# SIOG 231
# GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 13

The Magnetotelluric Method (and processing)

2/20/2024

**Recap of basic theory:**

Magnetotelluric impedance given by field ratios,

$$Z = \frac{E_x(\omega)}{H_y(\omega)} = \frac{\mu\omega}{k}$$

and is proportional to the complex wavenumber, which itself depends on conductivity

$$k = (i\sigma\mu\omega)^{\frac{1}{2}} = \sqrt{\frac{\sigma\mu\omega}{2}} + i\sqrt{\frac{\sigma\mu\omega}{2}}$$

so we can relate conductivity (or resistivity) to impedance

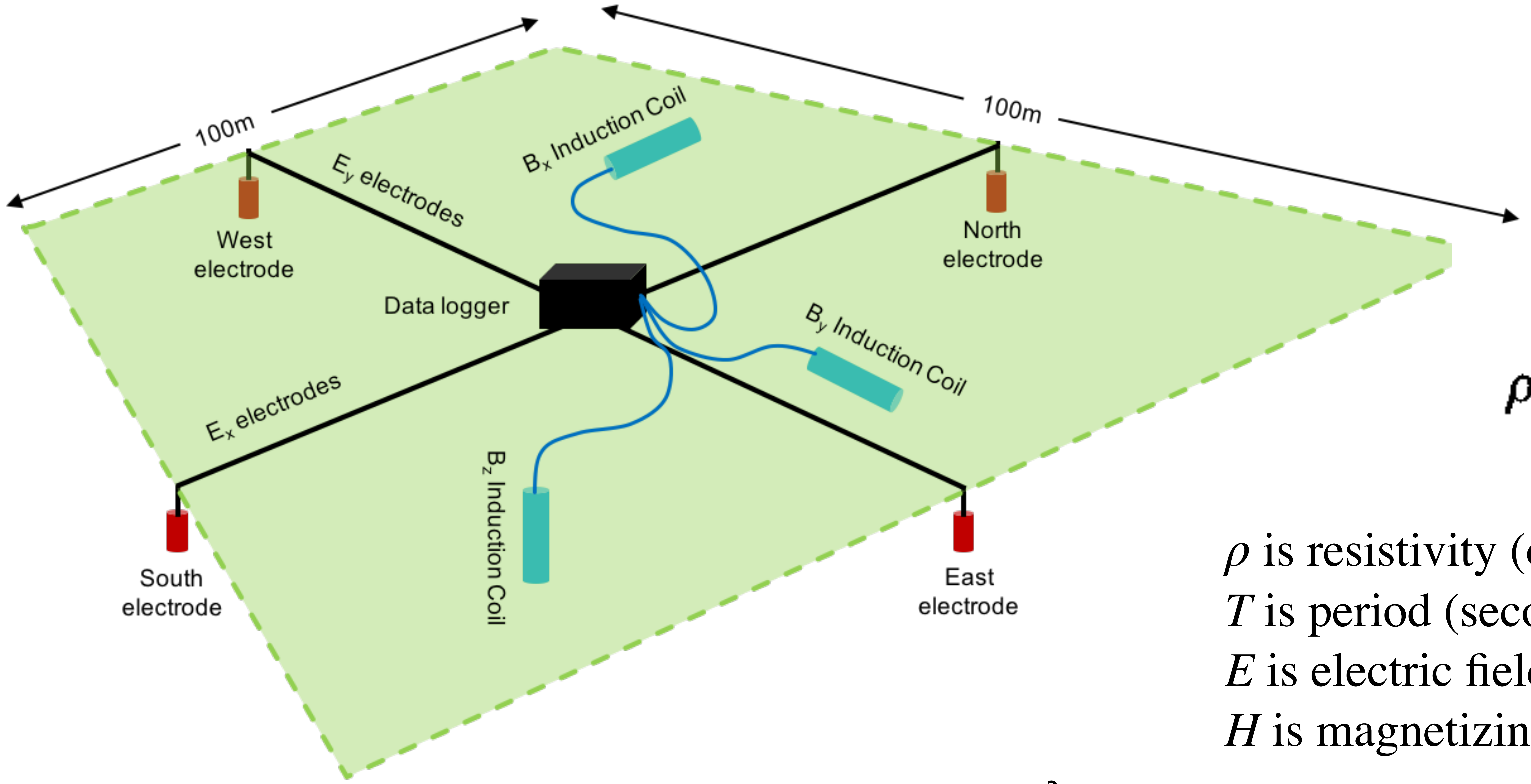
$$\rho = \frac{1}{\sigma} = \frac{1}{2\pi f\mu}\left|Z\right|^2 = \frac{T}{2\pi\mu}\left|\frac{E_x}{H_y}\right|^2$$

and comput a phase between *E* and *H*

$$\Phi = \arg(Z)$$

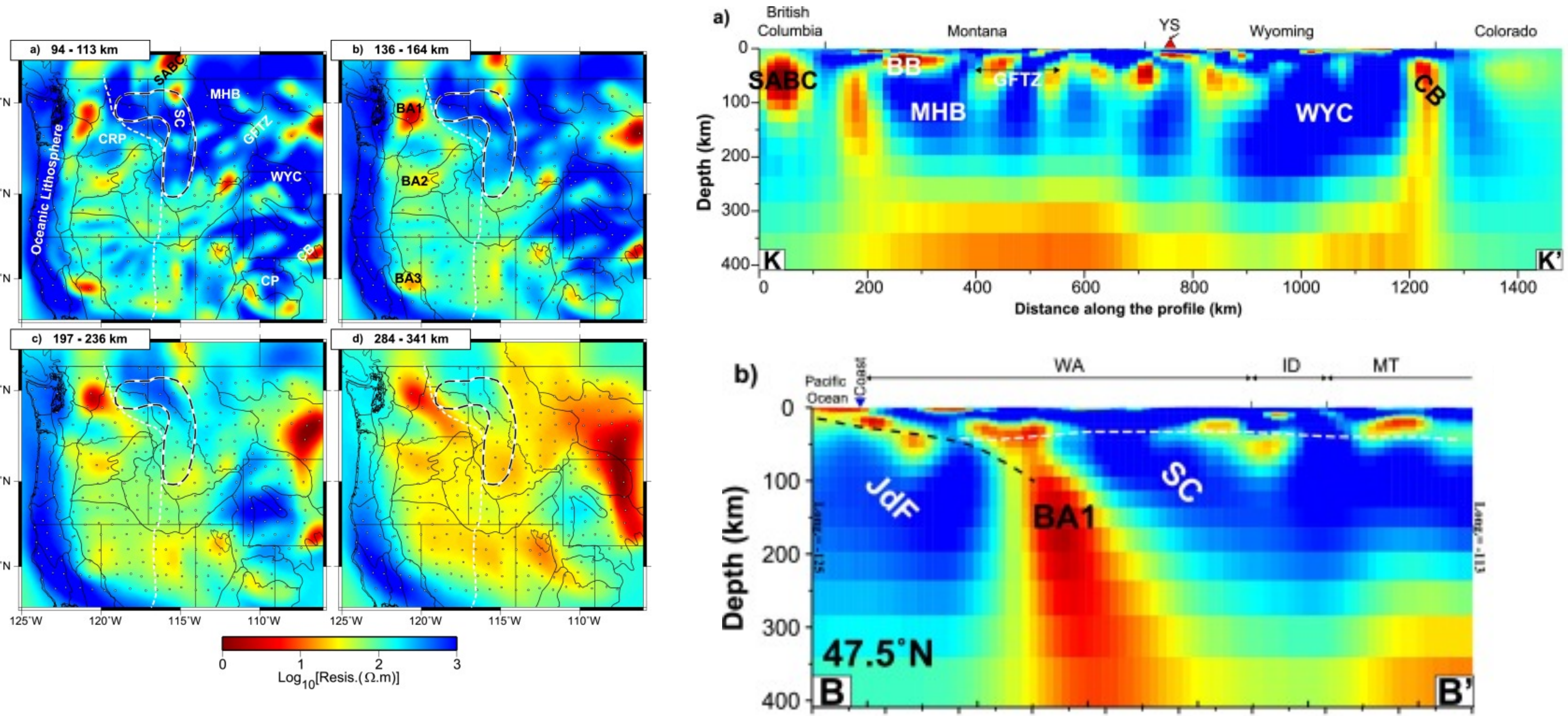
**1950**: A.N. Tikhonov, T. Rikitake, Y. Kato, and T. Kikuchi developed mathematical descriptions for the relationship between induced electric and magnetic fields.

**1953**: Louis Cagniard described a practical method to use measurements of magnetic and electric fields to estimate Earth conductivity, and called it the magnetotelluric method.

