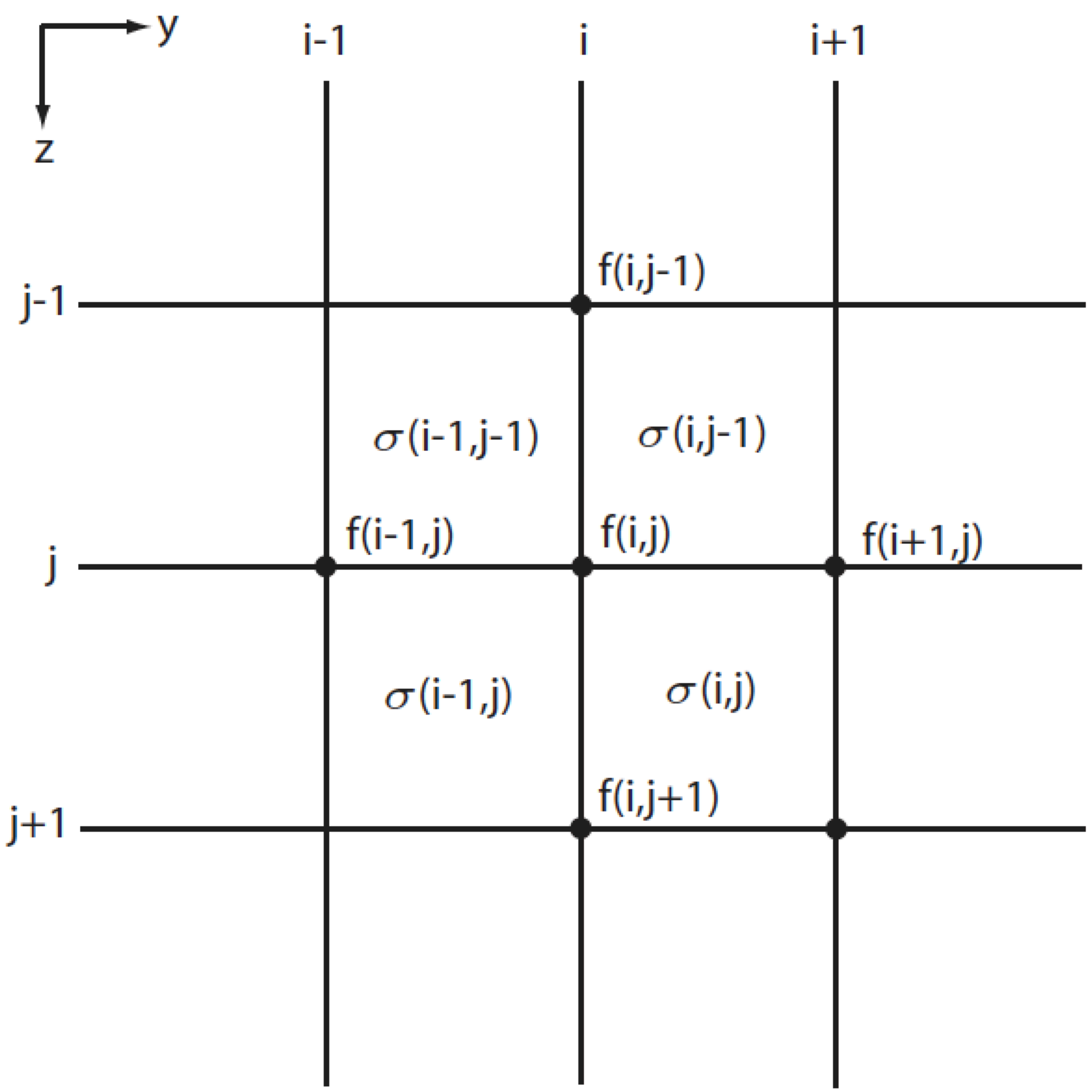
$$E_{i-1} - (2 + \Delta z^2 i\omega\mu_0\sigma_i)E_i + E_{i+1} = 0$$

in matrix form $\mathbf{A}x = b$

solved: $x = \mathbf{A}^{-1}b$

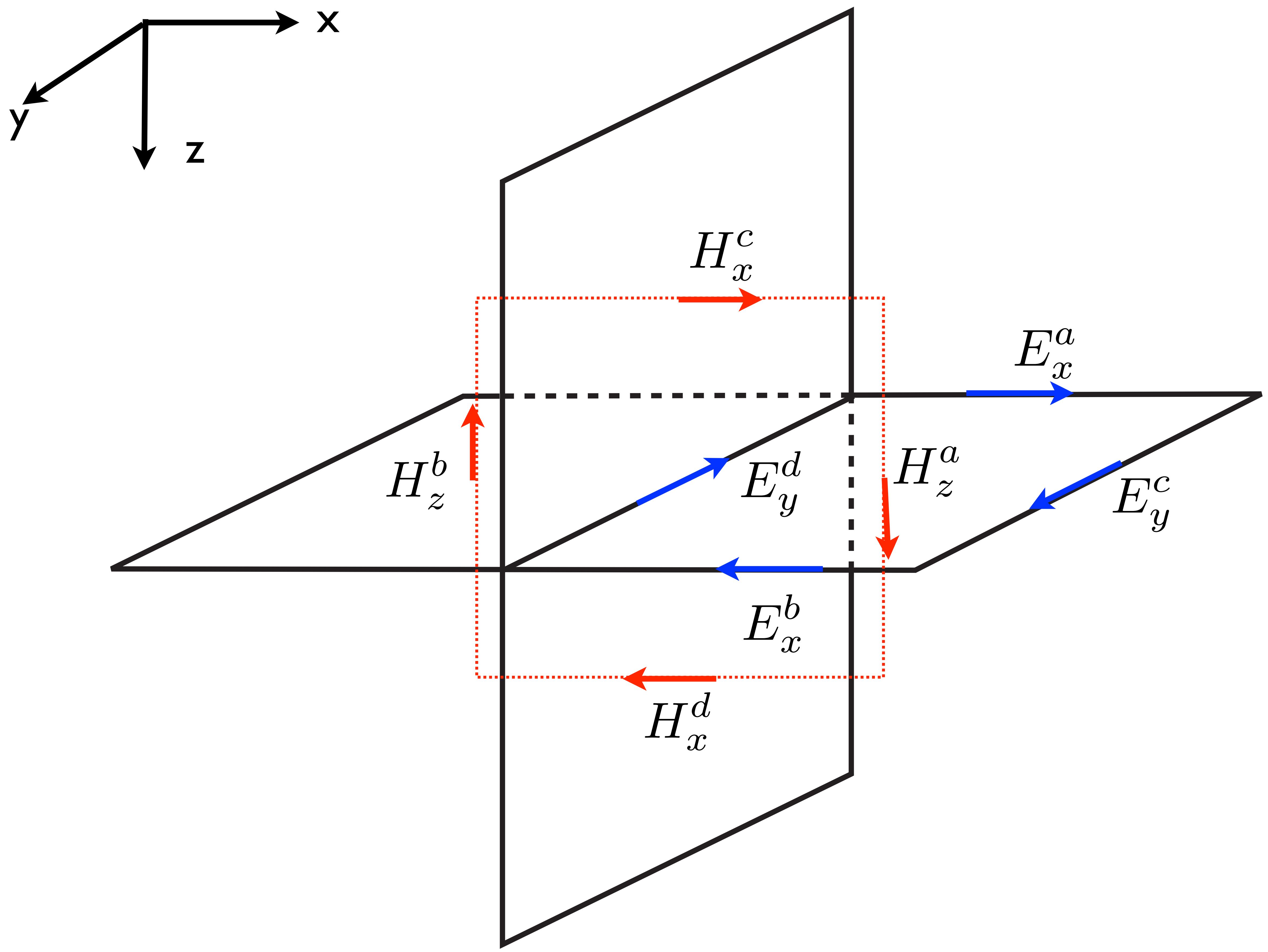
(in Homework #4).

A finite difference grid in 2D



Here conductivities are constant in the cells. For the electric field at i,j, the average of the 4 adjacent conductivities is used. This is clearly an approximation.

Finite difference in 3D. A staggered grid is often used.



$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z$$

$$\frac{\partial E_x}{\partial y} \approx \frac{E_x^a - E_x^b}{\Delta y}$$

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma\mathbf{E}$$

$$\mathbf{Le} = \mathbf{s}$$

difference operators are placed into the matrix L

e is the unknown electric field on cell edges

s contains scattering terms

Finite elements:

Consider a PDE $$u''(x) = f(x) \quad , \quad x = [0, 1]$$

Finite elements:

Consider a PDE $$u''(x) = f(x) \quad , \quad x = [0,1]$$

The "weak formulation": $$\int_0^1 f(x)v(x)dx = \int_0^1 u''(x)v(x)dx$$

for any smooth test function *v(x)* with boundary conditions *v(0) = v(1) = 0*

Finite elements:

Consider a PDE $$u''(x) = f(x) \quad , \quad x = [0,1]$$

The "weak formulation": $$\int_0^1 f(x)v(x)dx = \int_0^1 u''(x)v(x)dx$$

for any smooth test function $v(x)$ with boundary conditions $v(0) = v(1) = 0$

Integration by parts: $$\int_a^b v(x)u'(x)dx = v(x)u(x)\Big|_a^b - \int_a^b v'(x)u(x)dx$$

Reminder of where integration by parts comes from:

$$(uv)' = u'v + uv'$$

integrate from a to b

$$\int_a^b (uv)' dx = \int_a^b [u'v + uv'] dx$$

$$uv|_a^b = \int_a^b [u'v + uv'] dx$$

$$\int_a^b u'v = uv|_a^b - \int_a^b uv'$$

Finite elements:

Consider a PDE $$u''(x) = f(x) \quad , \quad x = [0,1]$$

The “weak formulation”: $$\int_0^1 f(x)v(x)dx = \int_0^1 u''(x)v(x)dx$$

for any smooth test function *v(x)* with boundary conditions *v(0) = v(1) = 0*

Integration by parts: $$\int_a^b v(x)u'(x)dx = v(x)u(x)\big|_a^b - \int_a^b v'(x)u(x)dx$$

applied to the RHS gives

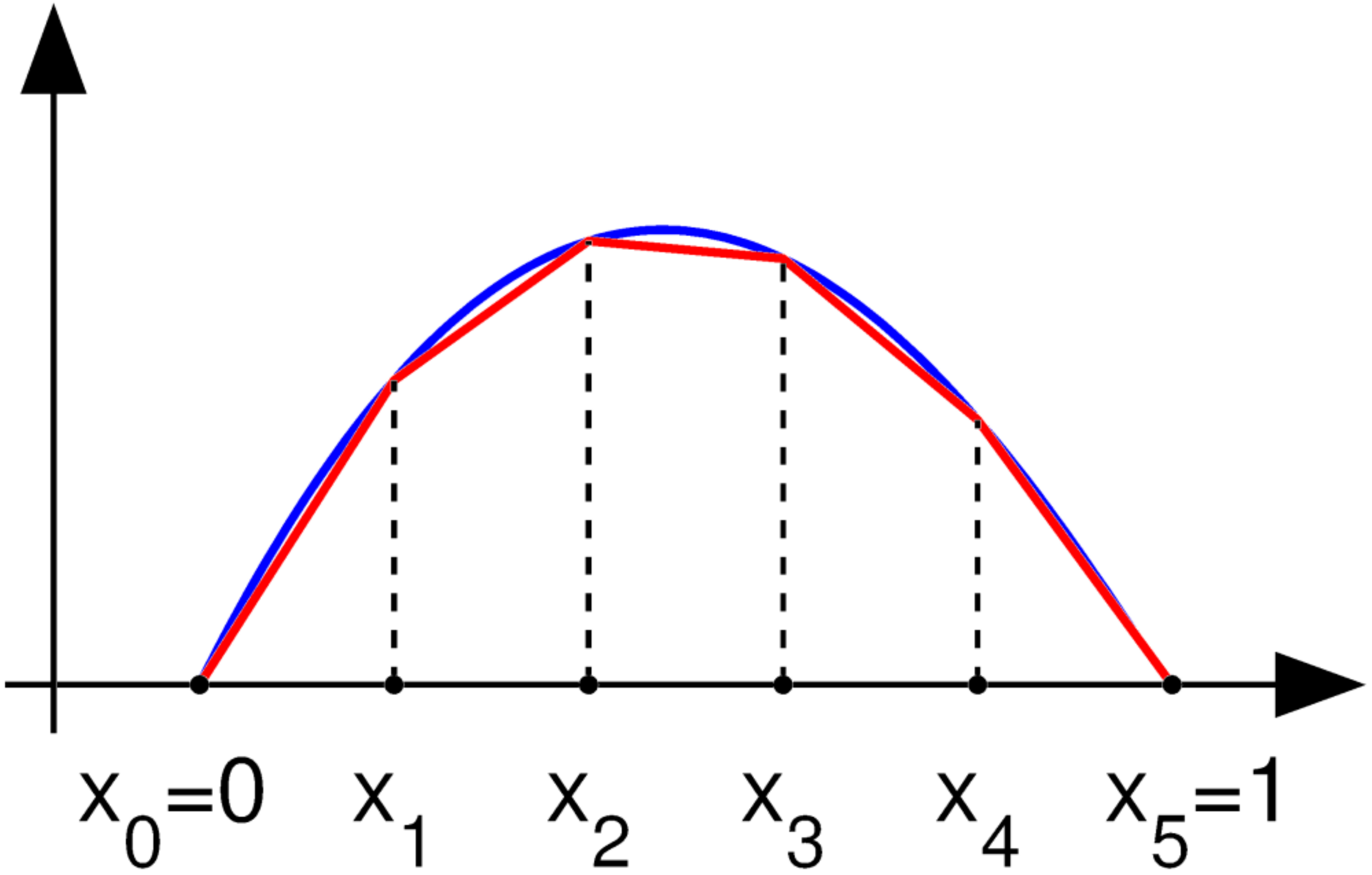
$$\int_0^1 u''(x)v(x)dx = u(x)'v(x)\big|_0^1 - \int_0^1 u'(x)v'(x)dx \equiv -\phi(u,v)$$

= zero because of boundary conditions on *v(x)*

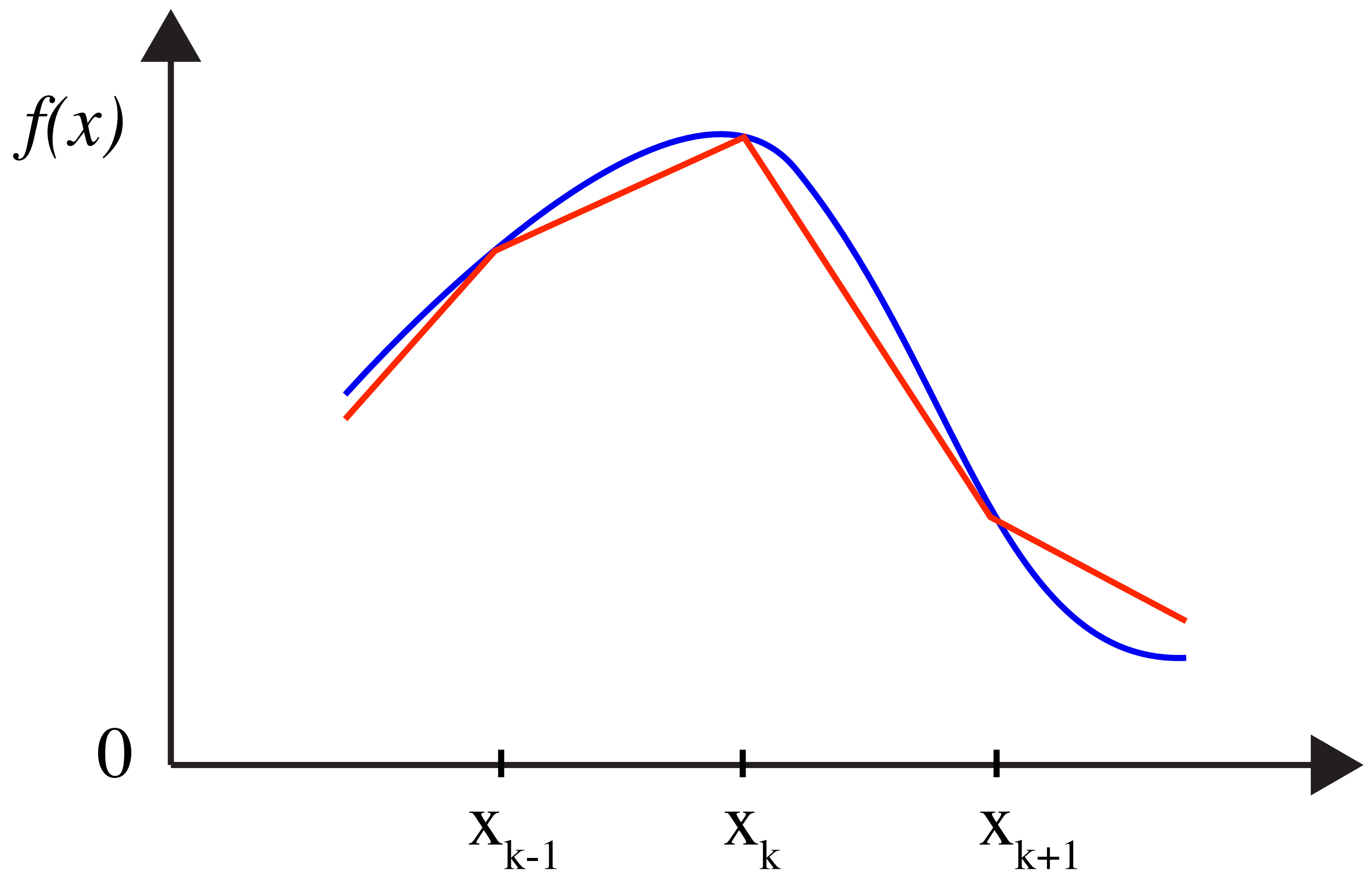
and our weak formulation is

$$\int_0^1 f(x)v(x)dx = -\int_0^1 u'(x)v'(x)dx \equiv -\phi(u,v)$$

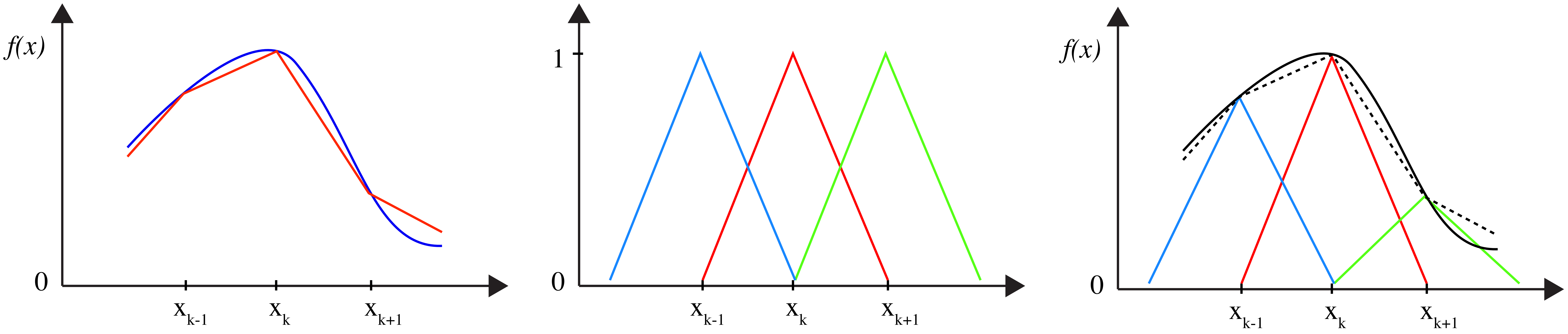
Divide *x* up into *n* elements defined by *n+1* nodes



Approximate our function f(x) by piecewise linear functions.



These have a basis of tent functions



given by

$$v_k(x) = \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}} & x \in [x_{k-1}, x_k] \\ \frac{x_{k+1} - x}{x_{k+1} - x_k} & x \in [x_k, x_{k+1}] \\ 0 & otherwise \end{cases}$$

Our function is given by

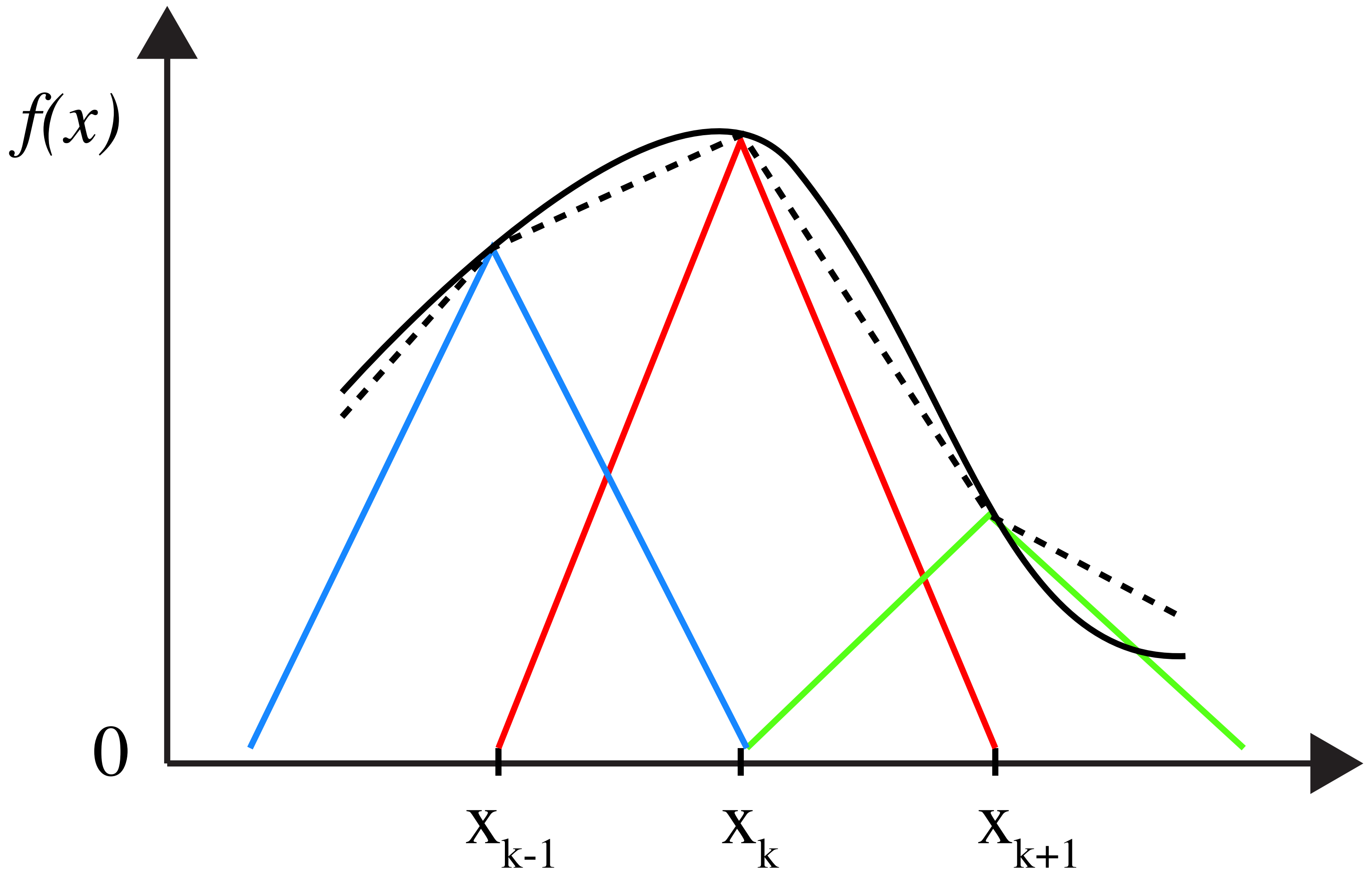
$$f(x) = \sum_{k=1}^{n} f_k v_k(x)$$

and our unknown solution

$$u(x) = \sum_{k=1}^{n} u_k v_k(x)$$

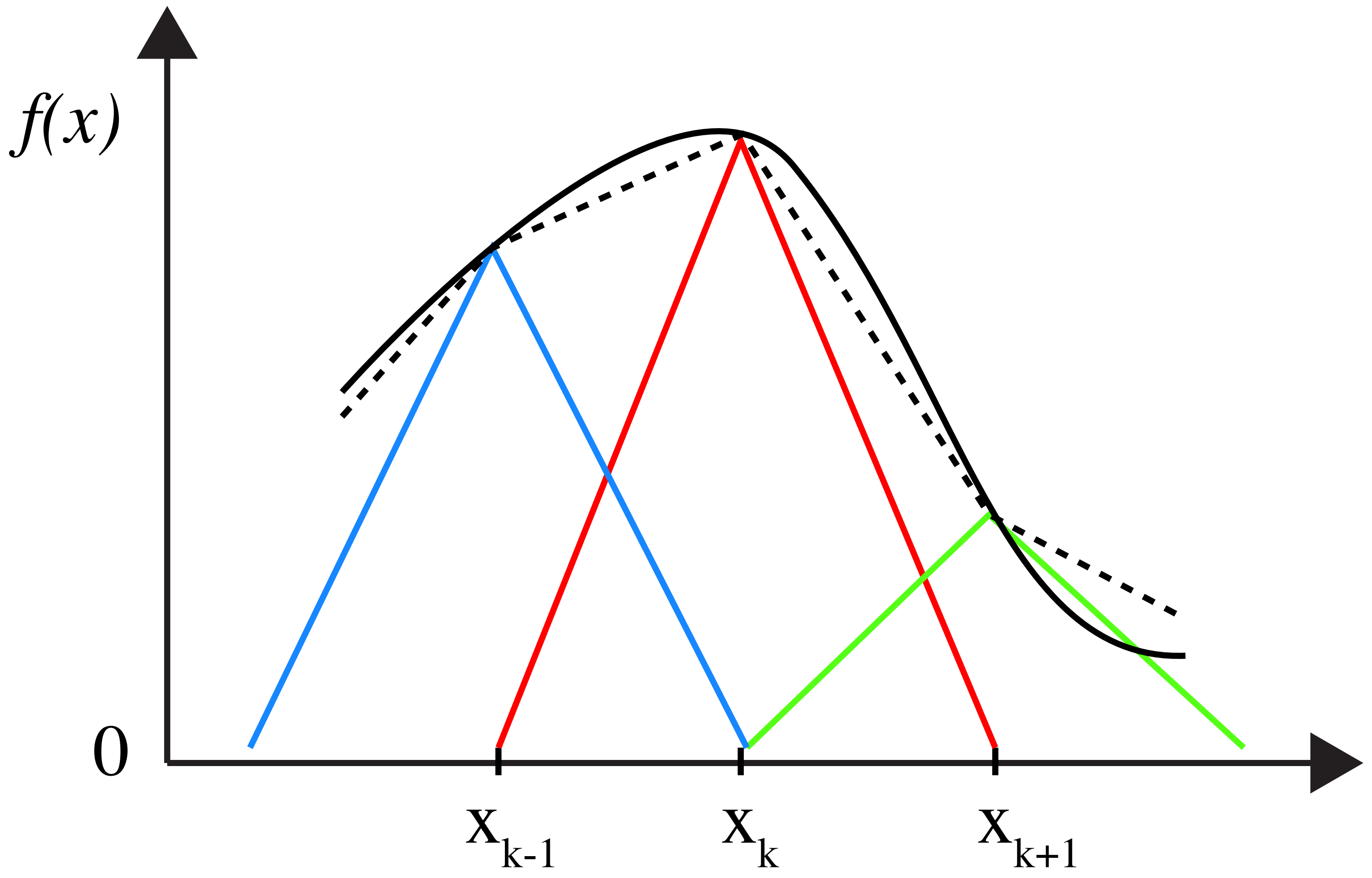
with derivatives

$$u'(x) = \sum_{k=1}^{n} u_k v'_k(x)$$



Our function is given by

$$f(x) = \sum_{k=1}^{n} f_k v_k(x)$$



and our unknown solution

$$u(x) = \sum_{k=1}^{n} u_k v_k(x)$$

with derivatives

$$u'(x) = \sum_{k=1}^{n} u_k v_k'(x)$$

The weak formulation

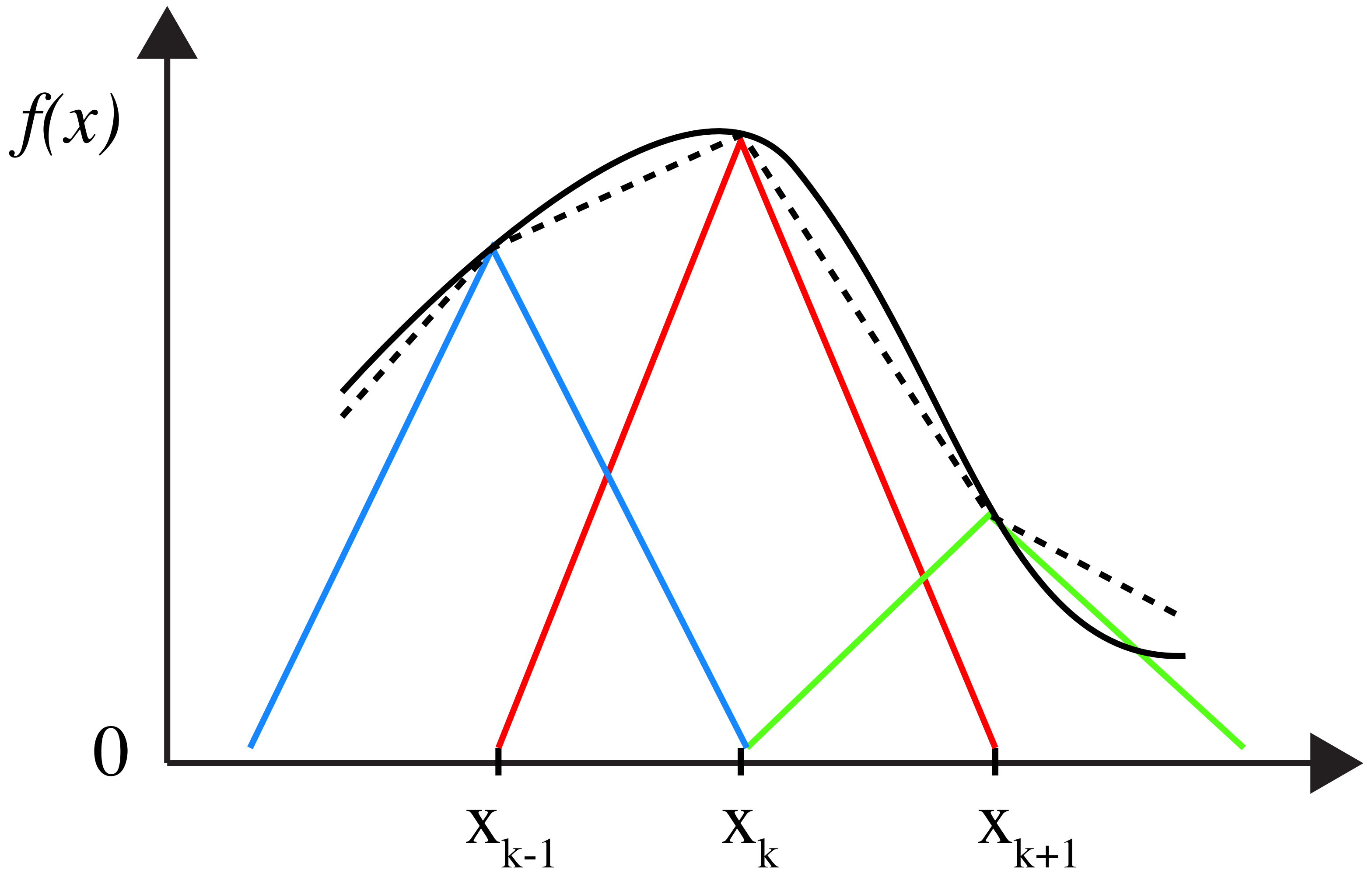
$$\int_0^1 f(x)v(x)dx = -\int_0^1 u'(x)v'(x)dx \equiv -\phi(u,v)$$

is now

$$-\sum_{k=1}^{n} u_k \phi(v_k, v) = \sum_{k=1}^{n} f_k \int_0^1 v_k v(x)dx$$

Our function is given by

$$f(x) = \sum_{k=1}^{n} f_k v_k(x)$$



and our unknown solution

$$u(x) = \sum_{k=1}^{n} u_k v_k(x)$$

with derivatives

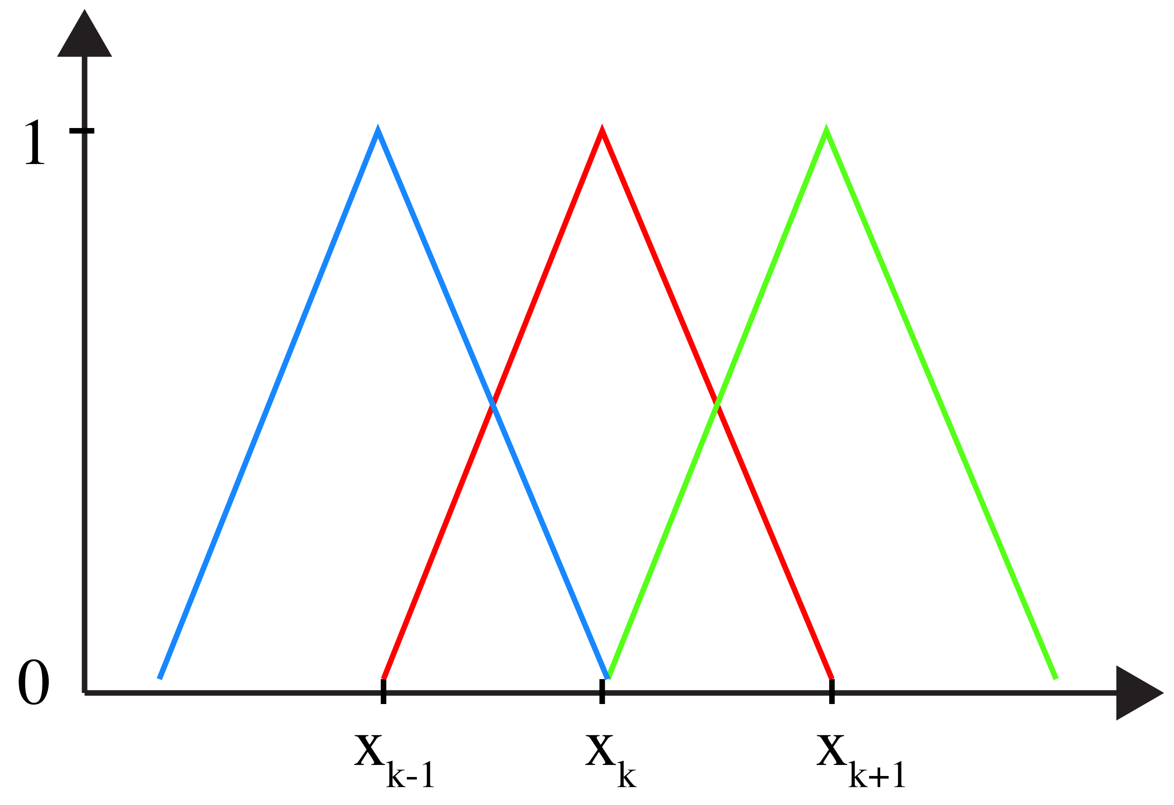
$$u'(x) = \sum_{k=1}^{n} u_k v_k'(x)$$

The weak formulation is now

$$-\sum_{k=1}^{n} u_k \phi(v_k, v) = \sum_{k=1}^{n} f_k \int_0^1 v_k v(x) dx$$

Bring in the test functions! Unit tents again:

$$-\sum_{k=1}^{n} u_k \phi(v_k, v_j) = \sum_{k=1}^{n} f_k \int_0^1 v_k v_j dx \quad \text{for } j = 1, ..., n$$



$$-\sum_{k=1}^{n} u_k \phi(v_k, v_j) = \sum_{k=1}^{n} f_k \int_0^1 v_k v_j dx \quad \text{for } j = 1, ..., n$$

This is just another linear system

$$-\mathbf{L}\mathbf{u} = \mathbf{b}$$

$$L_{ij} = \phi(v_i, v_j) = \int_0^1 v_i'(x) v_j'(x) dx$$

$$b_j = \sum_{k=1}^{n} f_k \int_0^1 v_k v_j dx$$

which we can solve for **u**

The integral is zero everywhere except

$$|j-k| < 1 \quad \text{or} \quad |j-i| < 1$$

Applied to 1D MT: $$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

is written as $$-E''(z) + i\omega\mu_o\sigma(z)E(z) = 0$$

Applied to 1D MT: $$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

is written as $$-E''(z) + i\omega\mu_o\sigma(z)E(z) = 0$$

and the weak form is $$-\int_0^Z E''(z)v(z)dz + \int_0^Z i\omega\mu_o\sigma(z)E(z)v(z)dz = 0$$

(Z is some large depth.)