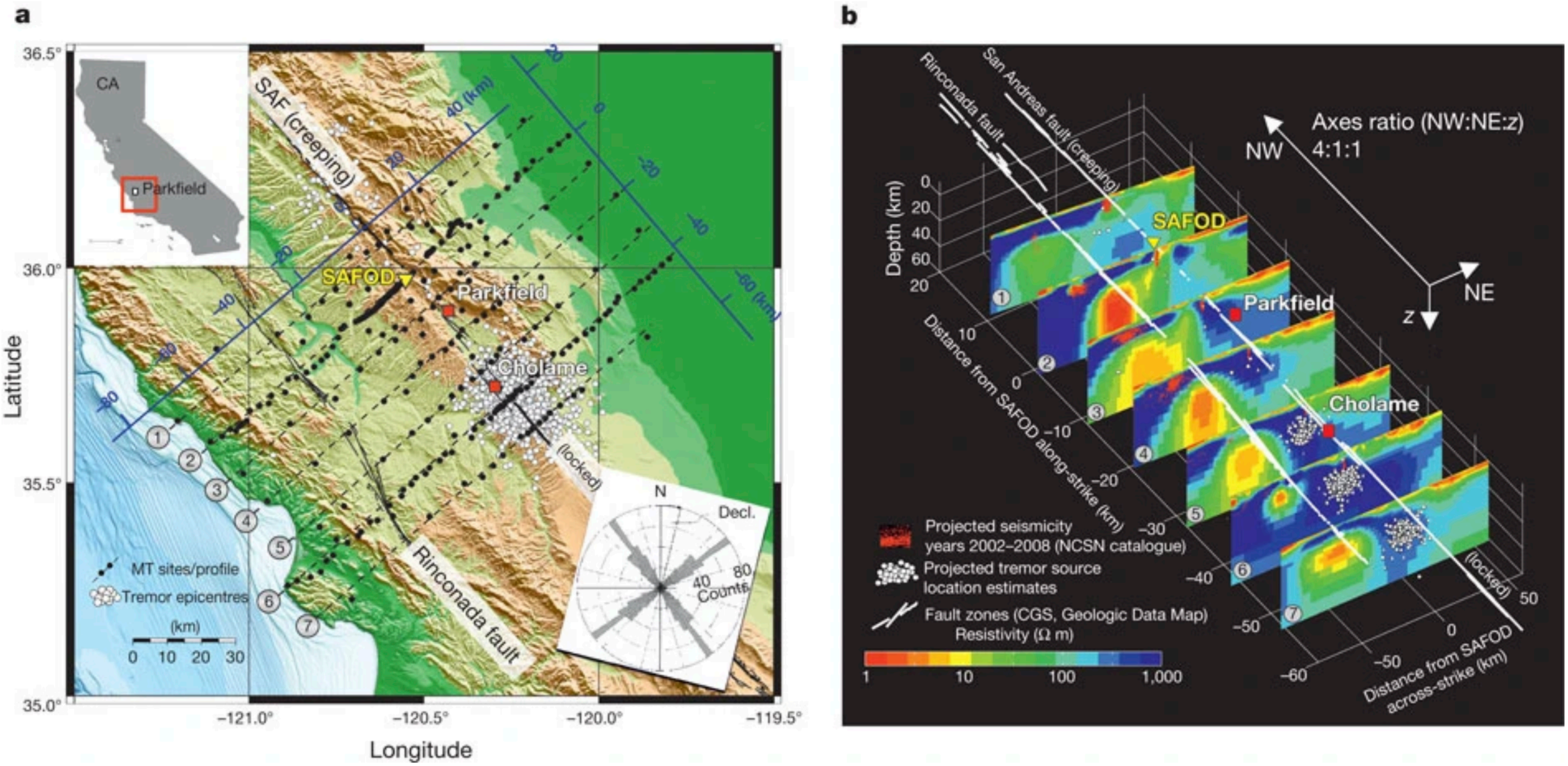



$$\rho = 0.2T\left(\frac{E}{H}\right)^2.$$

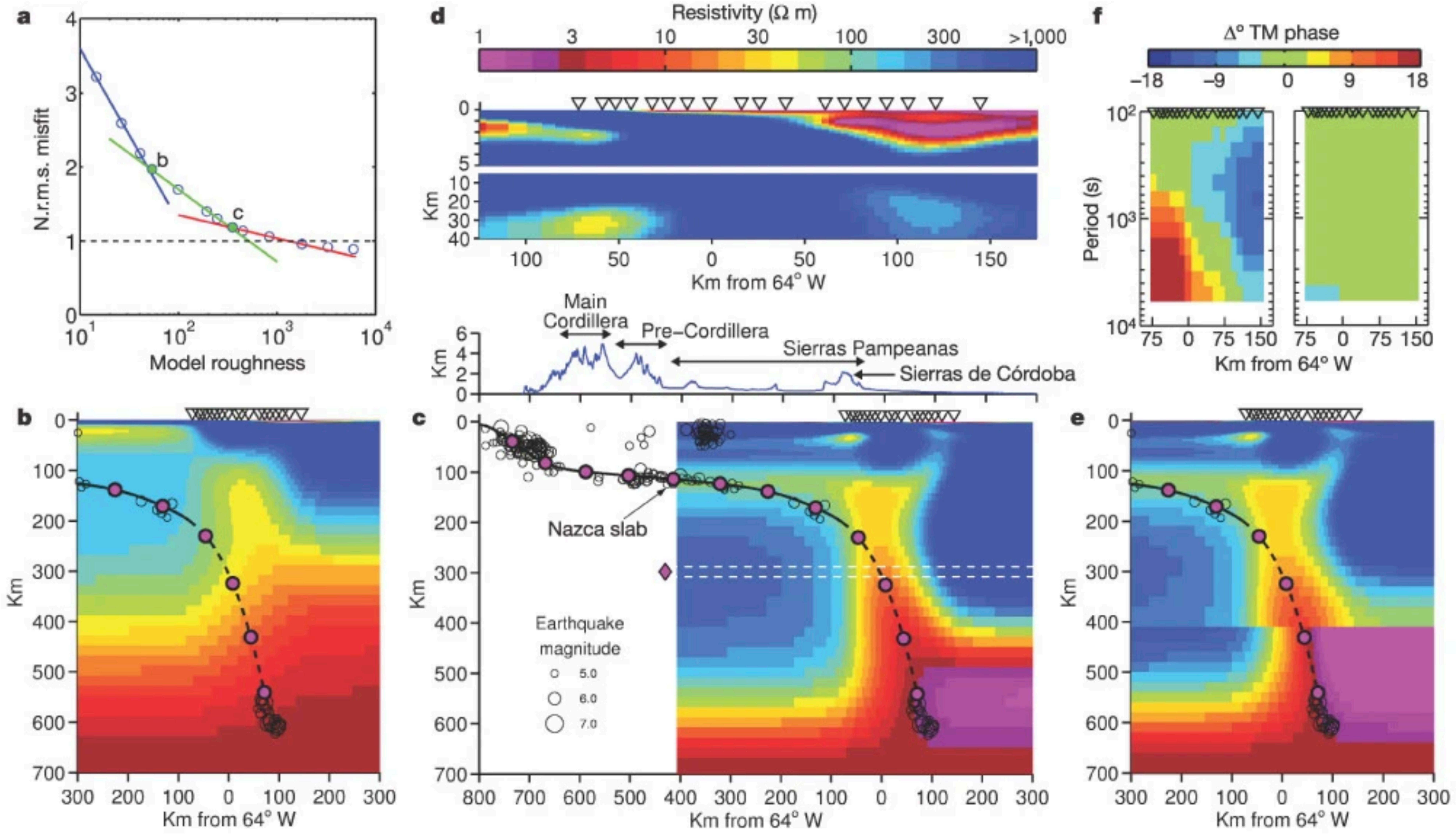
$\rho$ is resistivity (one over conductivity) (Ωm)
$T$ is period (seconds)
$E$ is electric field as period $T$ (V/m)
$H$ is magnetizing field at period $T$ (A/m)