Applied to 1D MT: $$\nabla^2\mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

is written as $$-E''(z) + i\omega\mu_o\sigma(z)E(z) = 0$$

and the weak form is $$-\int_0^Z E''(z)v(z)dz + \int_0^Z i\omega\mu_o\sigma(z)E(z)v(z)dz = 0$$

(Z is some large depth.) Integration by parts:

$$-\left[E'(Z)v(Z) - E'(0)v(0) - \int_0^Z E'(z)v'(z)dz\right] + \int_0^Z i\omega\mu_o\sigma(z)E(z)v(z)dz = 0$$

$$\int_0^Z \left[E'(z)v'(z) + i\omega\mu_o\sigma(z)E(z)v(z)\right]dz = 0$$

Applied to 1D MT: $$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

is written as $$-E''(z) + i\omega\mu_o\sigma(z)E(z) = 0$$

and the weak form is $$-\int_0^Z E''(z)v(z)dz + \int_0^Z i\omega\mu_o\sigma(z)E(z)v(z)dz = 0$$

(Z is some large depth.) Integration by parts:

$$-\left[E'(Z)v(Z) - E'(0)v(0) - \int_0^Z E'(z)v'(z)dz\right] + \int_0^Z i\omega\mu_o\sigma(z)E(z)v(z)dz = 0$$

$$\int_0^Z \left[E'(z)v'(z) + i\omega\mu_o\sigma(z)E(z)v(z)\right]dz = 0$$

E in terms of basis functions

$$E(z) = \sum_{k=1}^{n} E_k v_k(z) \longrightarrow \int_0^Z \left[\sum_{k=1}^{n} E_k v_k'(z)v'(z) + i\omega\mu_o\sigma(z)\sum_{k=1}^{n} E_k v_k(z)v(z)\right]dz = 0$$

or $$\sum_{k=1}^{n}\int_0^Z \left[v_k'(z)v'(z) + i\omega\mu_o\sigma(z)v_k(z)v(z)\right]dz E_k = 0$$

$$\sum_{k=1}^{n} \int_0^Z \left[v_k'(z) v'(z) + i\omega\mu_o \sigma(z) v_k(z) v(z) \right] dz E_k = 0$$

Introduce the test functions $v(z) = v_j(z),\ j = 1, ..., n$

gives

$$\sum_{k=1}^{n} \int_{\Omega_j} \left[v_k'(z) v_j'(z) + i\omega\mu_o \sigma(z) v_k(z) v_j(z) \right] dz E_k = 0 \quad \text{for } j = 1, ..., n$$

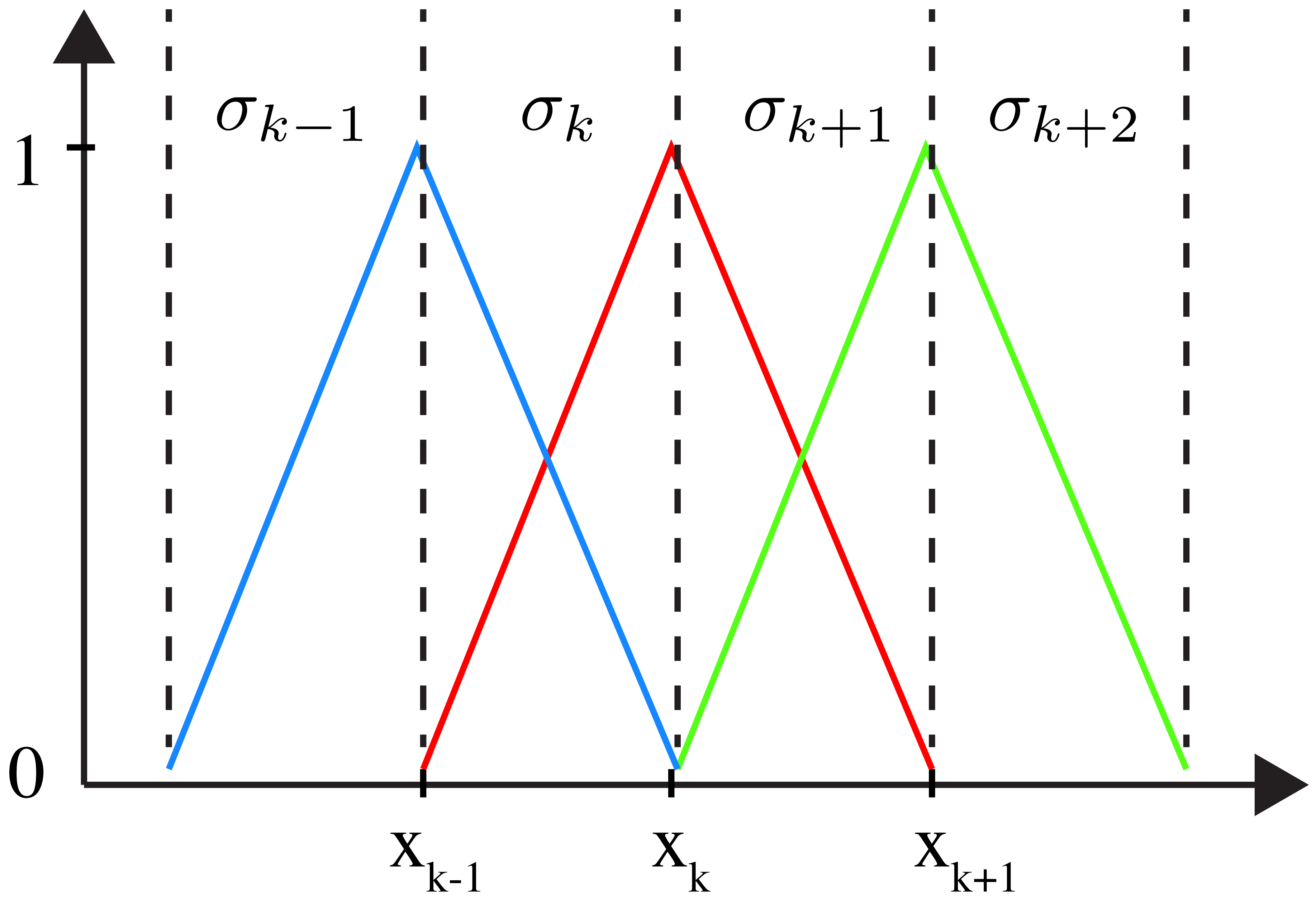
$$\sum_{k=1}^{n} \int_0^Z \left[v_k'(z) v'(z) + i\omega\mu_o \sigma(z) v_k(z) v(z) \right] dz E_k = 0$$

Introduce the test functions $v(z) = v_j(z),\ j = 1, ..., n$

gives

$$\sum_{k=1}^{n} \int_{\Omega_j} \left[v_k'(z) v_j'(z) + i\omega\mu_o \sigma(z) v_k(z) v_j(z) \right] dz E_k = 0 \quad \text{for } j = 1, ..., n$$

What is going on? The test functions sample the conductivity between nodes (inside the elements)



$$\sum_{k=1}^{n}\int_0^Z \left[v_k'(z)v'(z) + i\omega\mu_o\sigma(z)v_k(z)v(z)\right]dzE_k = 0$$

Introduce the test functions $v(z) = v_j(z),\ j = 1, ..., n$

gives

$$\sum_{k=1}^{n}\int_{\Omega_j} \left[v_k'(z)v_j'(z) + i\omega\mu_o\sigma(z)v_k(z)v_j(z)\right]dzE_k = 0 \quad \text{for } j = 1, ..., n$$

$$\mathbf{Ax} = \mathbf{b}$$

$$A_{jk} = \int_{z_{k-1}}^{z_{k+1}} \left[v_k'(z)v_j'(z) + i\omega\mu_o\sigma(z)v_k(z)v_j(z)\right]dz \qquad x_k = E_k \qquad \mathbf{b} = 0$$

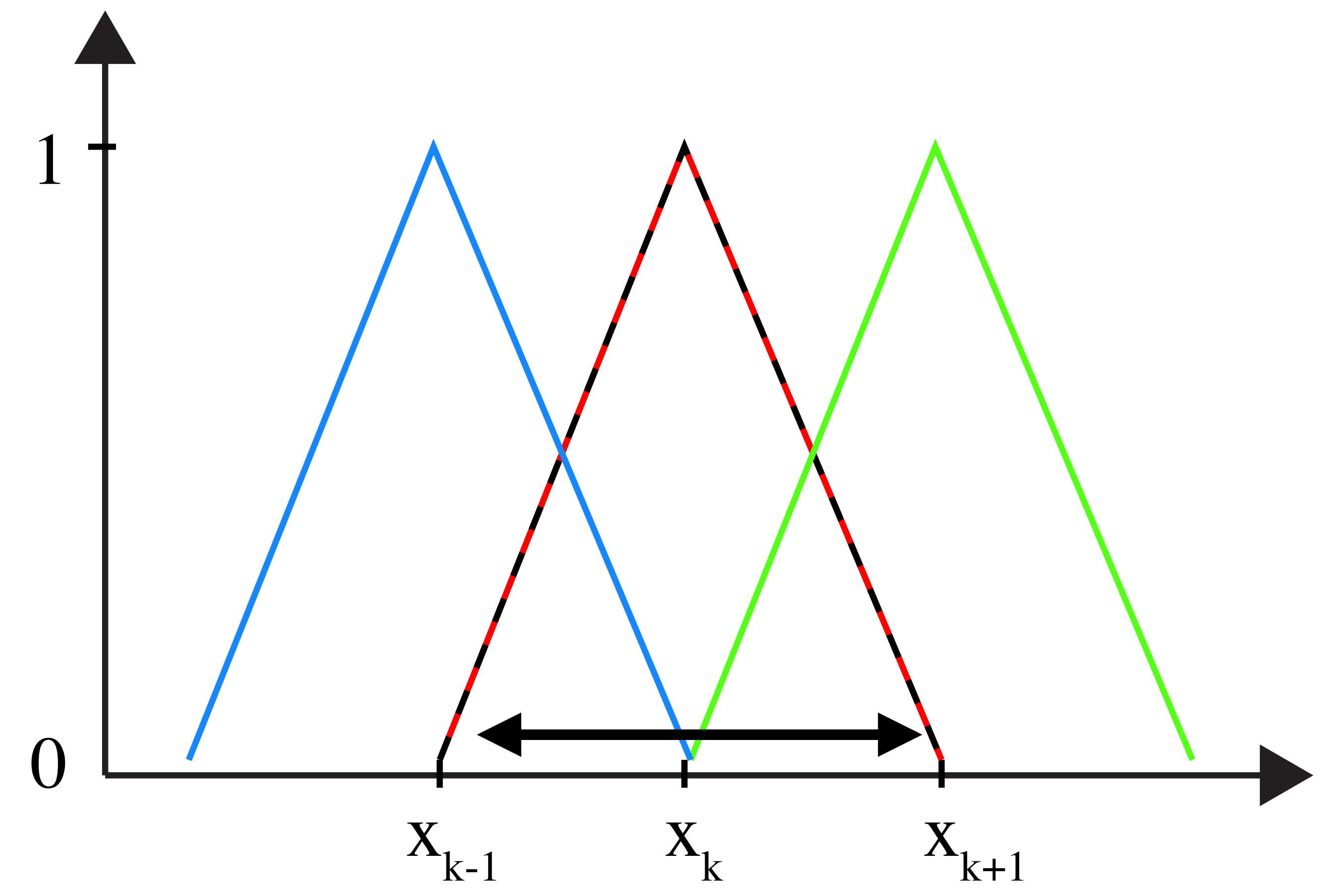
$$v_k(z) = \begin{cases} \frac{z - z_{k-1}}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{z_{k+1} - z}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & otherwise \end{cases} \qquad v_k'(z) = \begin{cases} \frac{1}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{-1}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & otherwise \end{cases}$$

$$A_{jk} = \int_{z_{k-1}}^{z_{k+1}} \left[v_k'(z) v_j'(z) + i\omega\mu_o \sigma(z) v_k(z) v_j(z) \right] dz$$

$$v_k(z) = \begin{cases} \frac{z - z_{k-1}}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{z_{k+1} - z}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & otherwise \end{cases} \qquad v_k'(z) = \begin{cases} \frac{1}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{-1}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & otherwise \end{cases}$$

$$A_{jk} = \int_{z_{k-1}}^{z_{k+1}} \left[v_k'(z) v_j'(z) + i\omega\mu_o \sigma(z) v_k(z) v_j(z) \right] dz$$

$$v_k(z) = \begin{cases} \frac{z - z_{k-1}}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{z_{k+1} - z}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & otherwise \end{cases} \qquad v_k'(z) = \begin{cases} \frac{1}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{-1}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & otherwise \end{cases}$$

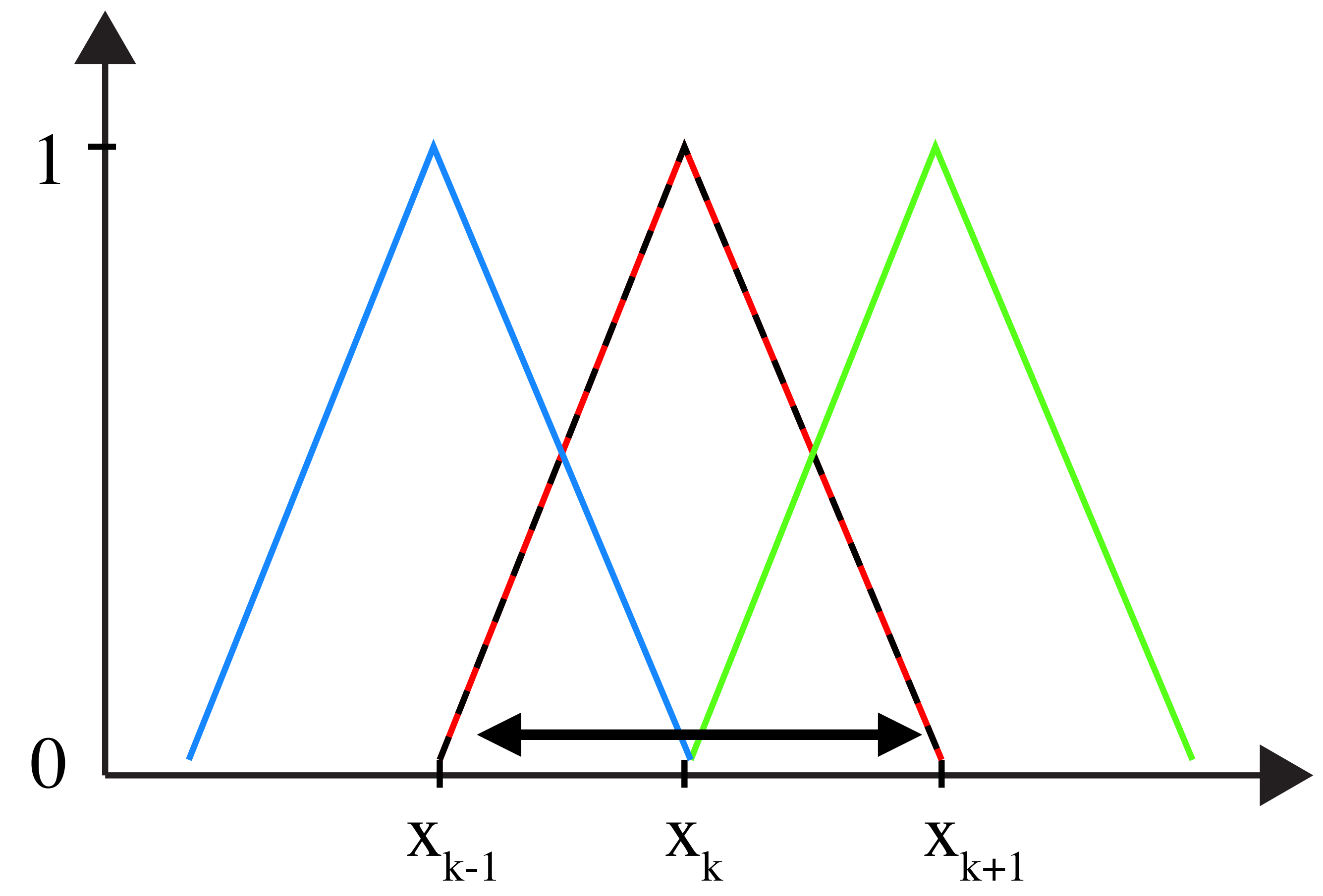


For $j = k$

$$A_{jk} = \int_{z_{k-1}}^{z_{k+1}} \frac{1}{\Delta z^2} dz + \int_{z_{k-1}}^{z_k} i\omega\mu_o \sigma_k \frac{(z - z_{k-1})^2}{\Delta z^2} dz + \int_{z_k}^{z_{k+1}} i\omega\mu_o \sigma_{k+1} \frac{(z_{k+1} - z)^2}{\Delta z^2} dz \quad j = k$$

$$A_{jk} = \int_{z_{k-1}}^{z_{k+1}} \left[v_k'(z) v_j'(z) + i\omega\mu_o \sigma(z) v_k(z) v_j(z) \right] dz$$

$$v_k(z) = \begin{cases} \frac{z - z_{k-1}}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{z_{k+1} - z}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & otherwise \end{cases} \qquad v_k'(z) = \begin{cases} \frac{1}{\Delta z} & z \in [z_{k-1}, z_k] \\ \frac{-1}{\Delta z} & z \in [z_k, z_{k+1}] \\ 0 & otherwise \end{cases}$$



For $j = k$

$$A_{jk} = \int_{z_{k-1}}^{z_{k+1}} \frac{1}{\Delta z^2} dz + \int_{z_{k-1}}^{z_k} i\omega\mu_o \sigma_k \frac{(z - z_{k-1})^2}{\Delta z^2} dz + \int_{z_k}^{z_{k+1}} i\omega\mu_o \sigma_{k+1} \frac{(z_{k+1} - z)^2}{\Delta z^2} dz \quad j = k$$

$$A_{jk} = \frac{1}{\Delta z^2} \left[z_{k+1} - z_{k-1} + i\omega\mu_o \sigma_k \left[\frac{1}{3}(z_k^3 - z_{k-1}^3) - z_{k-1} z_k^2 + z_{k-1}^2 z_k \right] + i\omega\mu_o \sigma_{k+1} \left[\frac{1}{3}(z_{k+1}^3 - z_k^3) + z_{k+1} z_k^2 - z_{k+1}^2 z_k \right] \right]$$

For $j = k$

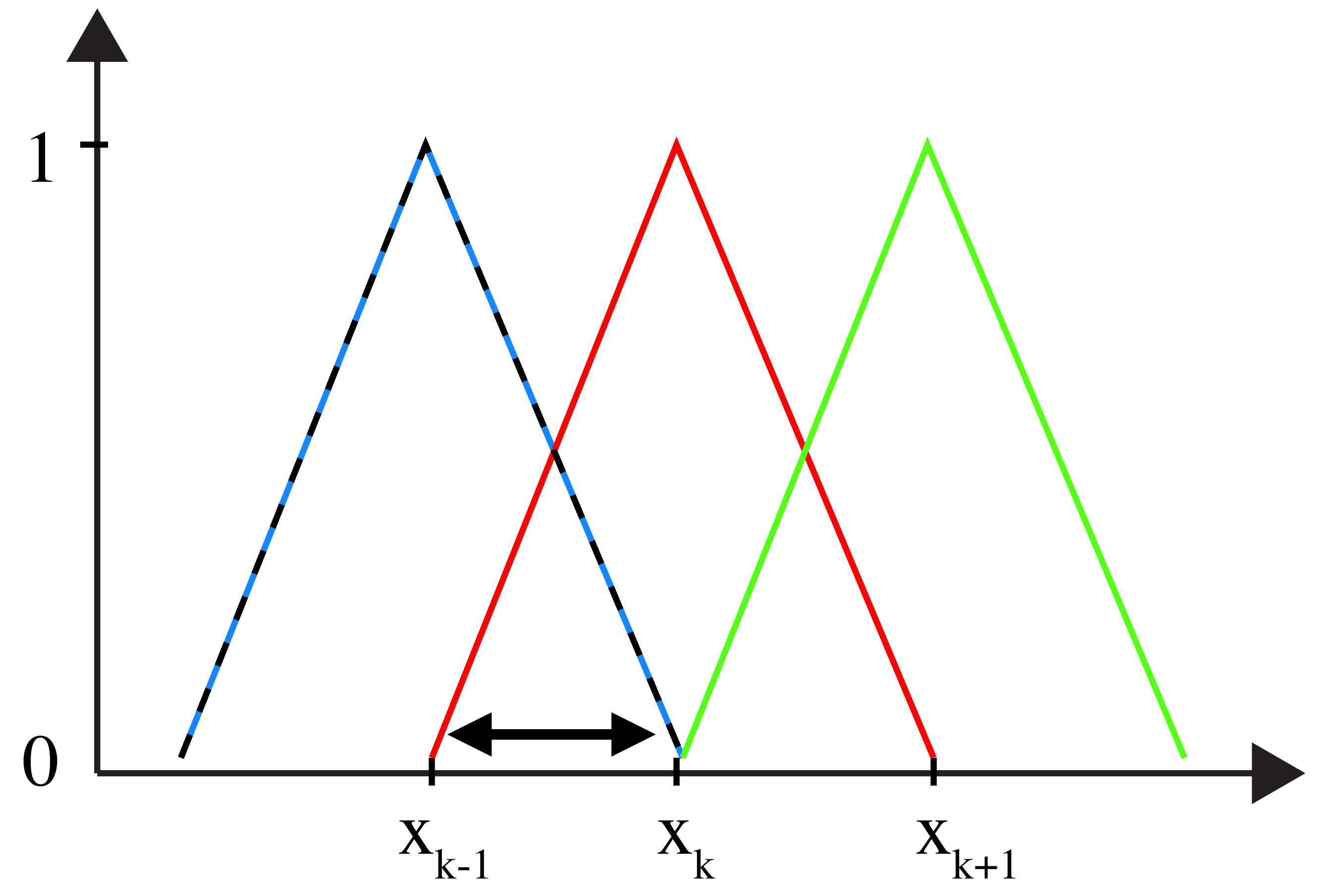
$$A_{jk} = \frac{1}{\Delta z^2}\left[z_{k+1} - z_{k-1} + i\omega\mu_o\sigma_k\big[\frac{1}{3}(z_k^3 - z_{k-1}^3) - z_{k-1}z_k^2 + z_{k-1}^2 z_k\big] + i\omega\mu_o\sigma_{k+1}\big[\frac{1}{3}(z_{k+1}^3 - z_k^3) + z_{k+1}z_k^2 - z_{k+1}^2 z_k\big]\right]$$

For $j = k$

$$A_{jk} = \frac{1}{\Delta z^2}\left[z_{k+1} - z_{k-1} + i\omega\mu_o\sigma_k\big[\frac{1}{3}(z_k^3 - z_{k-1}^3) - z_{k-1}z_k^2 + z_{k-1}^2 z_k\big] + i\omega\mu_o\sigma_{k+1}\big[\frac{1}{3}(z_{k+1}^3 - z_k^3) + z_{k+1}z_k^2 - z_{k+1}^2 z_k\big]\right]$$

For $j = k\text{-}1$

$$A_{jk} = \frac{1}{\Delta z^2}\left[z_{k-1} - z_k + i\omega\mu_o\sigma_k\big[\frac{1}{3}(z_{k-1}^3 - z_k^3) + z_{k-1}z_k^2 - z_{k-1}^2 z_k\big]\right]$$

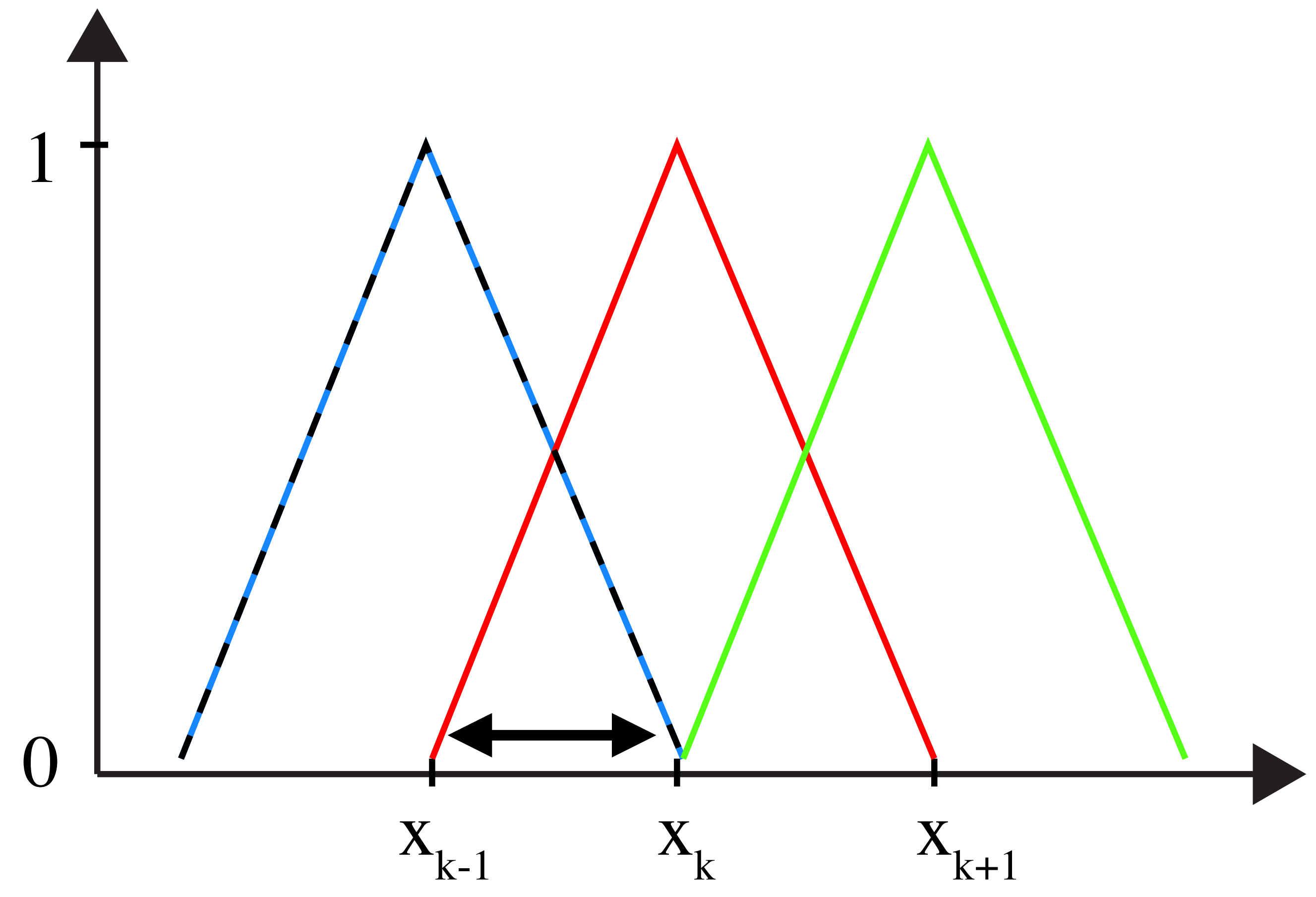


For $j = k$

$$A_{jk} = \frac{1}{\Delta z^2}\Big[z_{k+1} - z_{k-1} + i\omega\mu_o\sigma_k\big[\frac{1}{3}(z_k^3 - z_{k-1}^3) - z_{k-1}z_k^2 + z_{k-1}^2 z_k\big] + i\omega\mu_o\sigma_{k+1}\big[\frac{1}{3}(z_{k+1}^3 - z_k^3) + z_{k+1}z_k^2 - z_{k+1}^2 z_k\big]\Big]$$

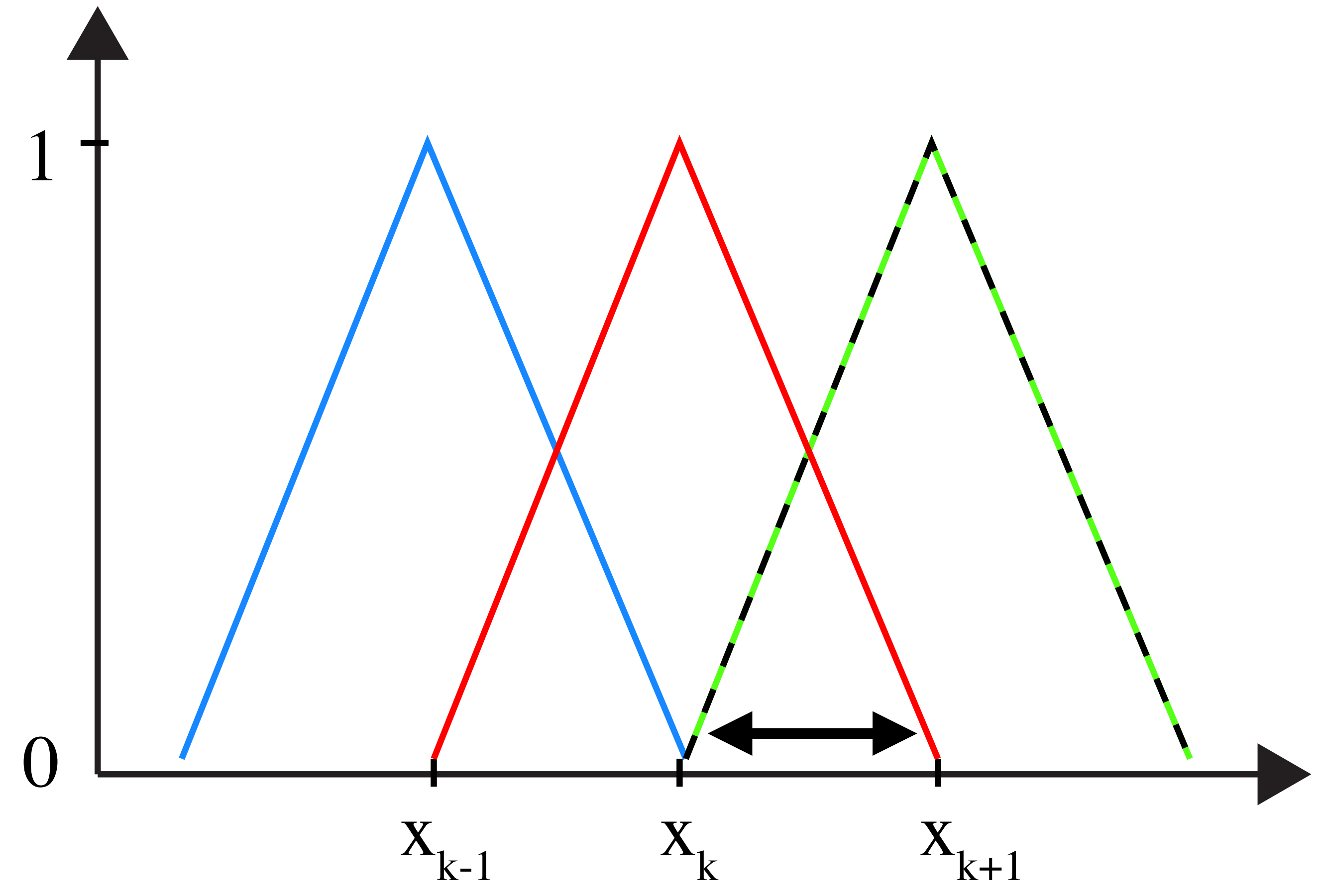
For $j = k\text{-}1$

$$A_{jk} = \frac{1}{\Delta z^2}\Big[z_{k-1} - z_k + i\omega\mu_o\sigma_k\big[\frac{1}{3}(z_{k-1}^3 - z_k^3) + z_{k-1}z_k^2 - z_{k-1}^2 z_k\big]\Big]$$



For $j = k+1$

$$A_{jk} = \frac{1}{\Delta z^2}\Big[z_k - z_{k+1} + i\omega\mu_o\sigma_{k+1}\big[\frac{1}{3}(z_k^3 - z_{k+1}^3) - z_{k+1}z_k^2 + z_{k+1}^2 z_k\big]\Big]$$

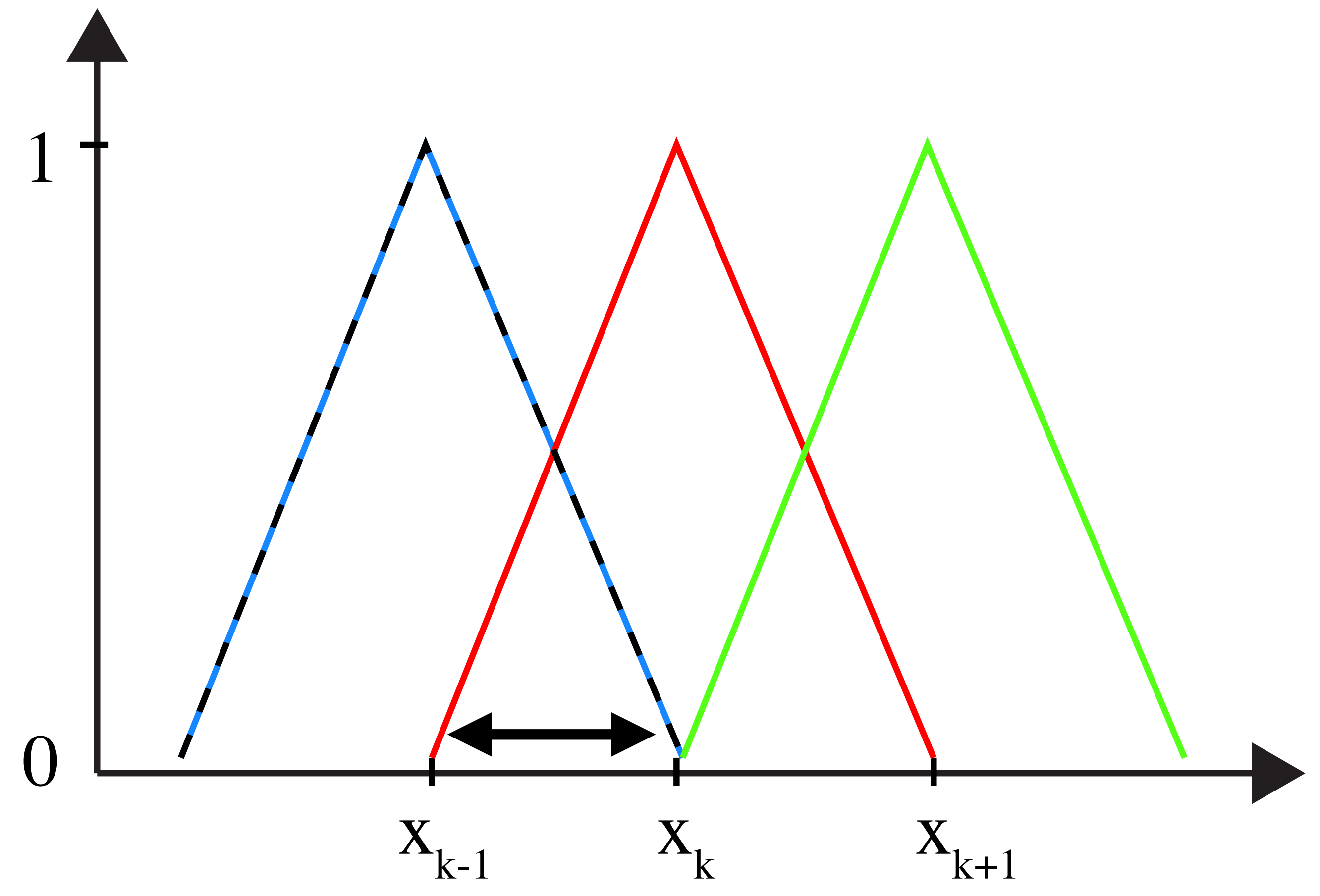


For $j = k$

$$A_{jk} = \frac{1}{\Delta z^2}\left[z_{k+1} - z_{k-1} + i\omega\mu_o\sigma_k\left[\frac{1}{3}(z_k^3 - z_{k-1}^3) - z_{k-1}z_k^2 + z_{k-1}^2 z_k\right] + i\omega\mu_o\sigma_{k+1}\left[\frac{1}{3}(z_{k+1}^3 - z_k^3) + z_{k+1}z_k^2 - z_{k+1}^2 z_k\right]\right]$$