Northwestern USA, from US Array: *Meqbel et al., EPSL, 2014*

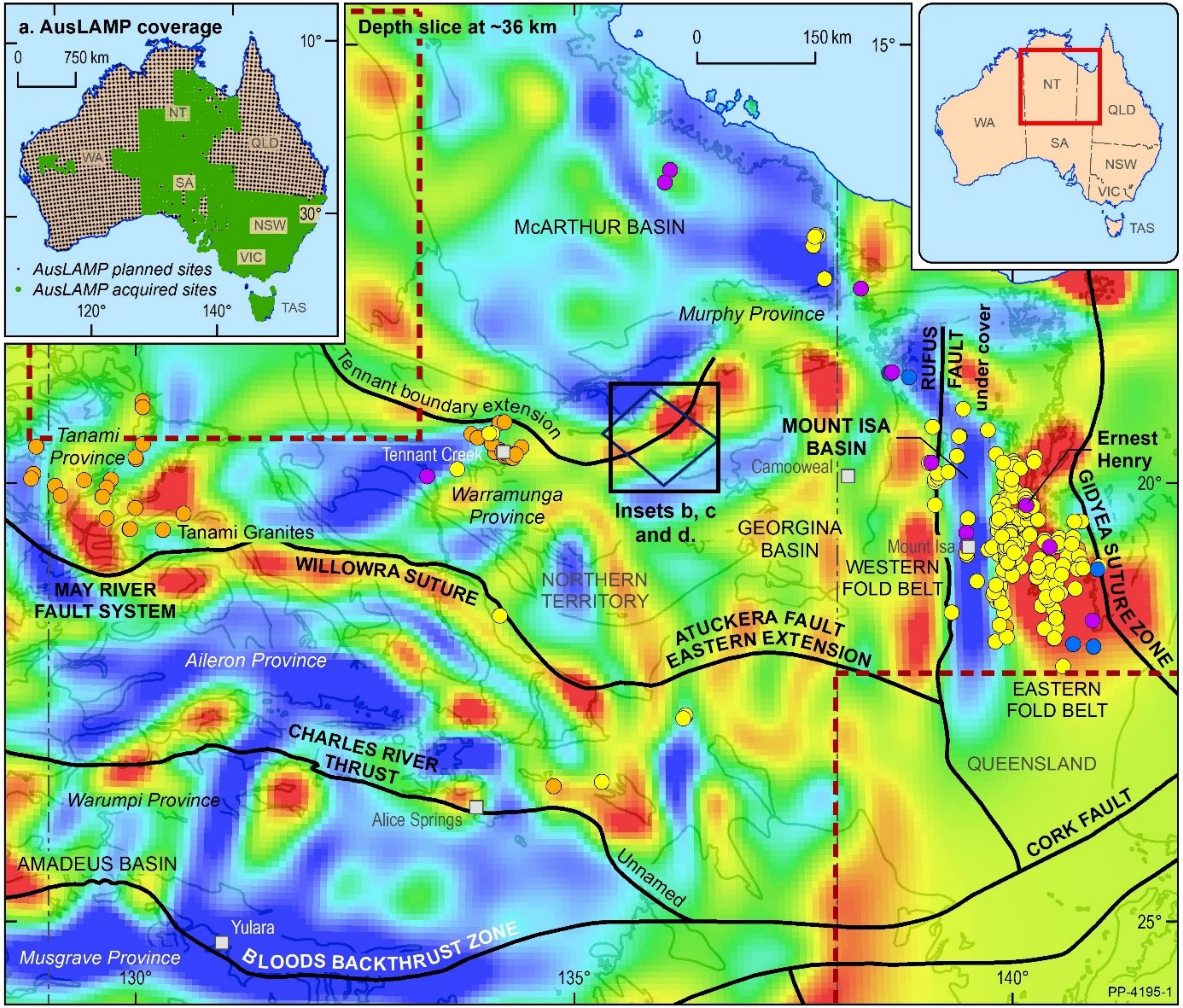



Central San Andreas Fault: *Becken et al., Nature, 2011*

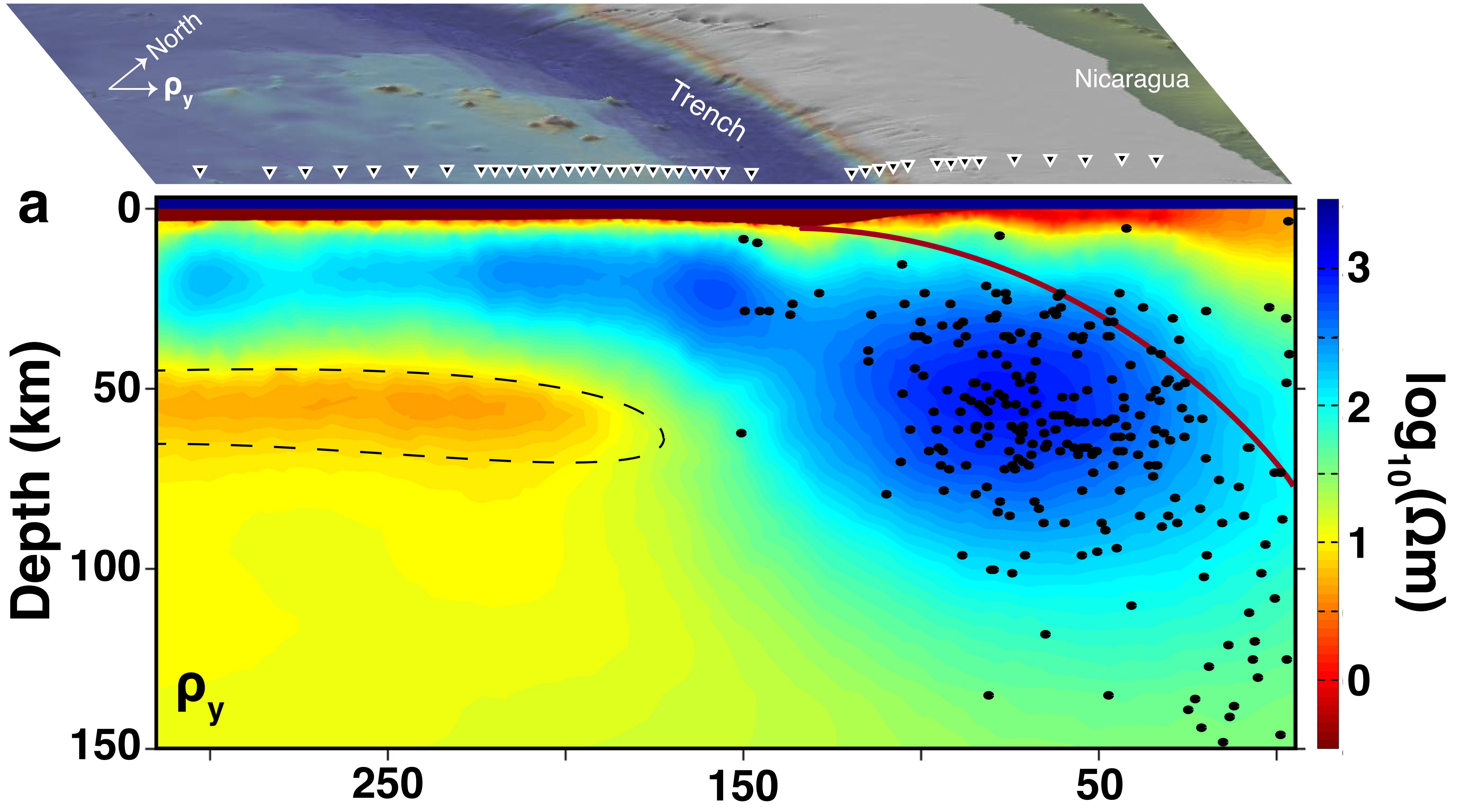


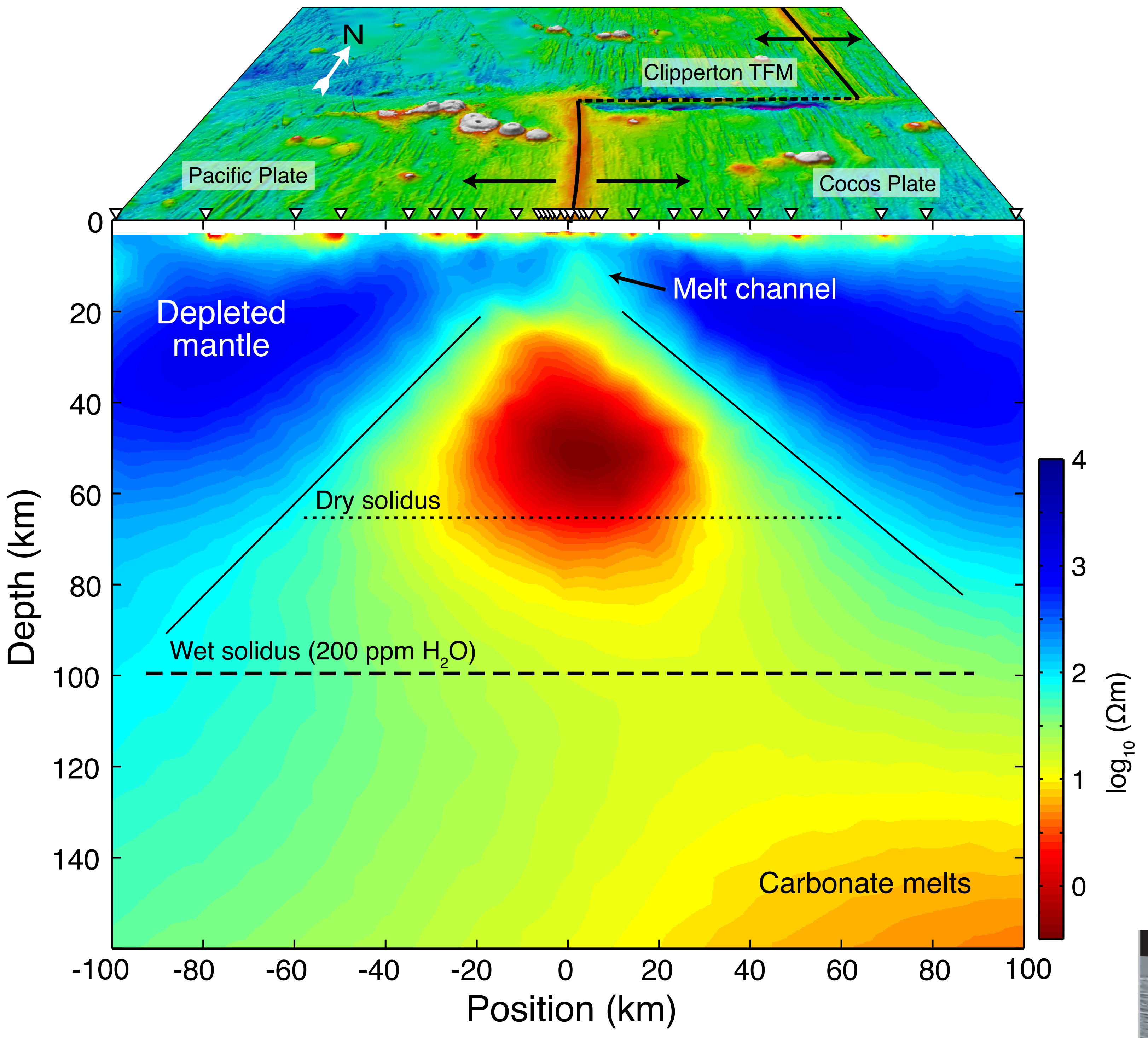

Subduction beneath Argentina: *Booker, Favetto, & Pomposiello, Nature, 2004*