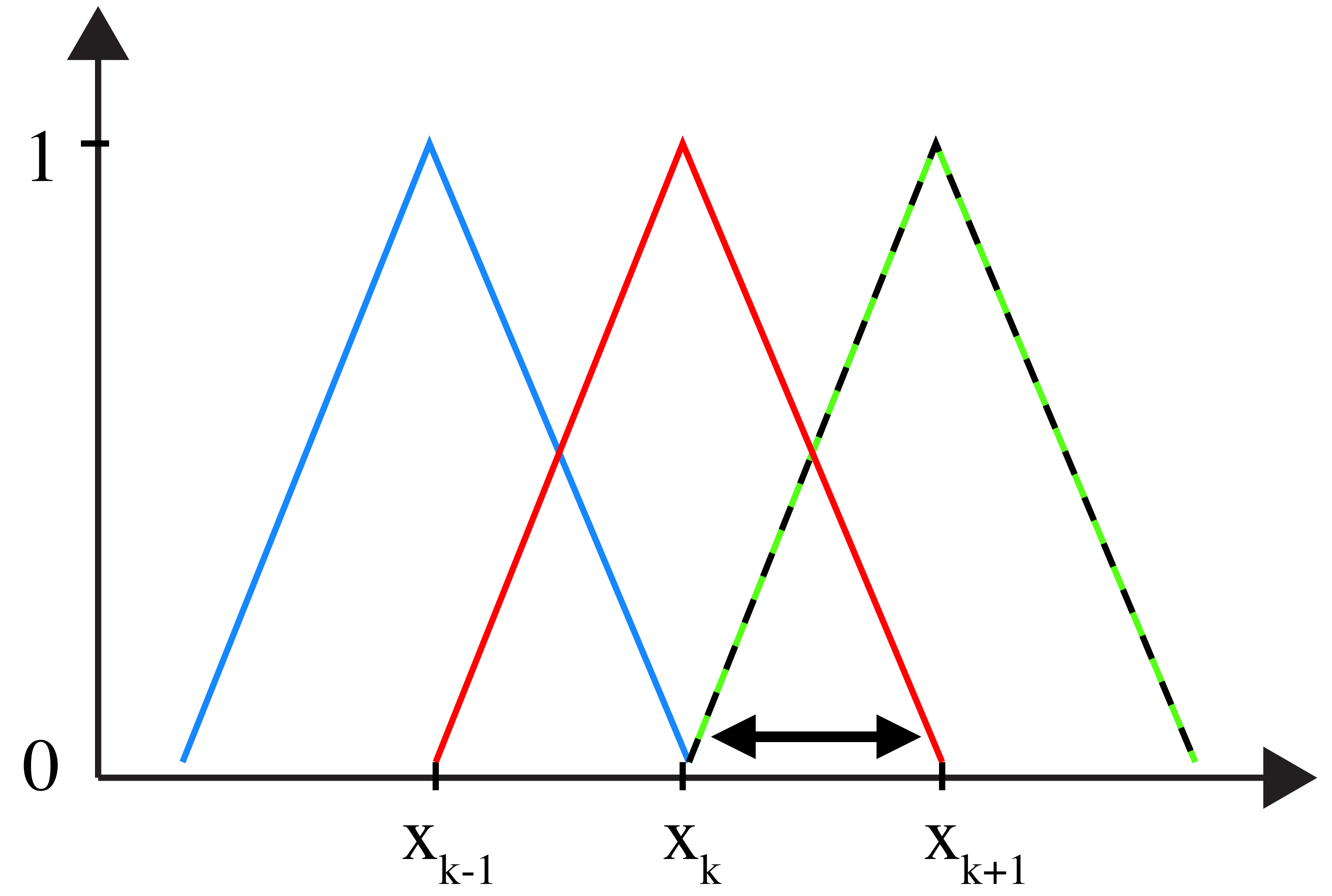
For $j = k\text{-}1$

$$A_{jk} = \frac{1}{\Delta z^2}\left[z_{k-1} - z_k + i\omega\mu_o\sigma_k\left[\frac{1}{3}(z_{k-1}^3 - z_k^3) + z_{k-1}z_k^2 - z_{k-1}^2 z_k\right]\right]$$



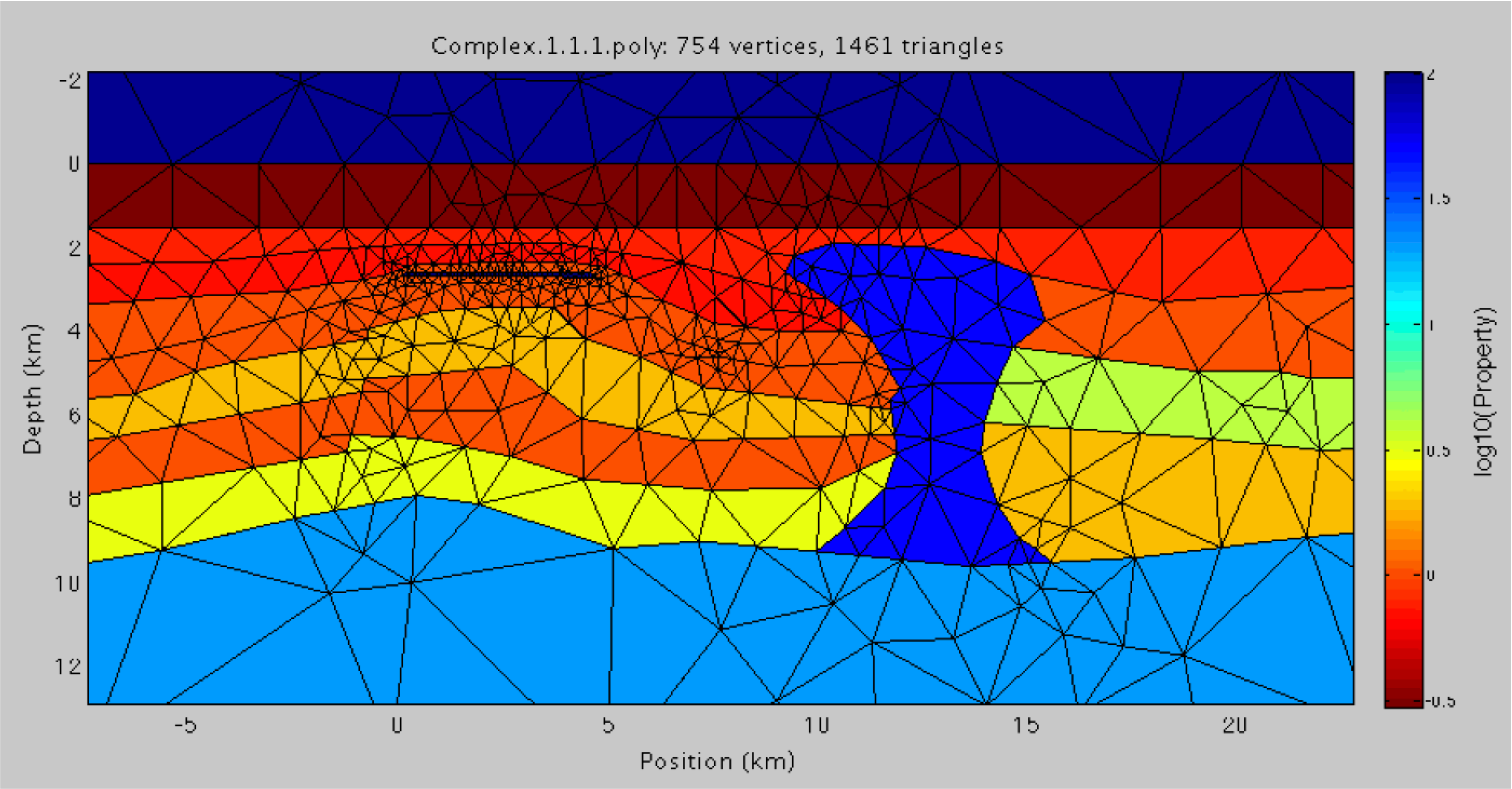
For $j = k\text{+}1$

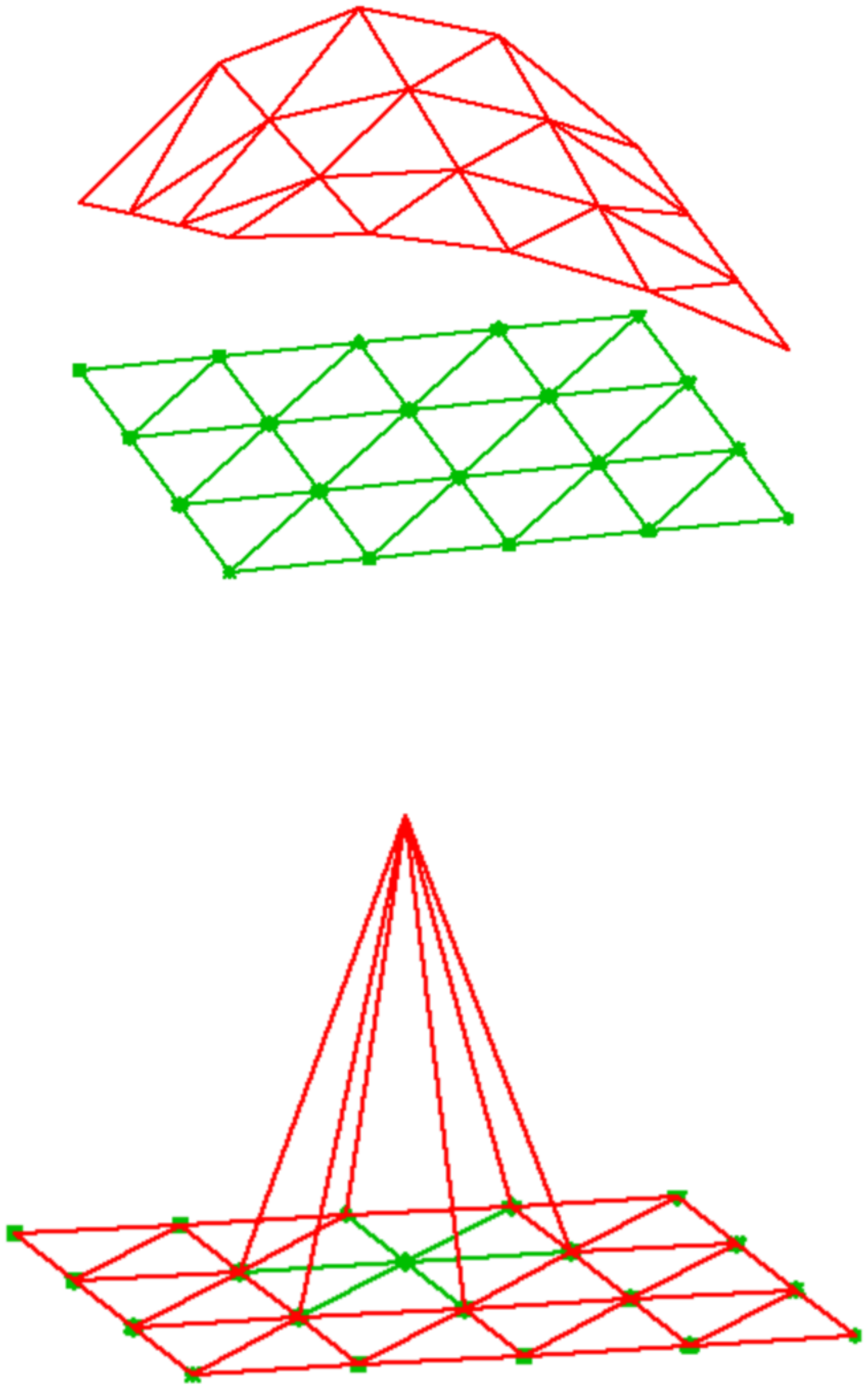
$$A_{jk} = \frac{1}{\Delta z^2}\left[z_k - z_{k+1} + i\omega\mu_o\sigma_{k+1}\left[\frac{1}{3}(z_k^3 - z_{k+1}^3) - z_{k+1}z_k^2 + z_{k+1}^2 z_k\right]\right]$$



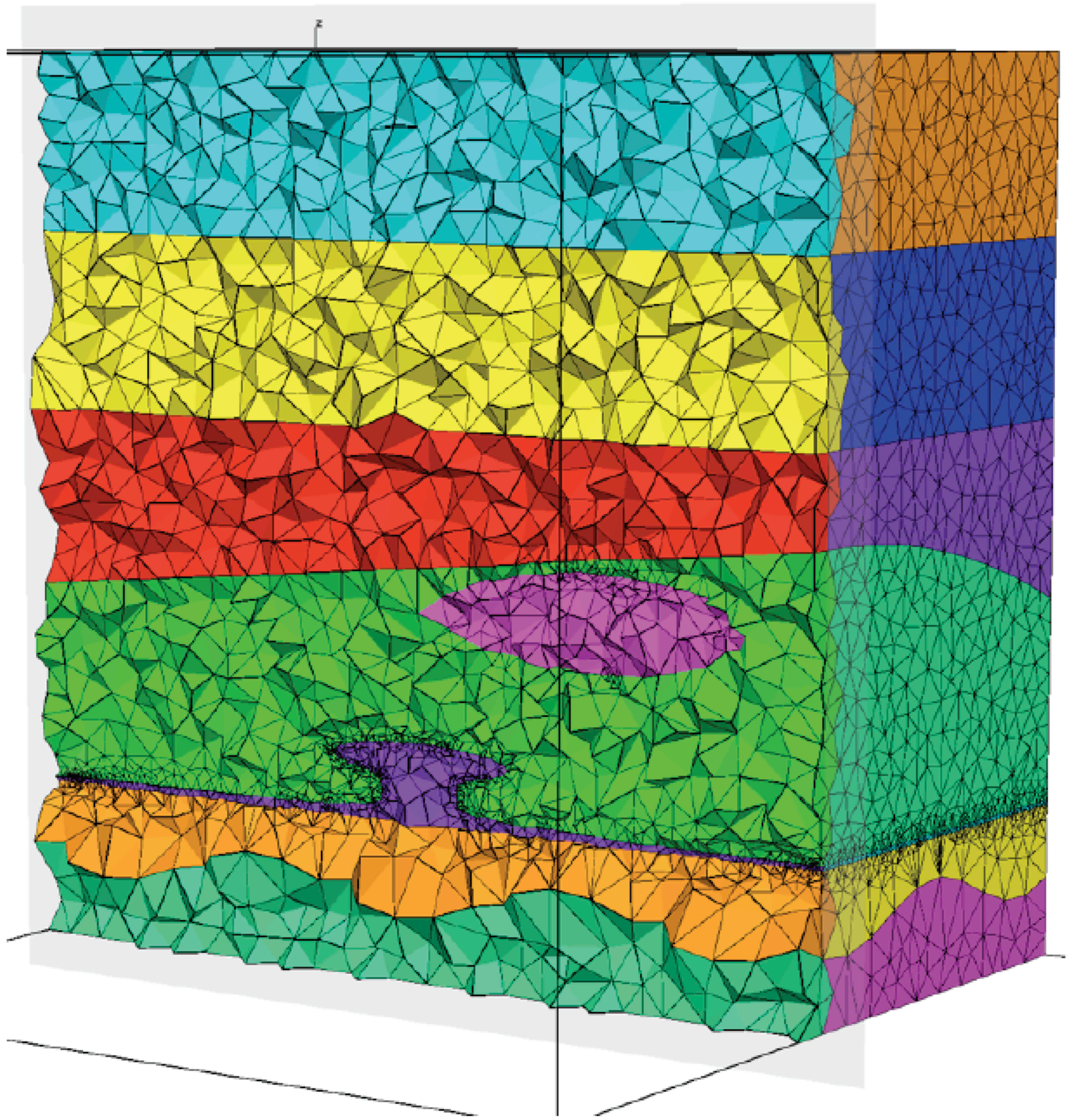
(Just add boundary conditions.)

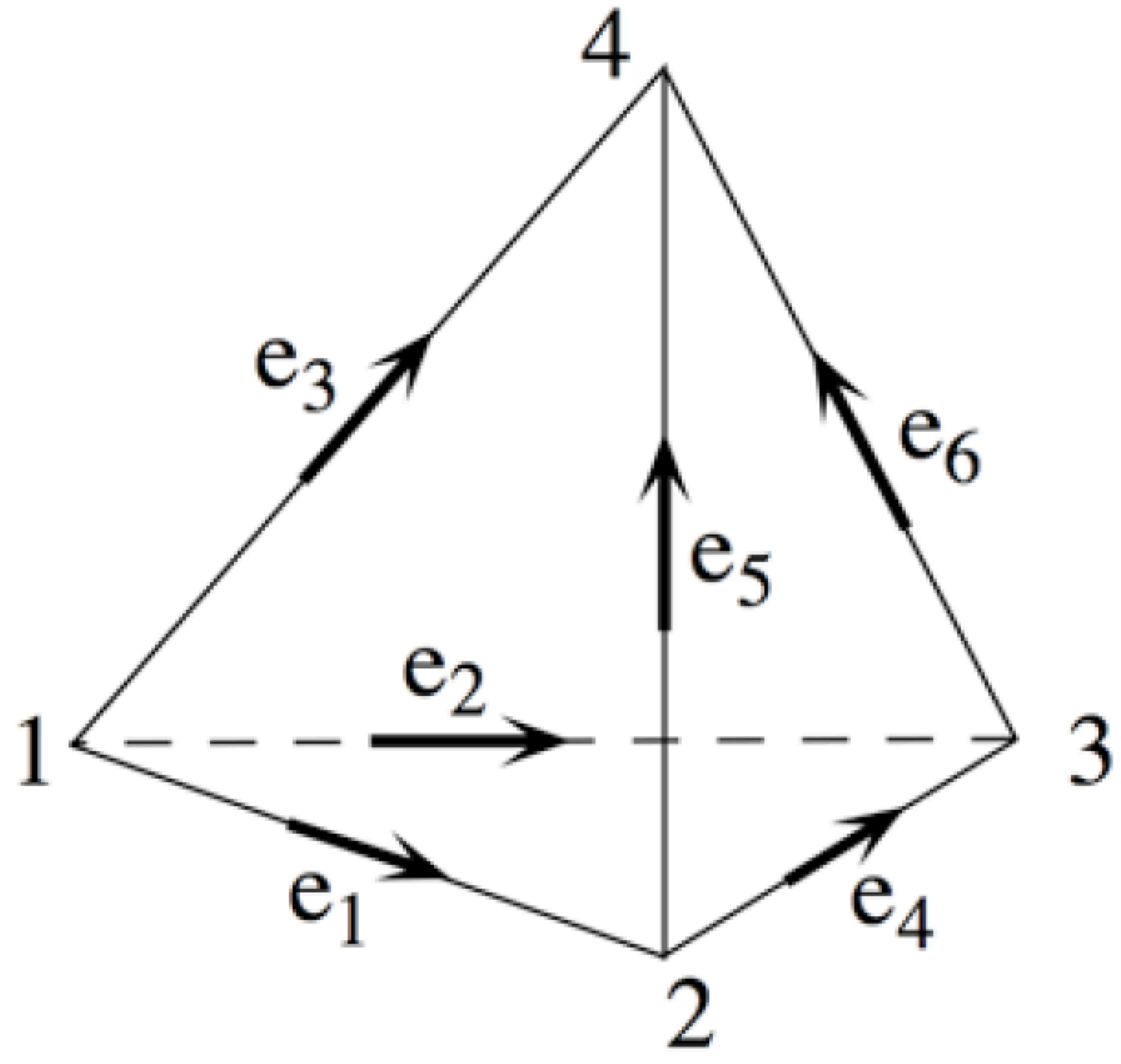
Finite elements in 2D. Electric fields defined on the nodes.

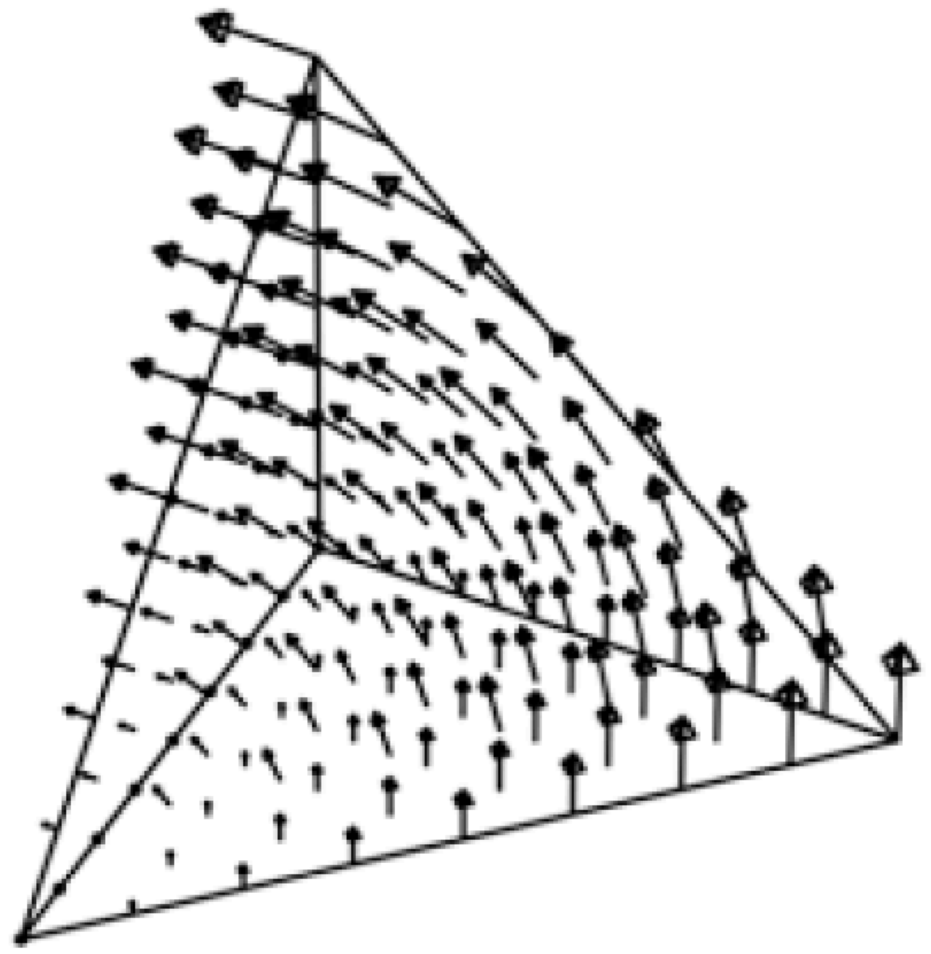




Finite elements in 3D. The electric fields are now defined on the element edges.





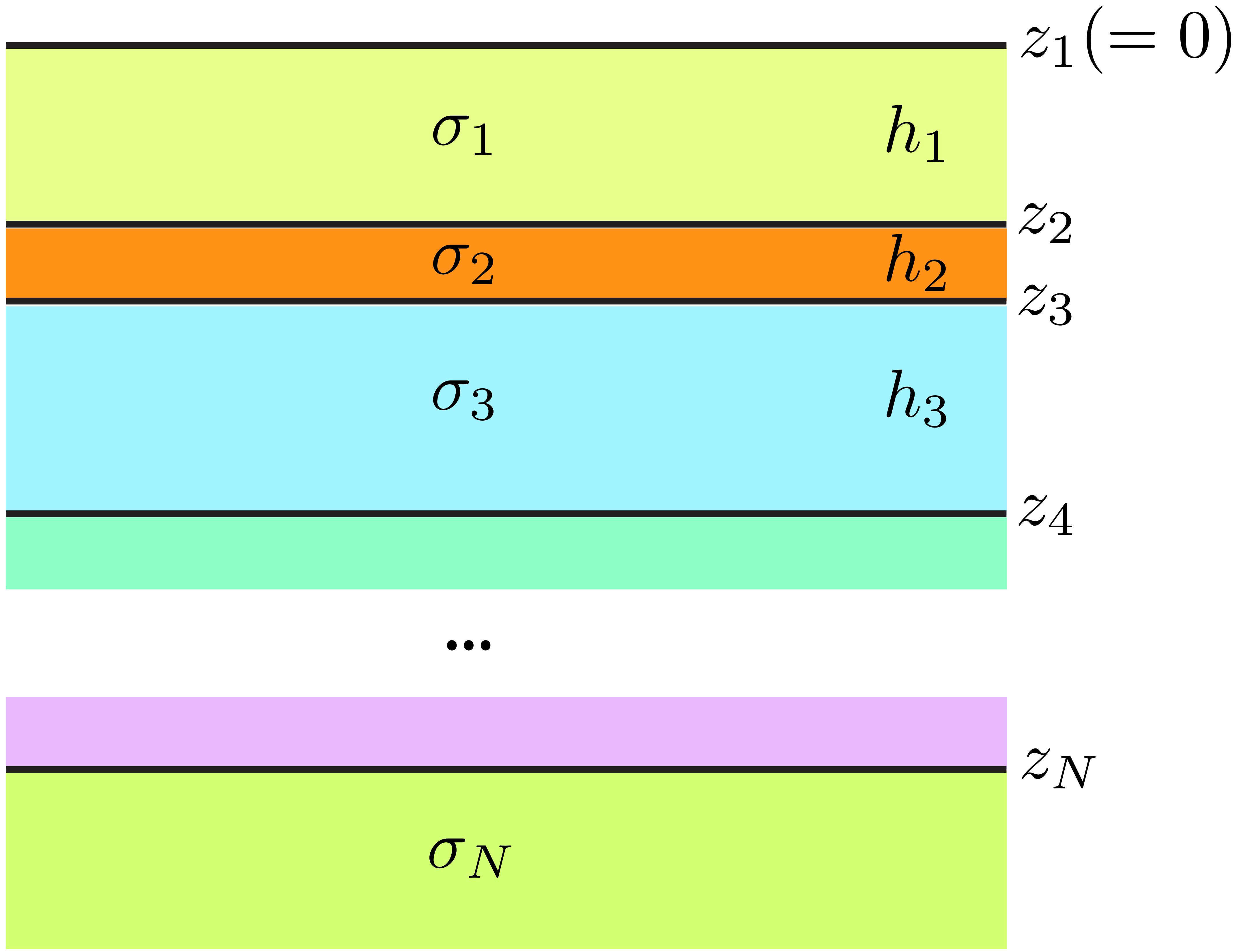


Linear basis for e6

The COMSOL Product Suite

Stack of layers:

FD and FE methods assume conductivity is a smooth function of z. They solve for $\mathbf{E}$ which allows one to compute impedance. But consider piece-wise constant conductivity - a stack of layers. It turns out we can solve for impedance directly.



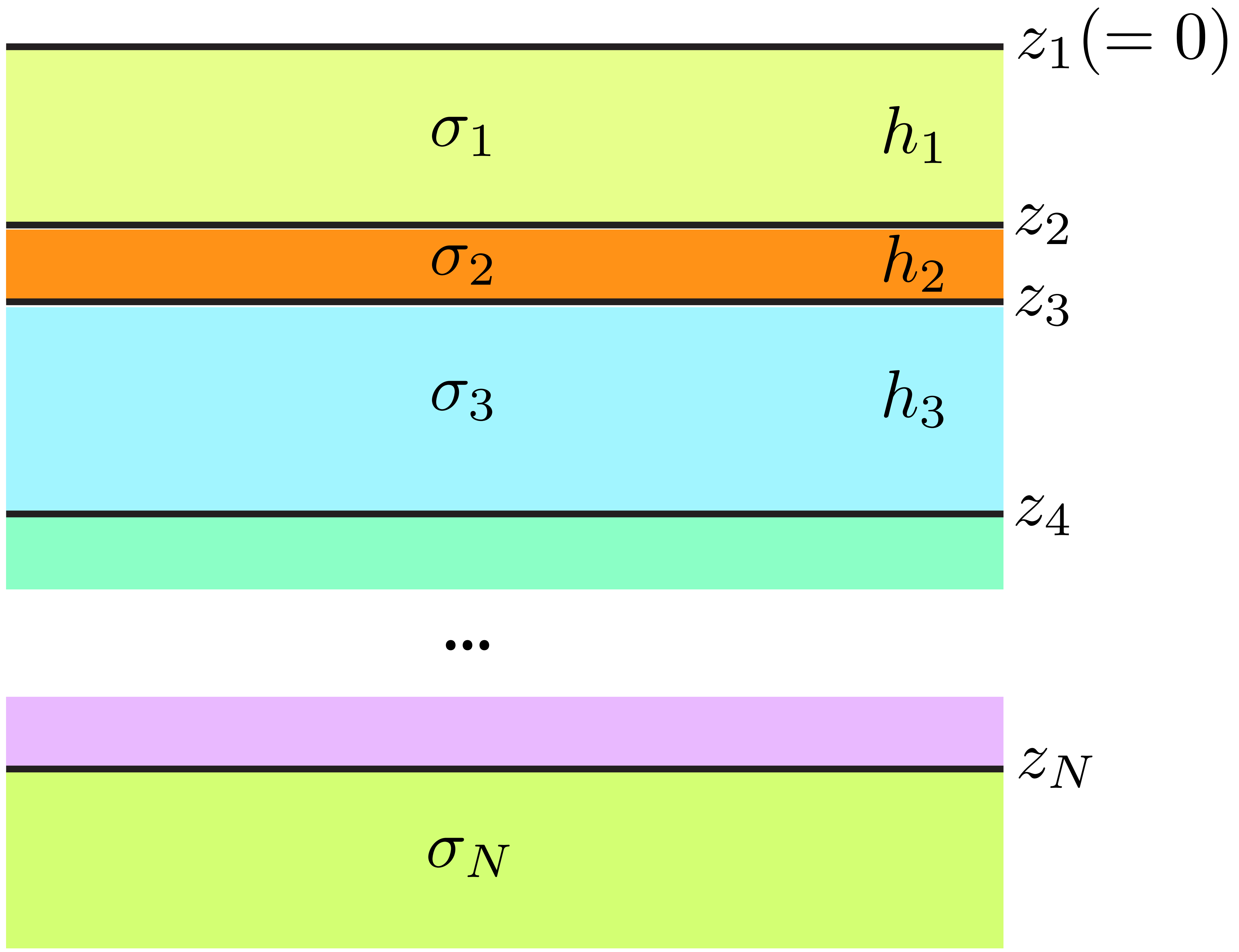
We showed that because there are no horizontal gradients in conductivity, our diffusion equation still holds in $\mathbf{E}$:

$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

with solutions as before for constant conductivity and wavenumber in each layer *n* (lecture 3)

$$E(z) = C_1 e^{k_n z} + C_2 e^{-k_n z}, \quad z_n \leq z \leq z_{n+1} \qquad k_n = (i\omega\mu_0\sigma_n)^{1/2}$$

(but this time C_1 does not go away within a layer)



$$E(z) = C_1 e^{k_n z} + C_2 e^{-k_n z}, \quad z_n \le z \le z_{n+1} \qquad\qquad k_n = (i\omega\mu_0\sigma_n)^{1/2}$$

A trick:

$$E(z) = A\cosh\big(k_n(z - z_{n+1})\big) + B\sinh\big(k_n(z - z_{n+1})\big), \quad z_n \le z \le z_{n+1}$$

$$E(z) = C_1 e^{k_n z} + C_2 e^{-k_n z}, \quad z_n \le z \le z_{n+1} \qquad\qquad k_n = (i\omega\mu_0\sigma_n)^{1/2}$$

A trick:

$$E(z) = A\cosh\big(k_n(z - z_{n+1})\big) + B\sinh\big(k_n(z - z_{n+1})\big), \quad z_n \le z \le z_{n+1}$$

This works because $\sinh x = \frac{1}{2}(e^x - e^{-x})$ $\qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$

Indeed, you can show $C_1 = \dfrac{A+B}{2e^{k_n z_{n+1}}}$ $\qquad C_2 = \dfrac{A-B}{2e^{k_n z_{n+1}}}$

$$E(z) = C_1 e^{k_n z} + C_2 e^{-k_n z}, \quad z_n \le z \le z_{n+1} \qquad\qquad k_n = (i\omega\mu_0\sigma_n)^{1/2}$$

A trick:

$$E(z) = A\cosh\big(k_n(z - z_{n+1})\big) + B\sinh\big(k_n(z - z_{n+1})\big), \quad z_n \le z \le z_{n+1}$$

This works because $\sinh x = \frac{1}{2}(e^x - e^{-x})$ $\qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$

Indeed, you can show $C_1 = \frac{A+B}{2e^{k_n z_{n+1}}}$ $\qquad C_2 = \frac{A-B}{2e^{k_n z_{n+1}}}$

Differentiating

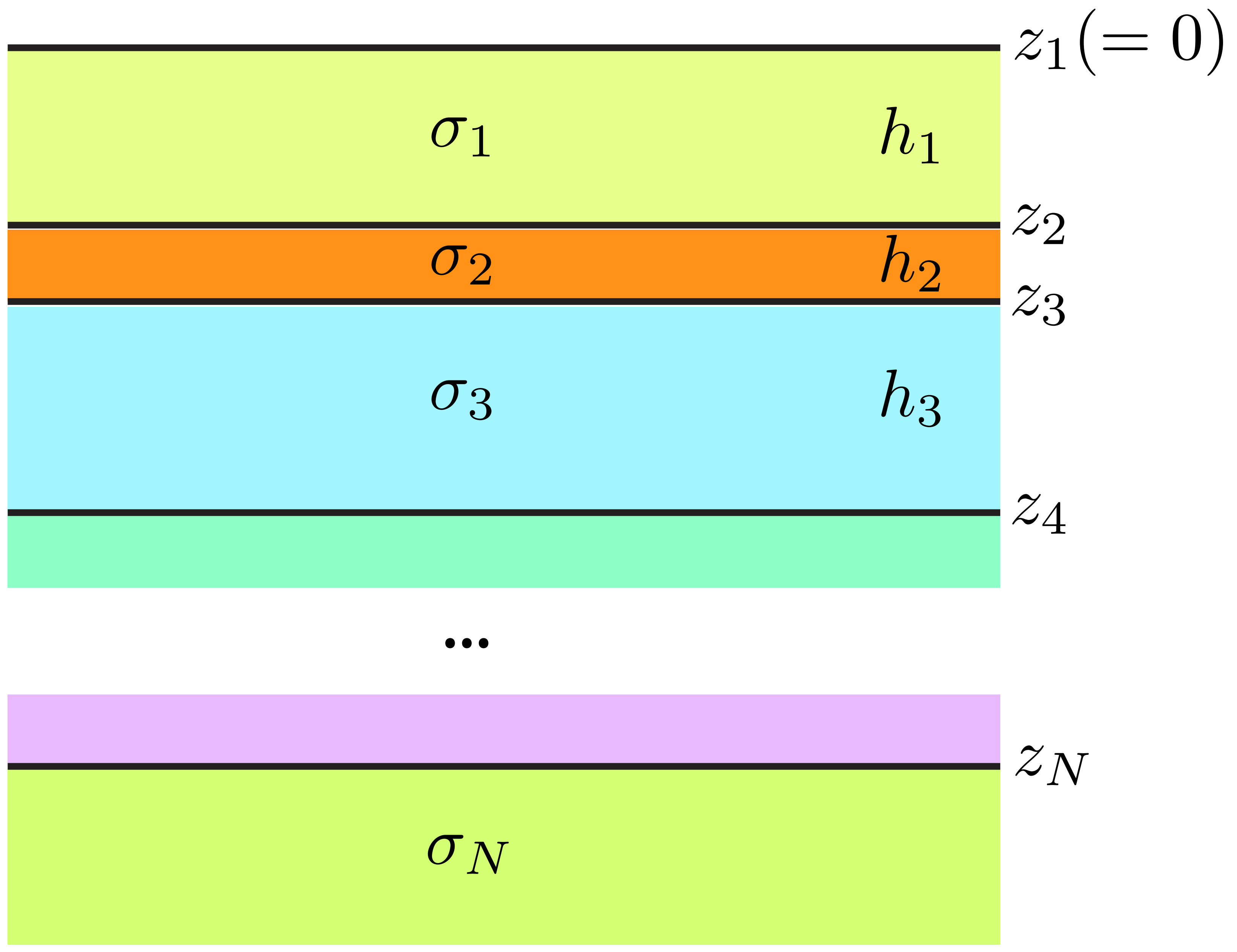
$$E'(z) = k_n[A\sinh\big(k_n(z - z_{n+1})\big) + B\cosh\big(k_n(z - z_{n+1})\big)], \quad z_n \le z \le z_{n+1}$$

allows us to calculate *c(z)* anywhere in our model

$$c(z) = -\frac{E(z)}{E'(z)}$$

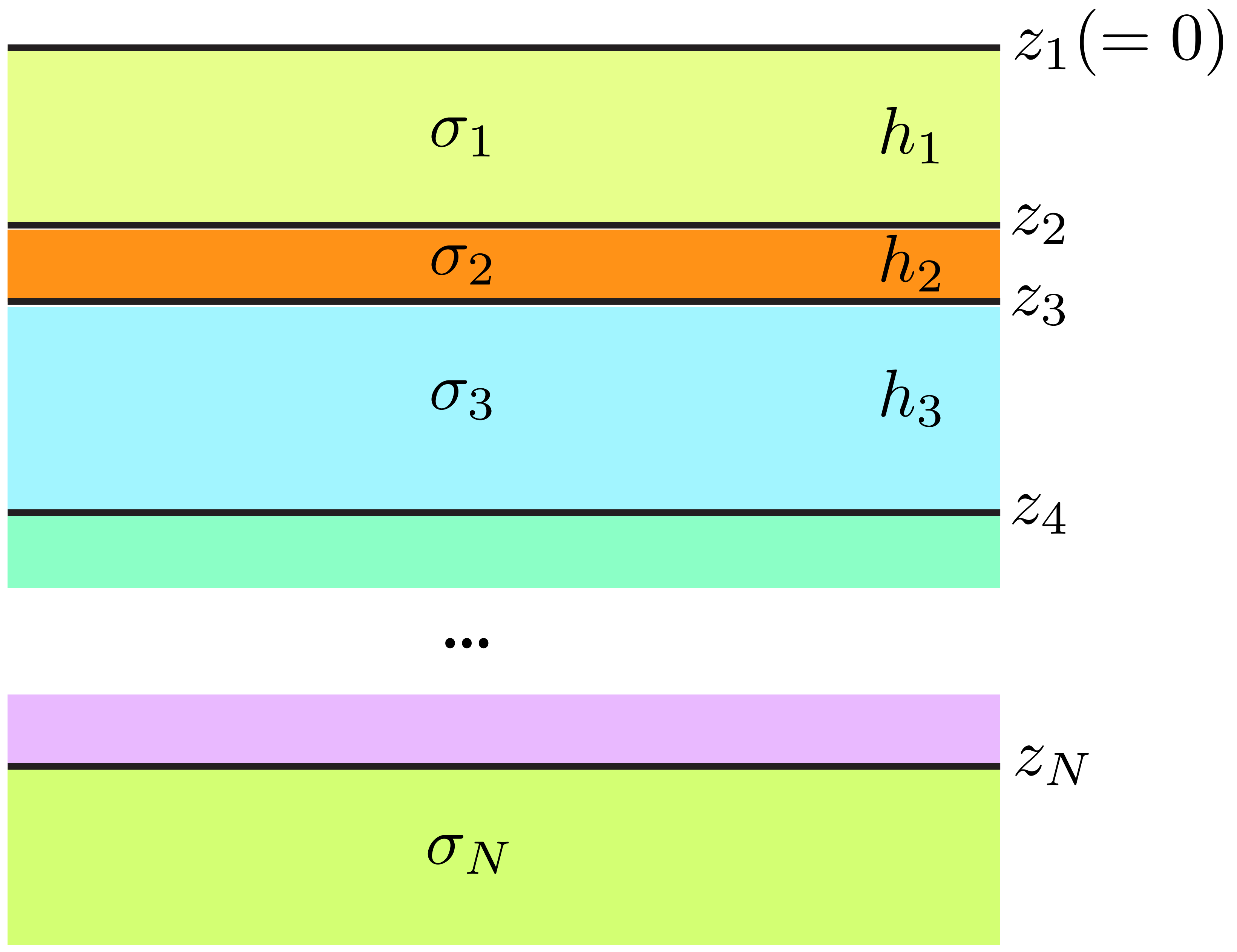
$$c(z) = -\frac{E(z)}{E'(z)} = -\frac{A\cosh(k_n(z - z_{n+1})) + B\sinh(k_n(z - z_{n+1}))}{k_n[A\sinh(k_n(z - z_{n+1})) + B\cosh(k_n(z - z_{n+1}))]}$$

$$z_n \leq z \leq z_{n+1}$$



$$c(z) = -\frac{E(z)}{E'(z)} = -\frac{A\cosh(k_n(z - z_{n+1})) + B\sinh(k_n(z - z_{n+1}))}{k_n[A\sinh(k_n(z - z_{n+1})) + B\cosh(k_n(z - z_{n+1}))]} \qquad z_n \le z \le z_{n+1}$$