AUSLAMP: *Duan et al., 2021 Geoscience Australia report*

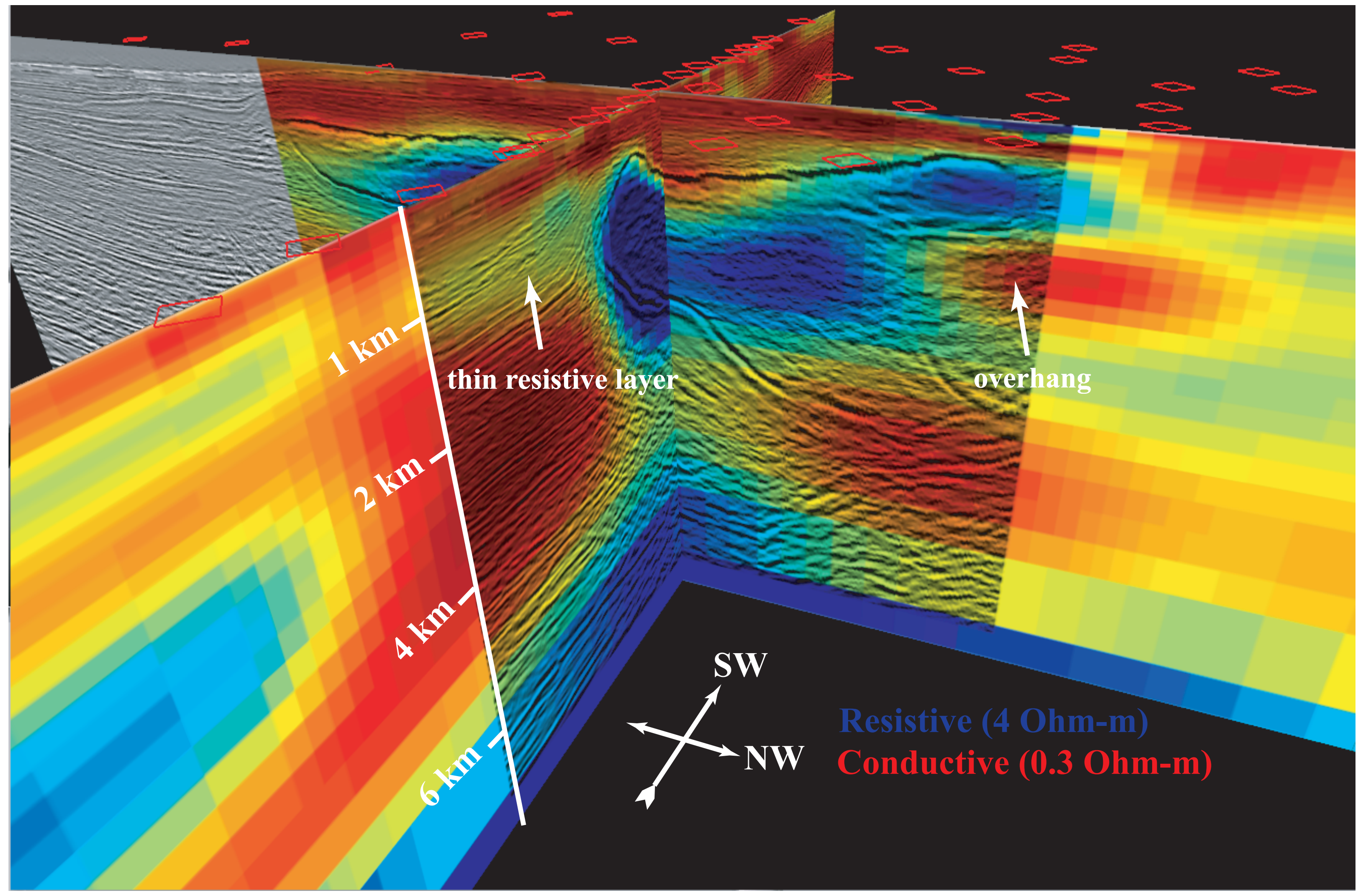


## Subduction off Nicaragua

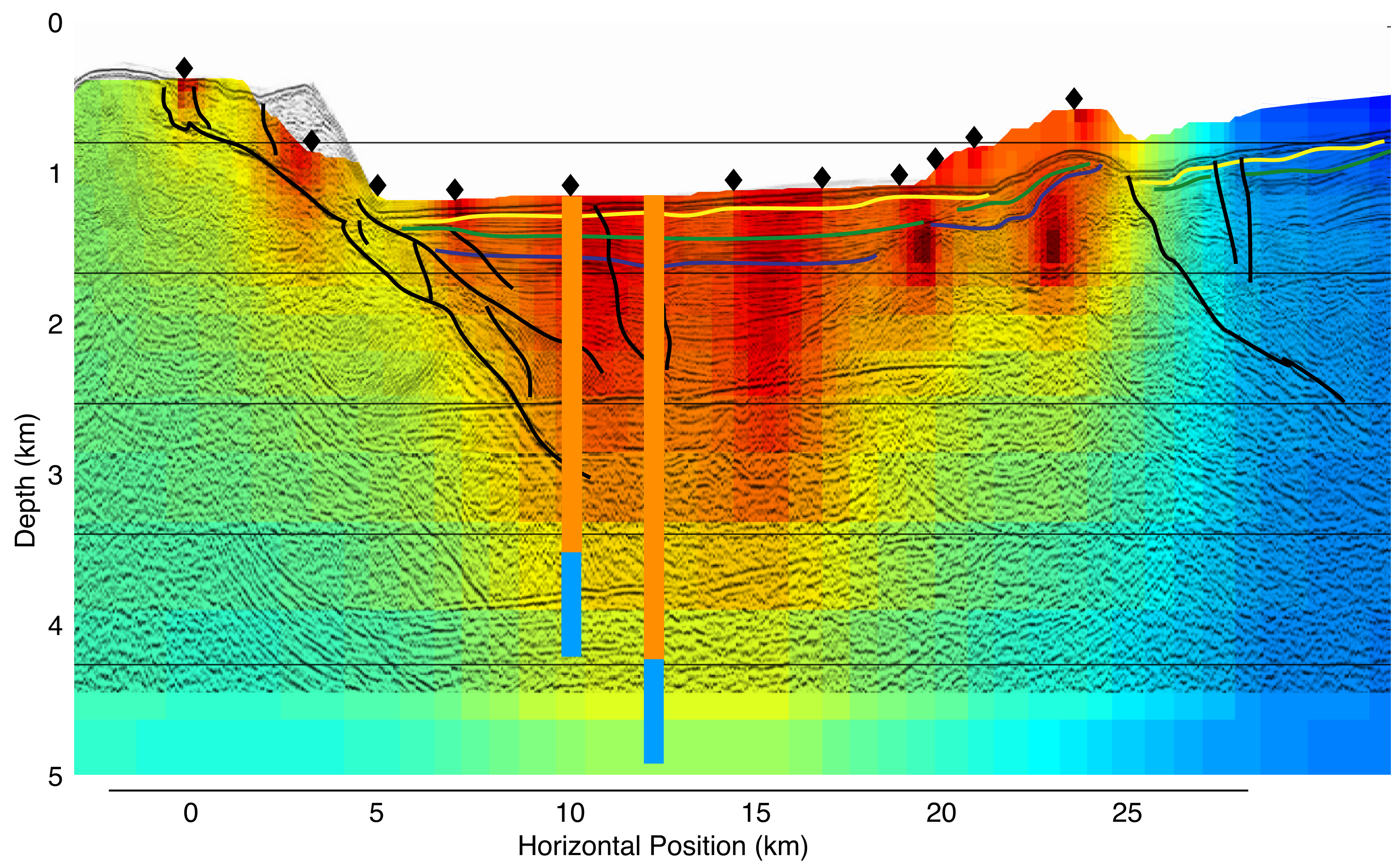



## East Pacific Rise




## Salt in Gulf of Mexico




## Basin structure off California

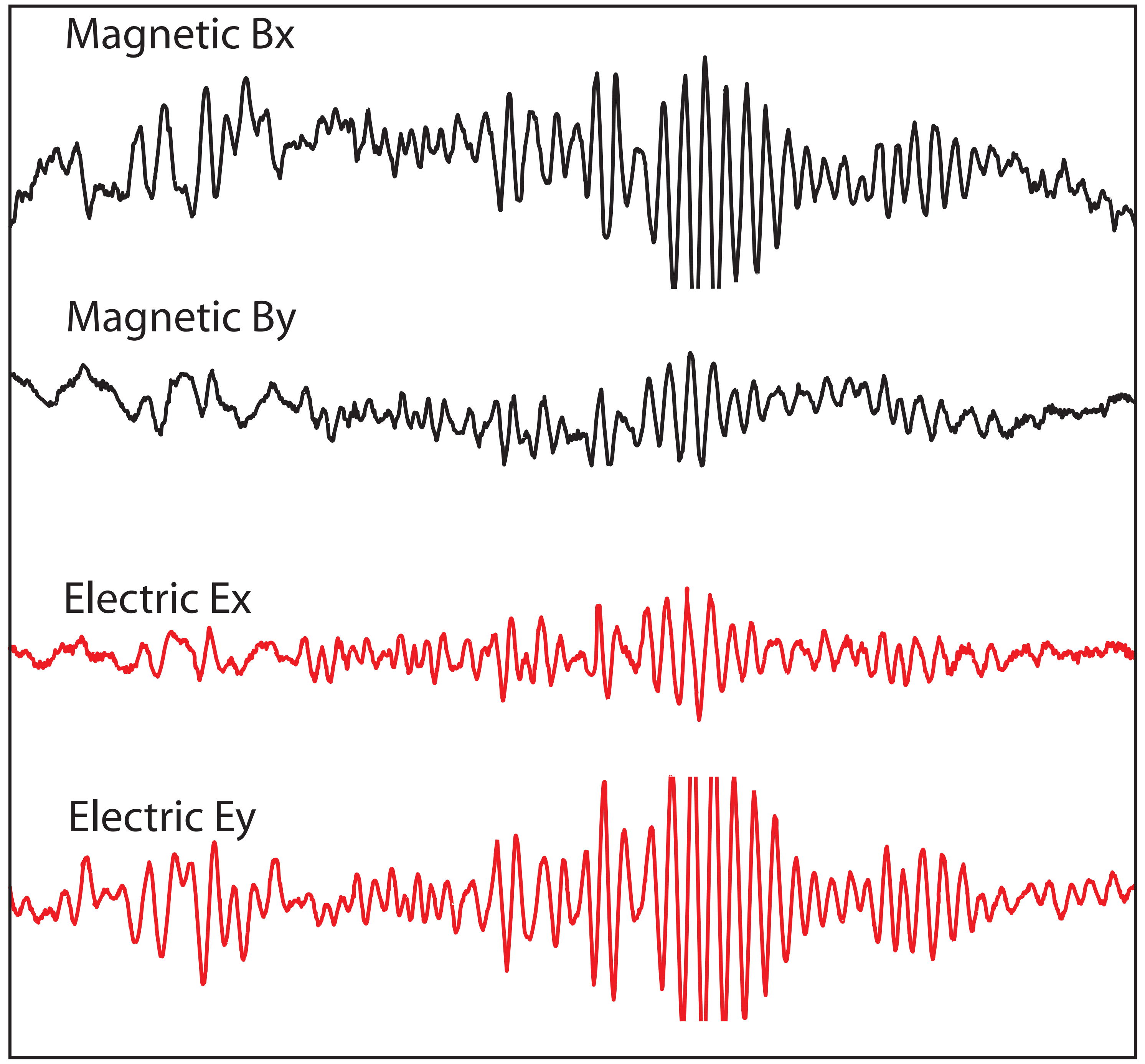