0 0 0 0

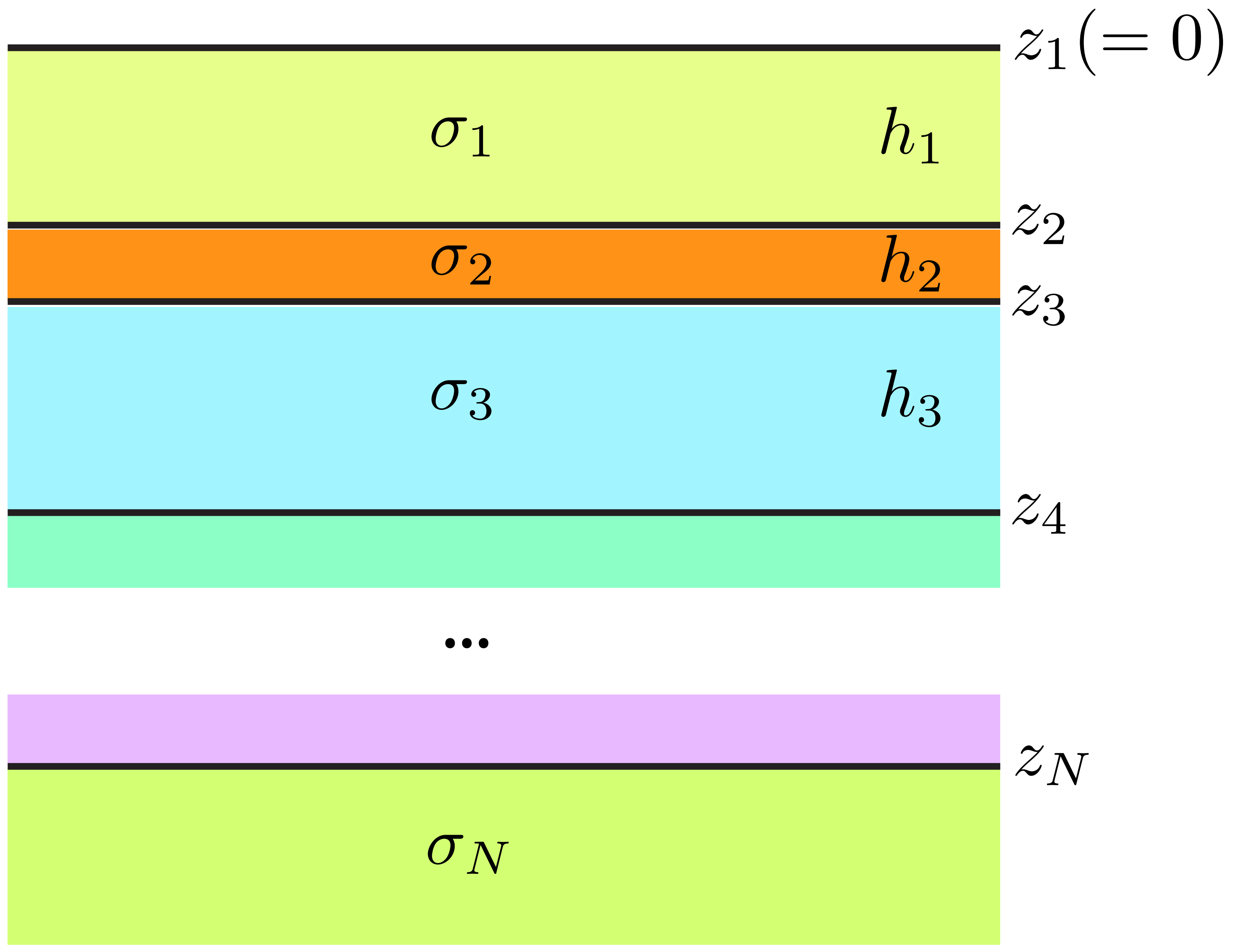


At the bottom of layer n, $z = z_{n+1}$ so $z - z_{n+1} = 0$

$\sinh(0) = 0$ and $\cosh(0) = 1$

$$c_{n+} = -\frac{E(z_{n+1})}{E'(z_{n+1})} = -\frac{A}{k_n B}$$

$$c(z) = -\frac{E(z)}{E'(z)} = -\frac{A\cosh(k_n(z - z_{n+1})) + B\sinh(k_n(z - z_{n+1}))}{k_n[A\sinh(k_n(z - z_{n+1})) + B\cosh(k_n(z - z_{n+1}))]} \qquad z_n \le z \le z_{n+1}$$



At the bottom of layer n, $z = z_{n+1}$ so $z - z_{n+1} = 0$

$\sinh(0) = 0$ and $\cosh(0) = 1$

$$c_{n+} = -\frac{E(z_{n+1})}{E'(z_{n+1})} = -\frac{A}{k_n B} = \boxed{c_{n+1}}$$

Because E and E' must be continuous, this is the same as the impedance at the top of layer $n+1$

$$c(z) = -\frac{E(z)}{E'(z)} = -\frac{A\cosh(k_n(z - z_{n+1})) + B\sinh(k_n(z - z_{n+1}))}{k_n[A\sinh(k_n(z - z_{n+1})) + B\cosh(k_n(z - z_{n+1}))]}$$

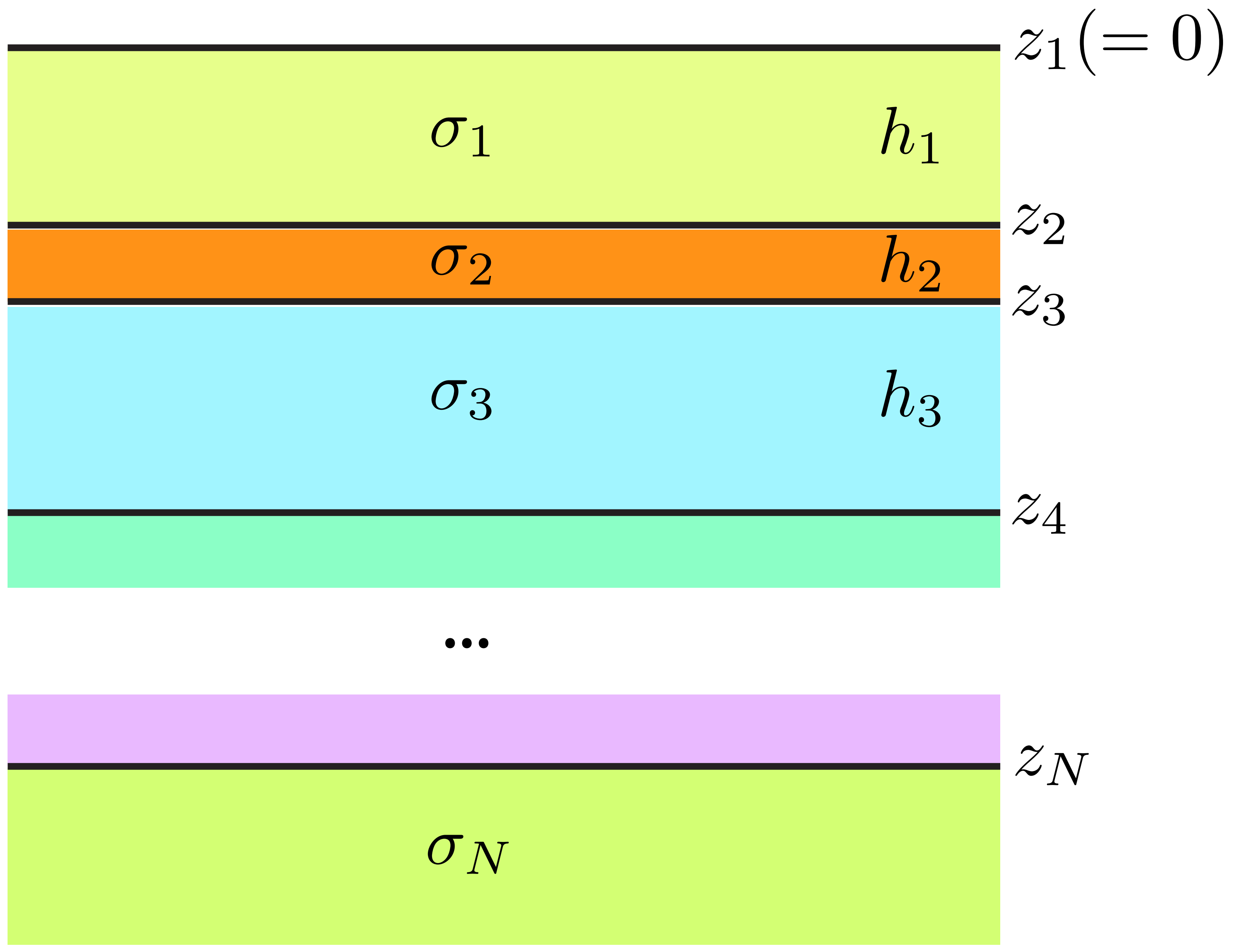
$-h_n \quad -h_n \qquad -h_n \quad -h_n$

$$z_n \leq z \leq z_{n+1}$$

At the top of layer n, $\quad z = z_n \quad$ so $\quad z - z_{n+1} = -h_n$

$$\sinh(-x) = -\sinh(x) \quad \text{and} \quad \cosh(-x) = \cosh(x)$$

$$c_n = -\frac{E(z_n)}{E'(z_n)} = -\frac{A\cosh(k_n h_n) - B\sinh(k_n h_n)}{k_n[-A\sinh(k_n h_n) + B\cosh(k_n h_n)]}$$



At the bottom of layer n, $\quad z = z_{n+1} \quad$ so $\quad z - z_{n+1} = 0$

$$\sinh(0) = 0 \quad \text{and} \quad \cosh(0) = 1$$

$$c_{n+} = -\frac{E(z_{n+1})}{E'(z_{n+1})} = -\frac{A}{k_n B} = c_{n+1}$$

$$c(z) = -\frac{E(z)}{E'(z)} = -\frac{A\cosh(k_n(z - z_{n+1})) + B\sinh(k_n(z - z_{n+1}))}{k_n[A\sinh(k_n(z - z_{n+1})) + B\cosh(k_n(z - z_{n+1}))]} \qquad z_n \le z \le z_{n+1}$$

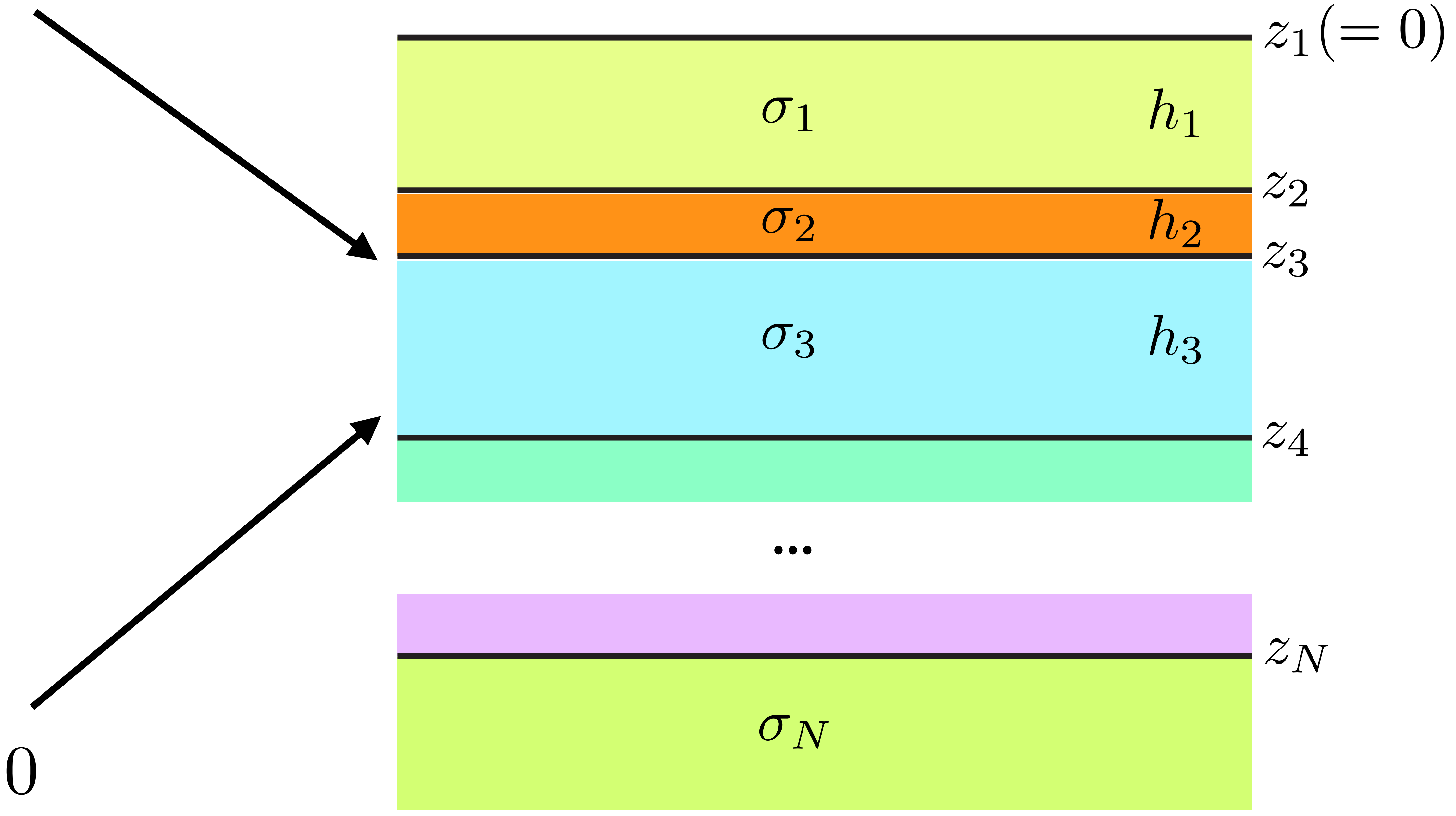
At the top of layer n, $\quad z = z_n \quad$ so $\quad z - z_{n+1} = -h_n$

$$\sinh(-x) = -\sinh(x) \quad \text{and} \quad \cosh(-x) = \cosh(x)$$

$$c_n = -\frac{E(z_n)}{E'(z_n)} = -\frac{A\cosh(k_n h_n) - B\sinh(k_n h_n)}{k_n[-A\sinh(k_n h_n) + B\cosh(k_n h_n)]}$$

divide everything by $B\sinh(k_n h_n)$

$$c_n = \frac{-(A/B)\coth(k_n h_n) + 1}{k_n[-A/B + \coth(k_n h_n)]}$$



At the bottom of layer n, $\quad z = z_{n+1} \quad$ so $\quad z - z_{n+1} = 0$

$$\sinh(0) = 0 \quad \text{and} \quad \cosh(0) = 1$$

$$c_{n+} = -\frac{E(z_{n+1})}{E'(z_{n+1})} = -\frac{A}{k_n B} = c_{n+1}$$

$$c(z) = -\frac{E(z)}{E'(z)} = -\frac{A\cosh(k_n(z-z_{n+1})) + B\sinh(k_n(z-z_{n+1}))}{k_n[A\sinh(k_n(z-z_{n+1})) + B\cosh(k_n(z-z_{n+1}))]}$$

$$z_n \leq z \leq z_{n+1}$$

At the top of layer n, $z = z_n$ so $z - z_{n+1} = -h_n$

$$\sinh(-x) = -\sinh(x) \quad \text{and} \quad \cosh(-x) = \cosh(x)$$

$$c_n = -\frac{E(z_n)}{E'(z_n)} = -\frac{A\cosh(k_n h_n) - B\sinh(k_n h_n)}{k_n[-A\sinh(k_n h_n) + B\cosh(k_n h_n)]}$$