# Work Flow

**Processing** →

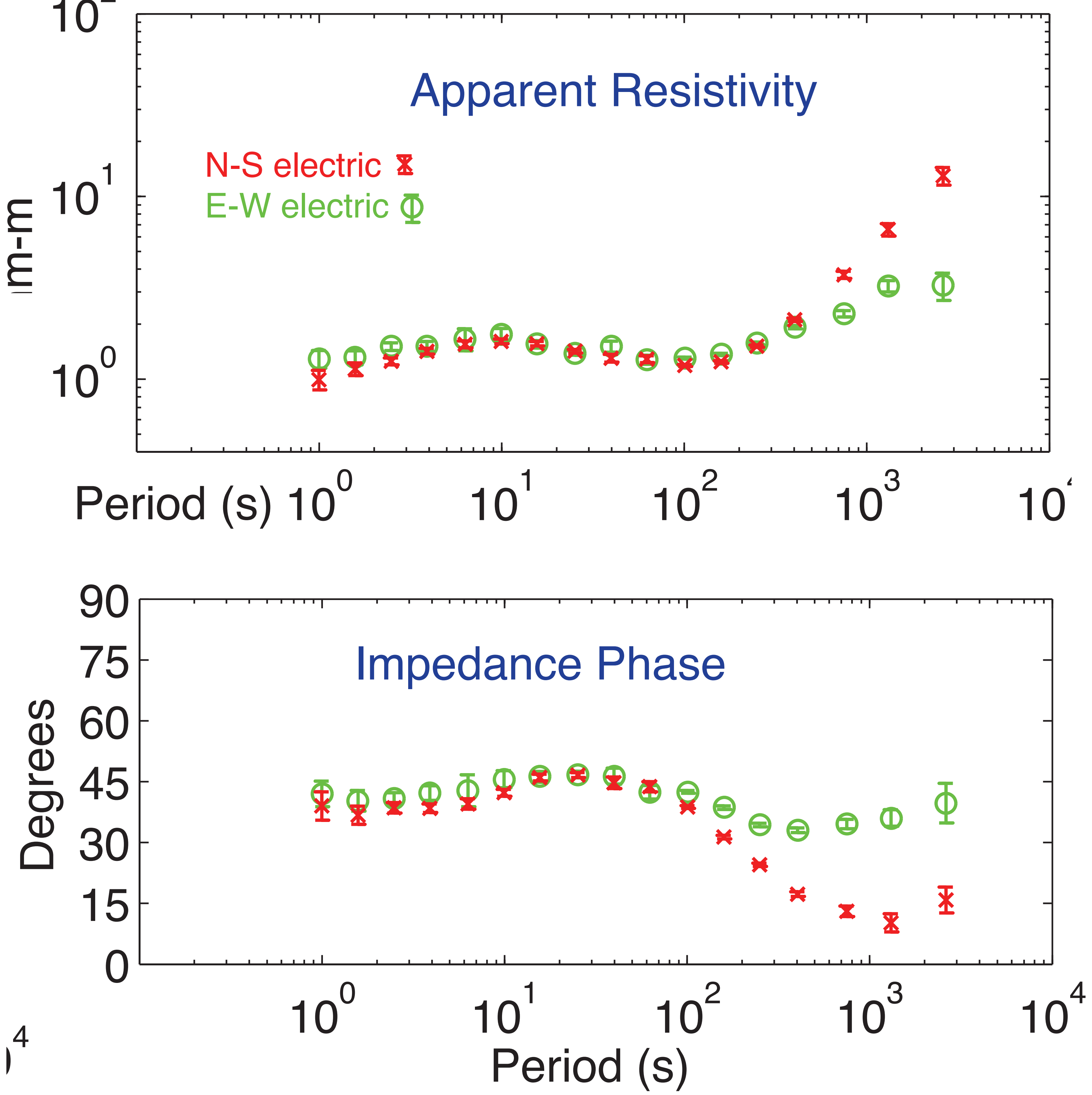
Inversion

**Inversion** →

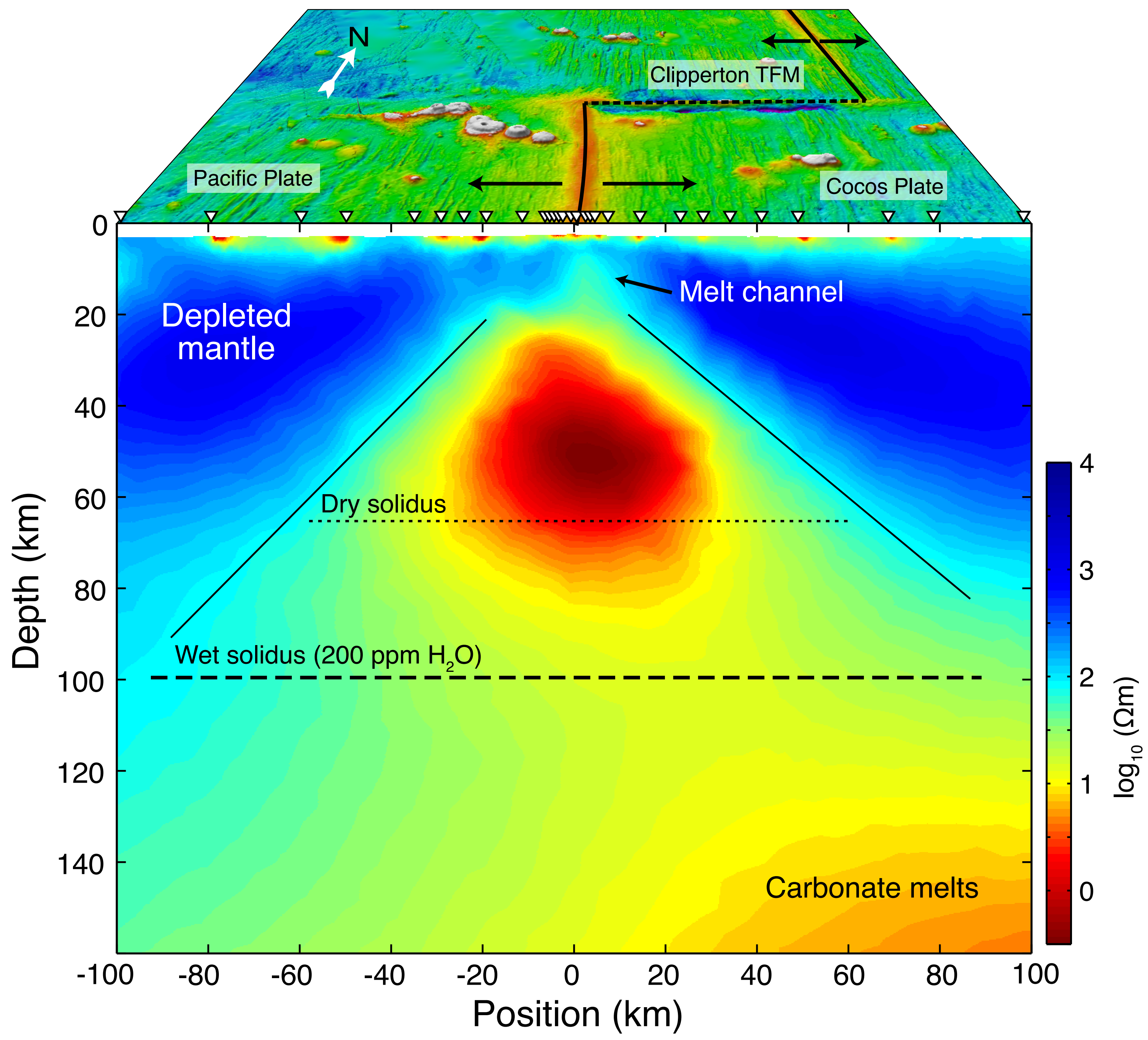
**MT timeseries**




**MT response functions**




**Resistivity model**




**Lab data/models**

↓

**Melt fraction**

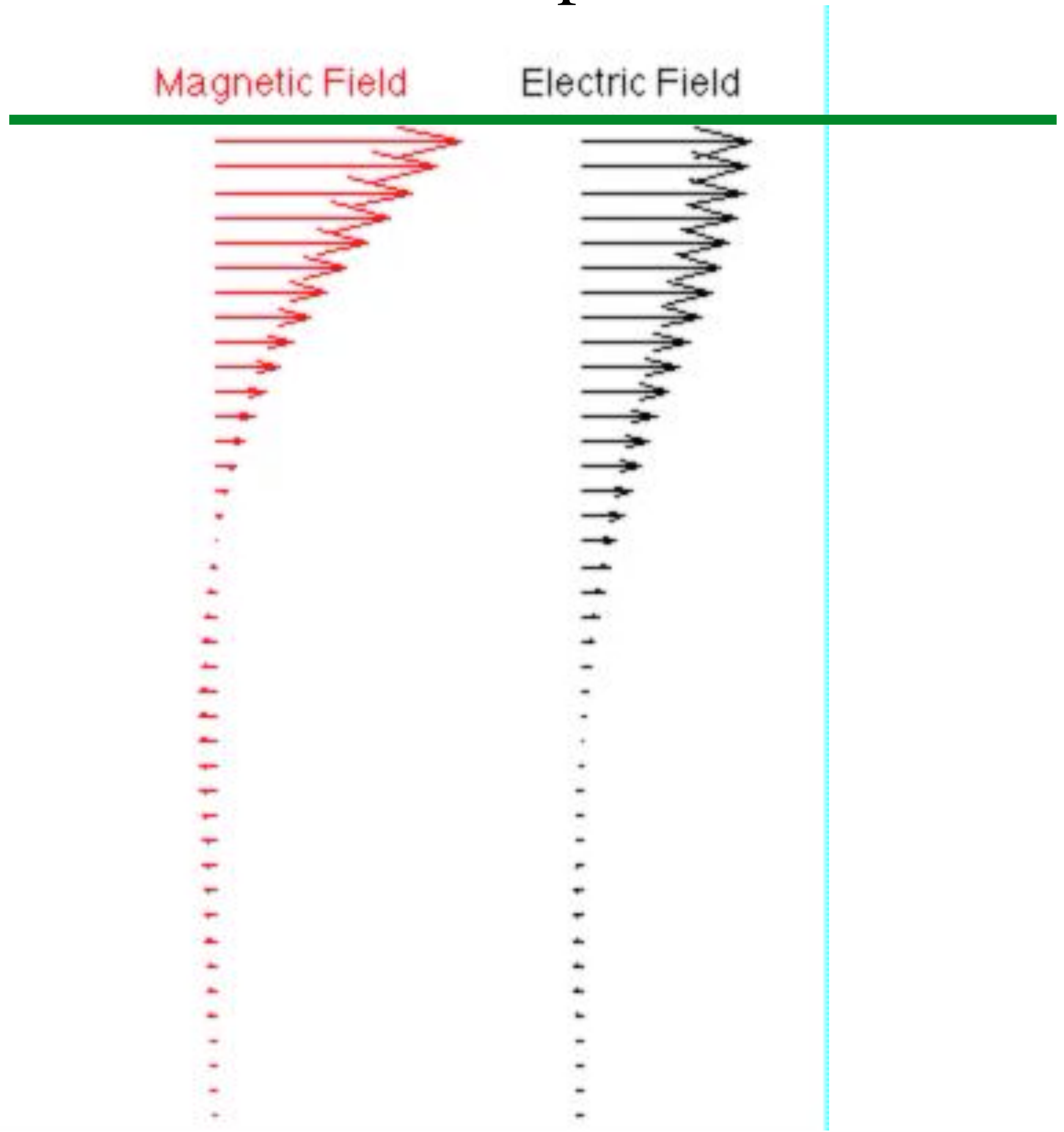
**Water salinity/porosity**

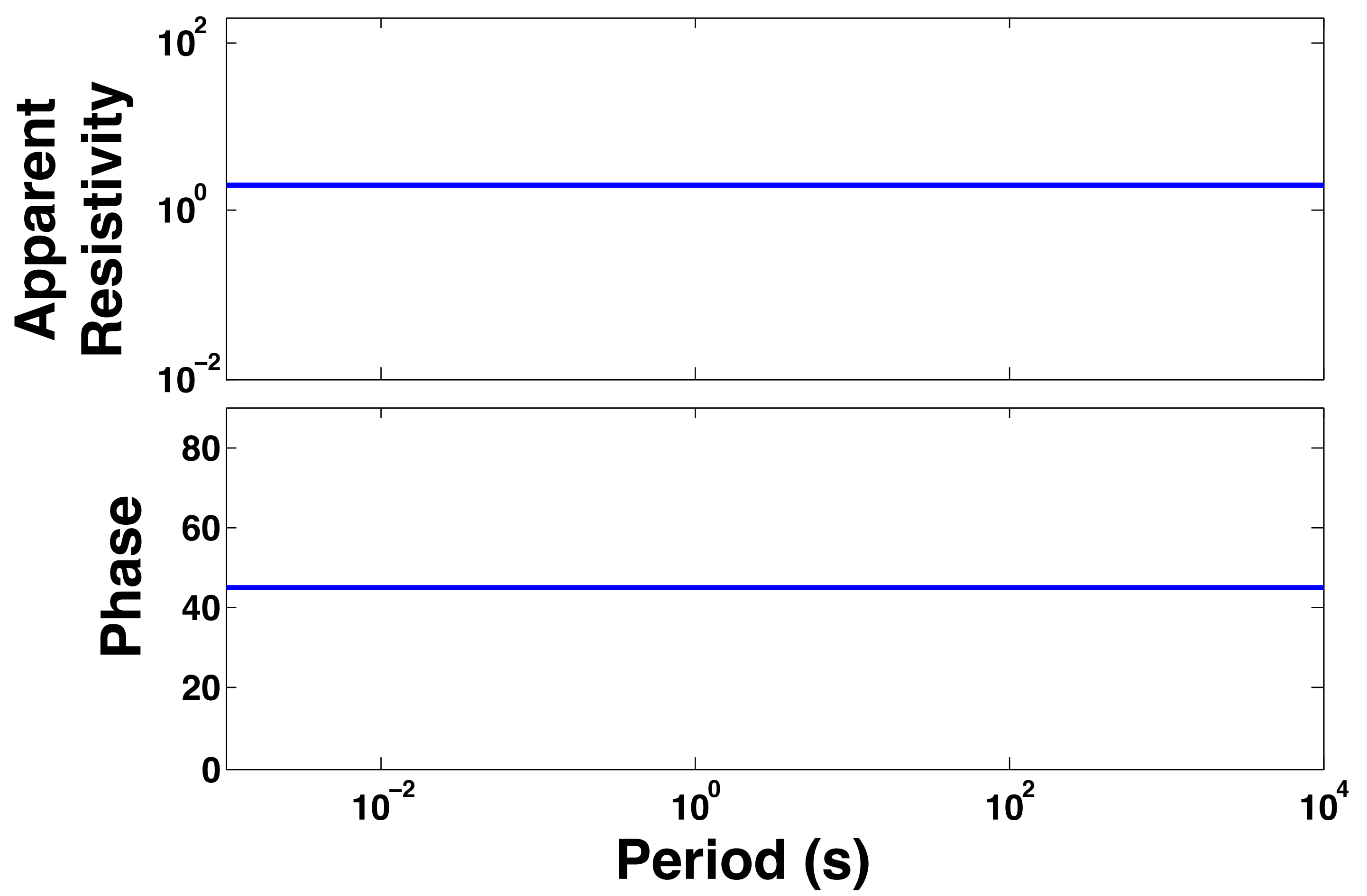
**Temperature**

**Gas/hydrate fraction**


digging in a magnetometer
programming logger
electrode wires

Half-space




Here is what the MT fields look like in a uniform conductor. The fields decay exponentially.