$$c_n = \frac{-(A/B)\coth(k_n h_n) + 1}{k_n[-A/B + \coth(k_n h_n)]}$$

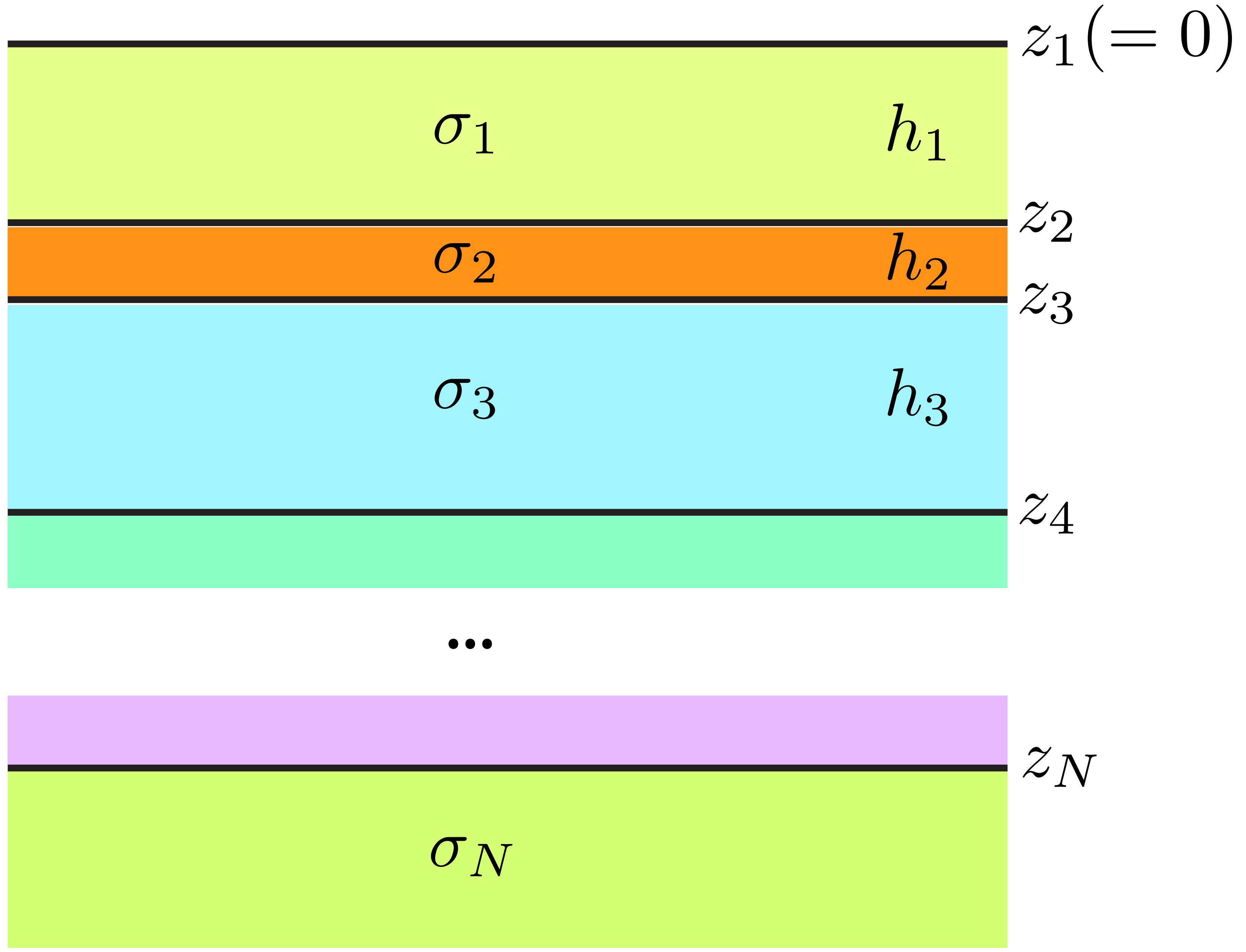
substitute A/B with c_{n+1} and divide by k_n

$$c_n = \frac{c_{n+1}\coth(k_n h_n) + 1/k_n}{k_n c_{n+1} + \coth(k_n h_n)}$$

At the bottom of layer n, $z = z_{n+1}$ so $z - z_{n+1} = 0$

$$\sinh(0) = 0 \quad \text{and} \quad \cosh(0) = 1$$

$$c_{n+} = -\frac{E(z_{n+1})}{E'(z_{n+1})} = -\frac{A}{k_n B} = c_{n+1}$$



Now we have an expression for c at the top of every layer that uses only that layer's conductivity and thickness and the c at the top of the layer below.

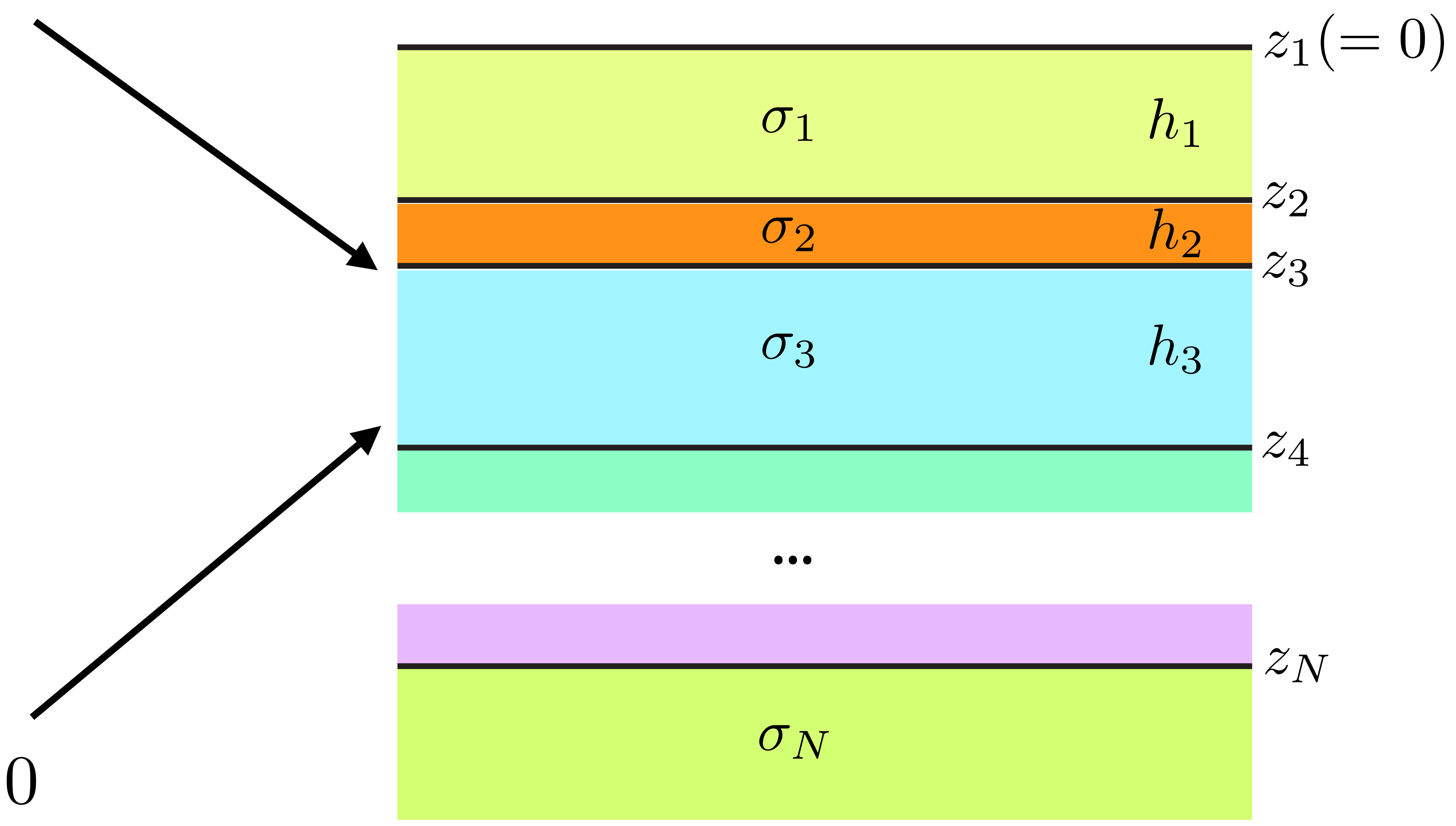
At the top of layer n, $z = z_n$ so $z - z_{n+1} = -h_n$

$$\sinh(-x) = -\sinh(x) \quad \text{and} \quad \cosh(-x) = \cosh(x)$$

$$c_n = -\frac{E(z_n)}{E'(z_n)} = -\frac{A\cosh(k_n h_n) - B\sinh(k_n h_n)}{k_n[-A\sinh(k_n h_n) + B\cosh(k_n h_n)]}$$

$$c_n = \frac{-(A/B)\coth(k_n h_n) + 1}{k_n[-A/B + \coth(k_n h_n)]}$$

$$c_n = \frac{c_{n+1}\coth(k_n h_n) + 1/k_n}{k_n c_{n+1} + \coth(k_n h_n)}$$



At the bottom of layer n, $z = z_{n+1}$ so $z - z_{n+1} = 0$

$$\sinh(0) = 0 \quad \text{and} \quad \cosh(0) = 1$$

$$c_{n+} = -\frac{E(z_{n+1})}{E'(z_{n+1})} = -\frac{A}{k_n B} = c_{n+1}$$

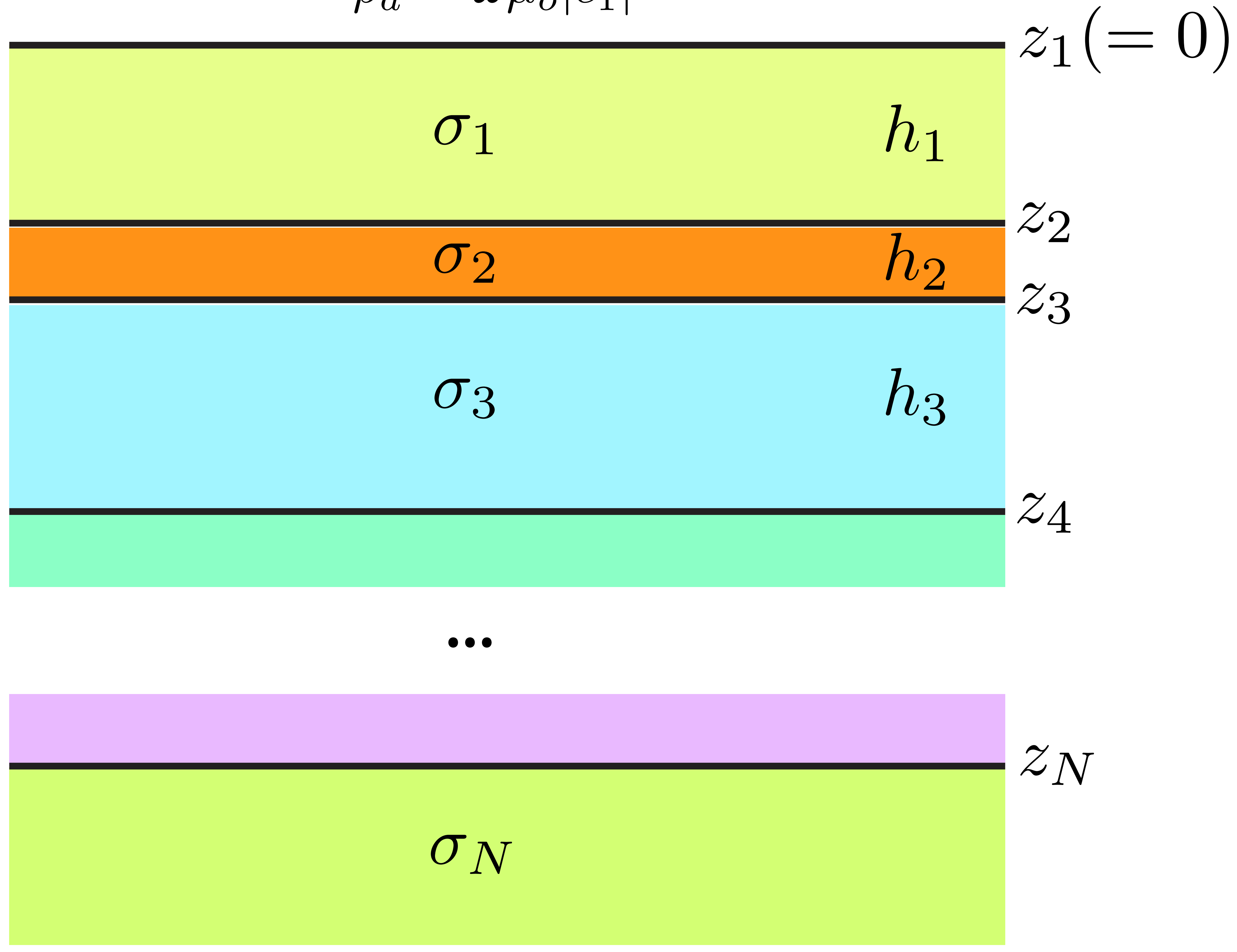
We have built a *recursion relationship*. We can start with the bottom half-space, for which c is analytical, and move up through the layers one by one to the surface, which is where we want our measurement..

$$c(z=0) = c_1 = \frac{c_2 \coth(k_1 h_1) + 1/k_1}{k_1 c_2 + \coth(k_1 h_1)}$$

$$\rho_a = \omega \mu_o |c_1|^2$$

$$c_n = \frac{c_{n+1} \coth(k_n h_n) + 1/k_n}{k_n c_{n+1} + \coth(k_n h_n)}$$

$$c_N = \frac{1}{i\omega}\frac{E}{B} = \frac{1}{i}\sqrt{\frac{1}{\omega \mu_o \sigma_N}}$$

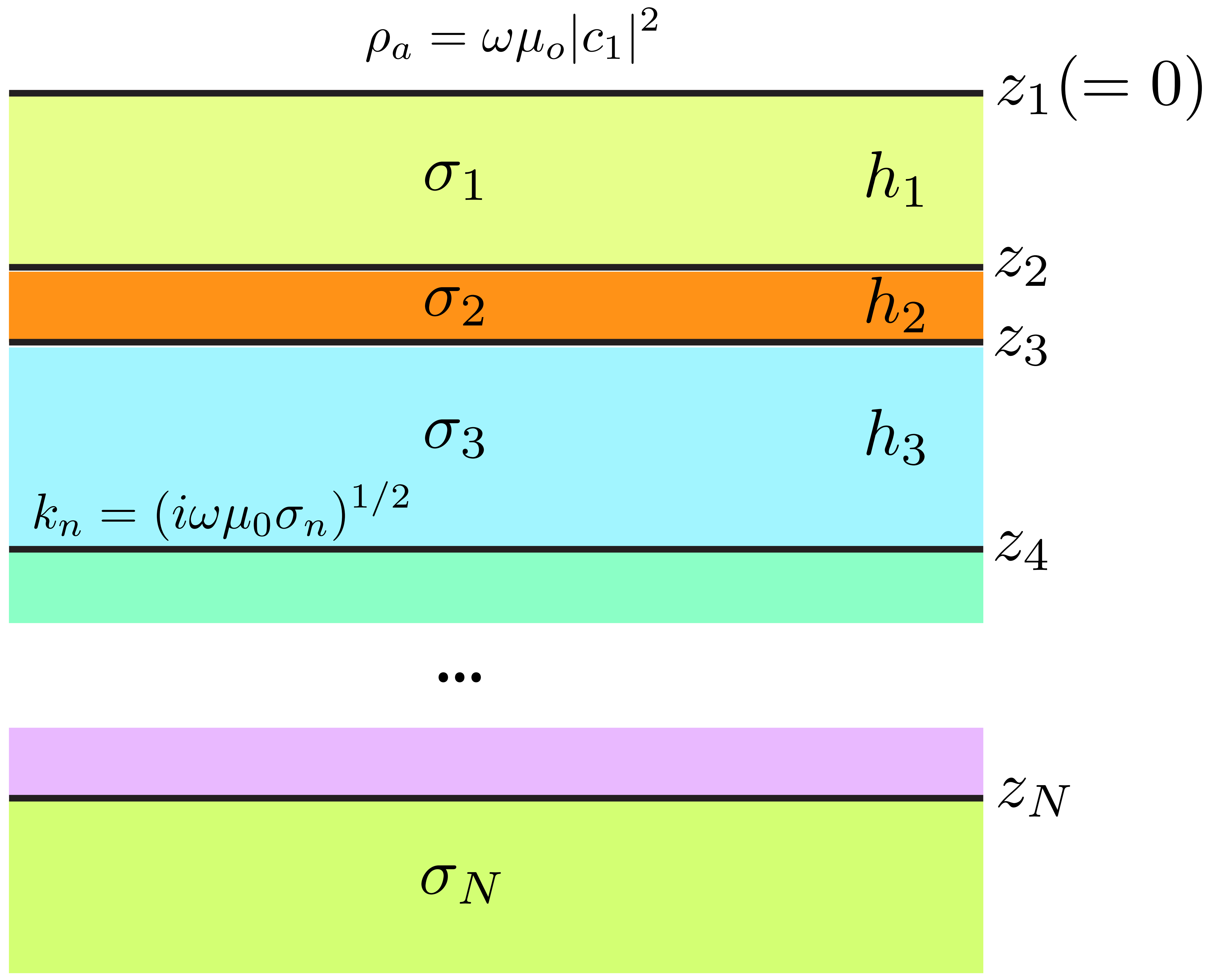


The impedance at any depth only depends on the layers beneath, not above!

$$c(z=0) = c_1 = \frac{c_2 \coth(k_1 h_1) + 1/k_1}{k_1 c_2 + \coth(k_1 h_1)}$$

$$c_n = \frac{c_{n+1} \coth(k_n h_n) + 1/k_n}{k_n c_{n+1} + \coth(k_n h_n)}$$

$$c_N = \frac{1}{i\omega}\frac{E}{B} = \frac{1}{i}\sqrt{\frac{1}{\omega \mu_o \sigma_N}}$$



Similar recursion relations:

Schmucker (1970):

$$G_i = \frac{k_{i+1}G_{i+1} + k_i \tanh(k_i h_i)}{k_i + k_{i+1}G_{i+1}\tanh(k_i h_i)} \qquad G_N = 1 \qquad \rho_a = \frac{\omega\mu_o}{|k_1 G_1|^2} \qquad \phi = \arctan\left(\frac{\text{Imag}(k_1 G_1)}{\text{Real}(k_1 G_1)}\right)$$

$$\frac{E_{j+1}}{E_j} = \cosh(k_j h_j) - G_j \sinh(k_j h_j) \qquad \frac{B_{j+1}}{B_j} = \frac{k_{j+1}G_{j+1}}{k_j G_j}\big(\cosh(k_j h_j) - G_j \sinh(k_j h_j)\big)$$

Ward and Hohmann (1988):

$$Z_j = L_j \frac{Z_{j+1} + L_j \tanh\left(ik_j h_j\right)}{L_j + Z_{j+1}\tanh\left(ik_j h_j\right)} \qquad L_j = \frac{\omega\mu_o}{k_j} \qquad k_j = \sqrt{-i\omega\mu_o\sigma_j}$$

Weidelt's transformation:

Analytical solutions exist for a layered, spherically symmetric Earth, but Weidelt (1972) derived a way to go from a flat-earth layered solution $\bar{\sigma}(z)$ into a spherical Earth solution. For spherical harmonic degree 1

$$\sigma(r) = f^{-4}(r/R)\bar{\sigma}\left(R\frac{(R/r) - (r/R)^2}{3f(r/R)}\right)$$

$$f(r/R) = \frac{2R/r + (r/R)^2}{3}$$