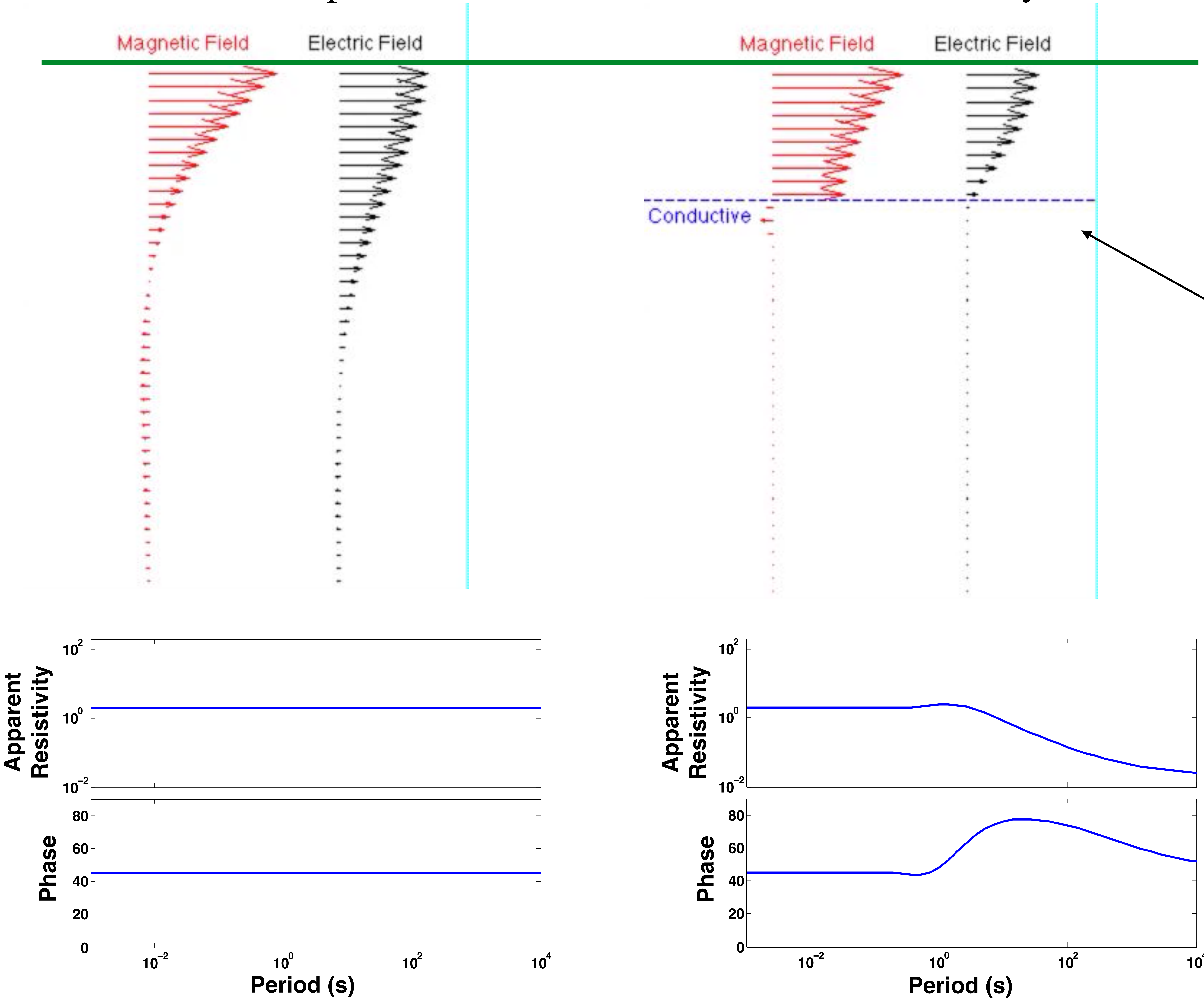
The induced electric field is 45° out of phase with the primary magnetic field.

We can compute a half-space equivalent electrical resistivity (apparent resistivity) at each frequency:

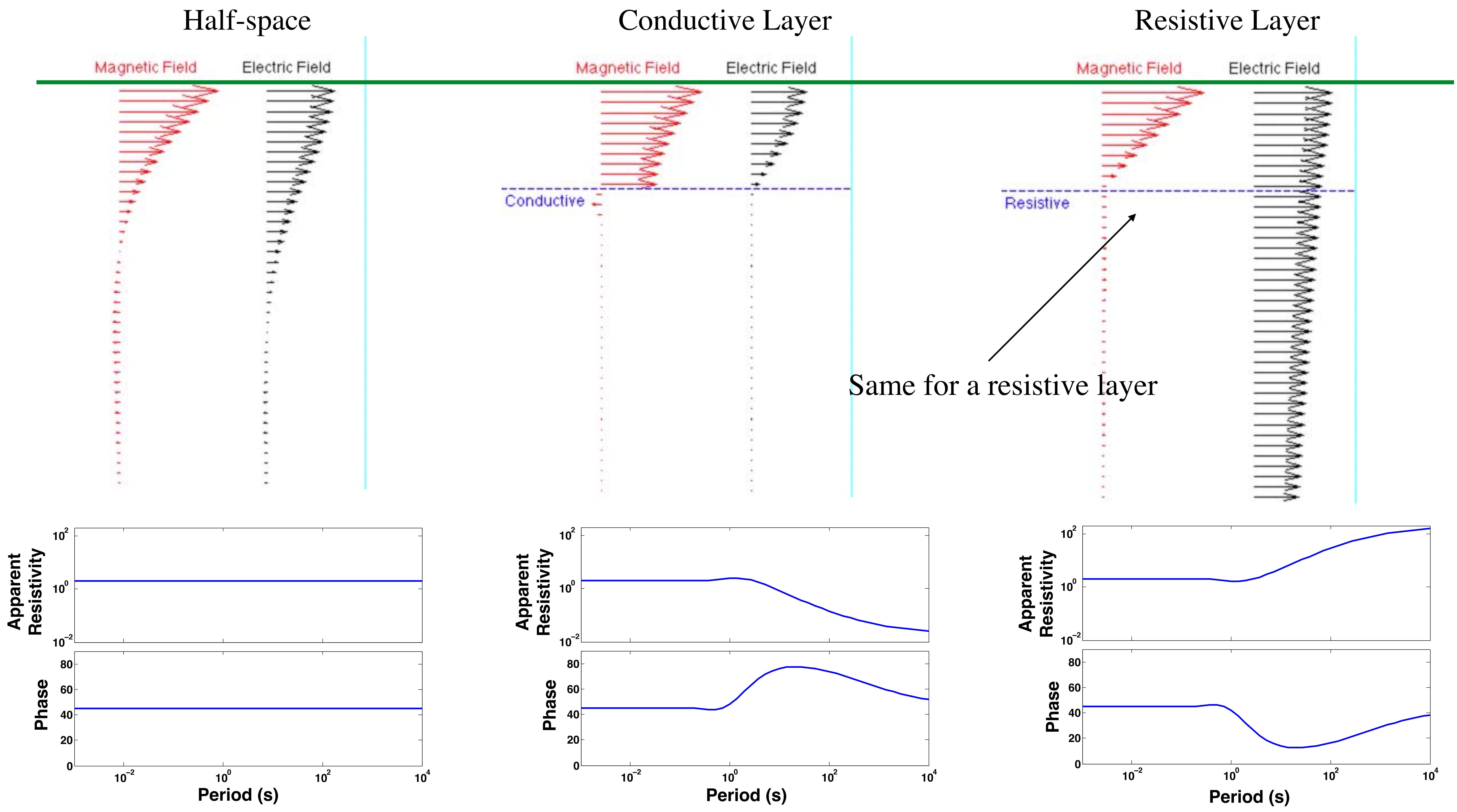
$$\rho_{\mathrm{a}}(\omega) = \frac{\mu_o}{\omega} \left| \frac{E(\omega)}{B(\omega)} \right|^2$$

We can also compute the phase difference between *E* and *B*. These become the MT sounding curves.

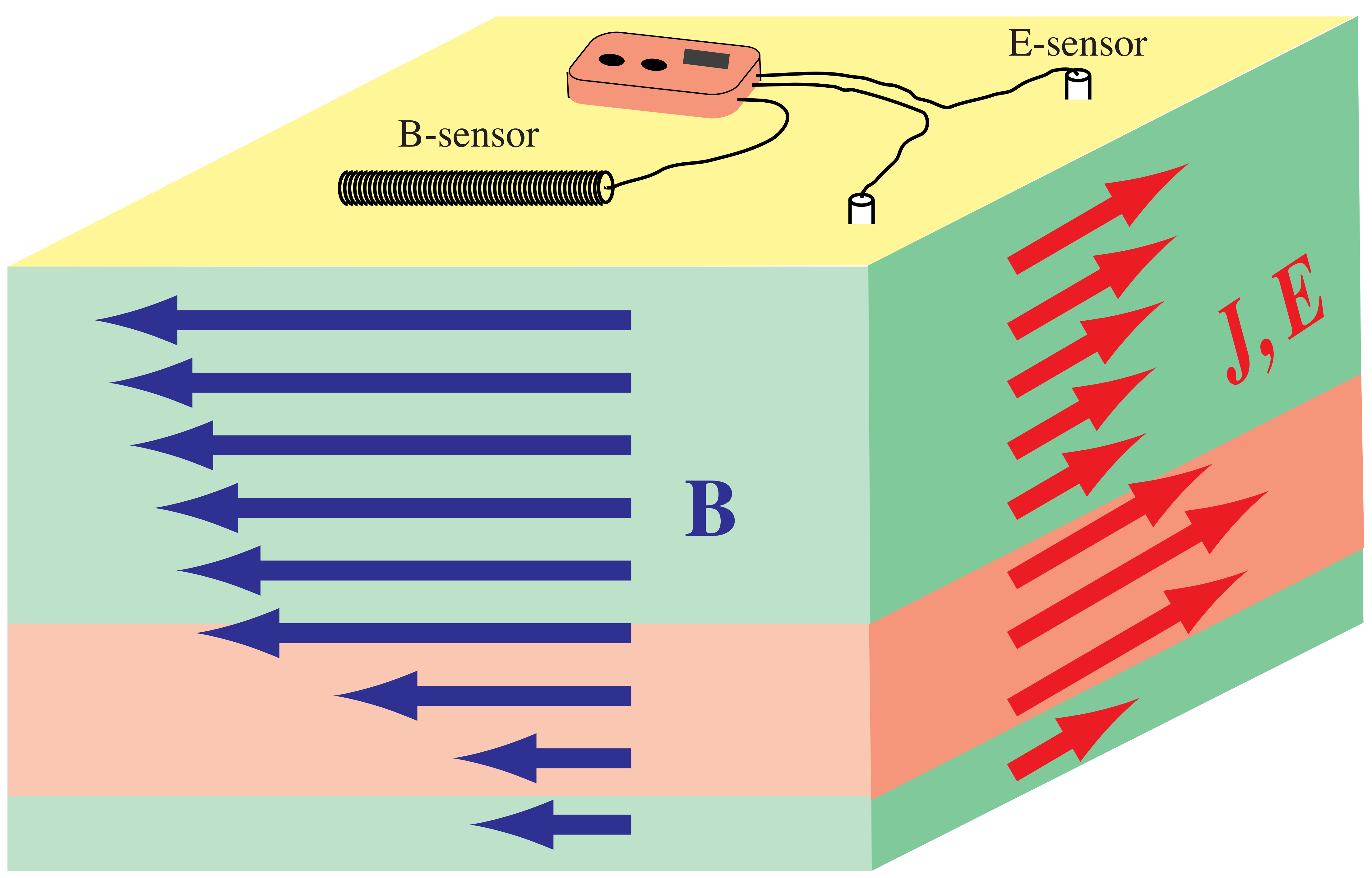
Half-space Conductive Layer

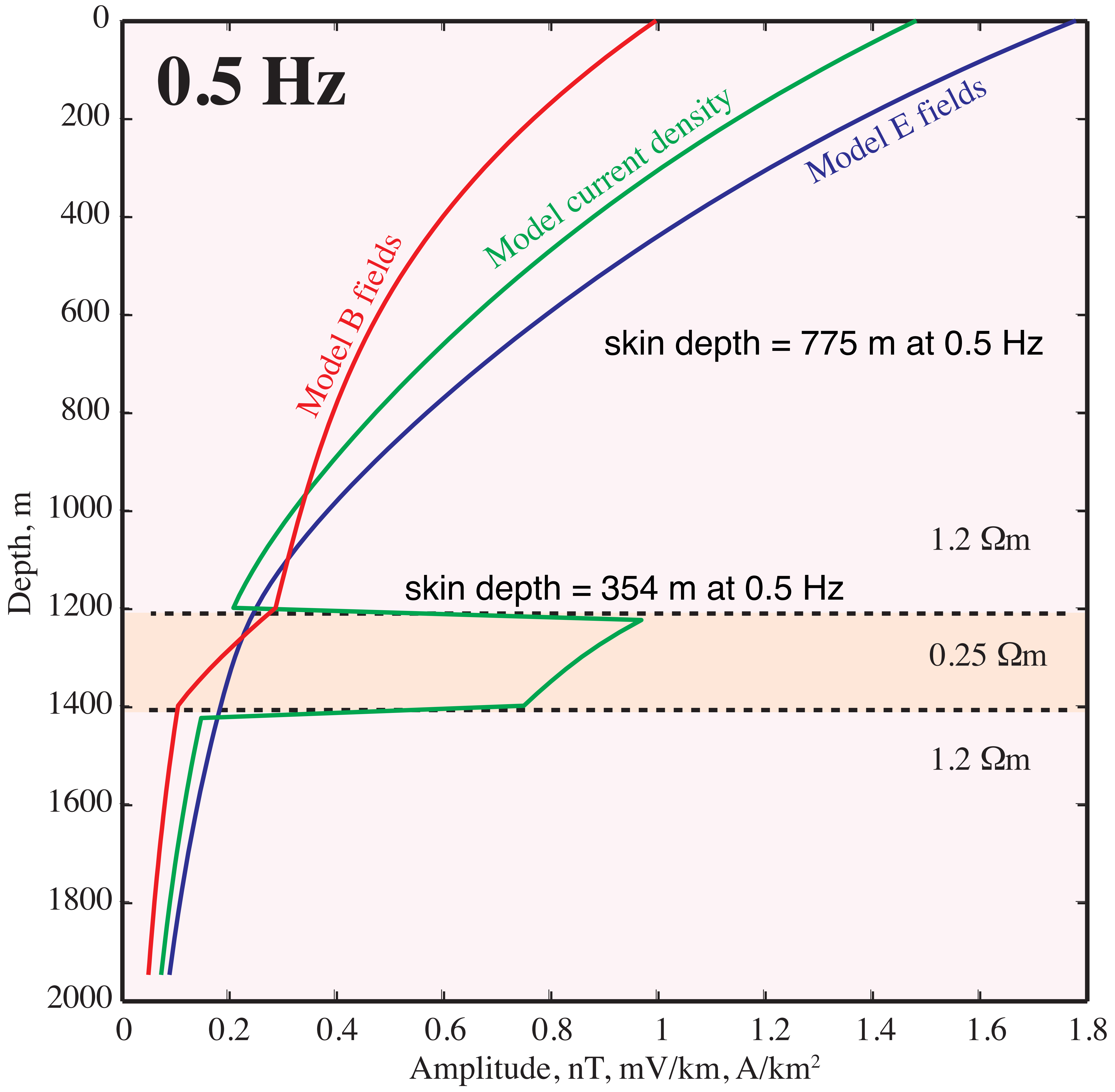



We can add a conductive layer at depth and things change at the surface

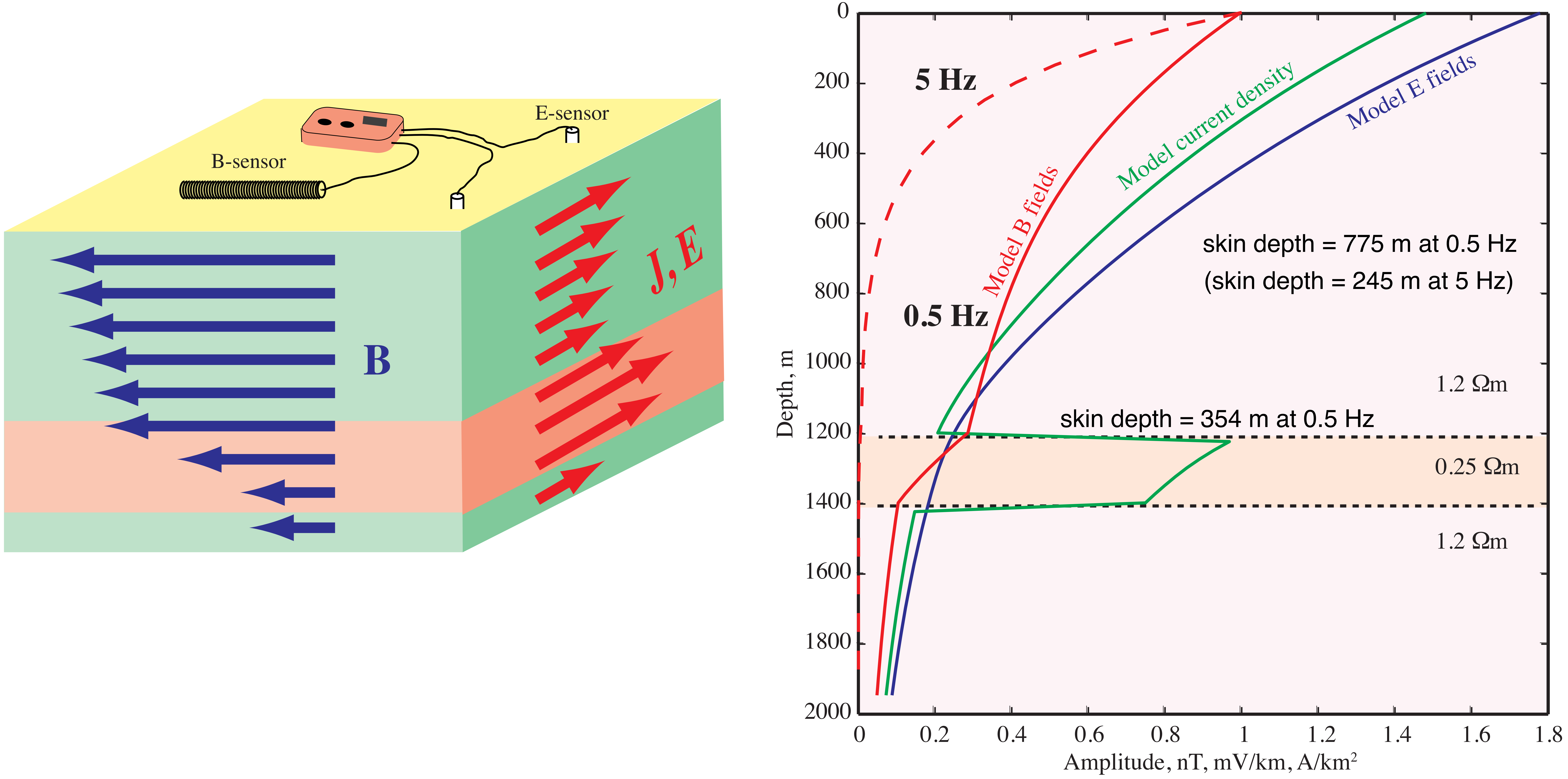

Half-space
Conductive Layer
Resistive Layer
Magnetic Field
Electric Field
Conductive
Resistive
Same for a resistive layer
Apparent Resistivity
Phase
Period (s)

# Magnetotelluric fields:

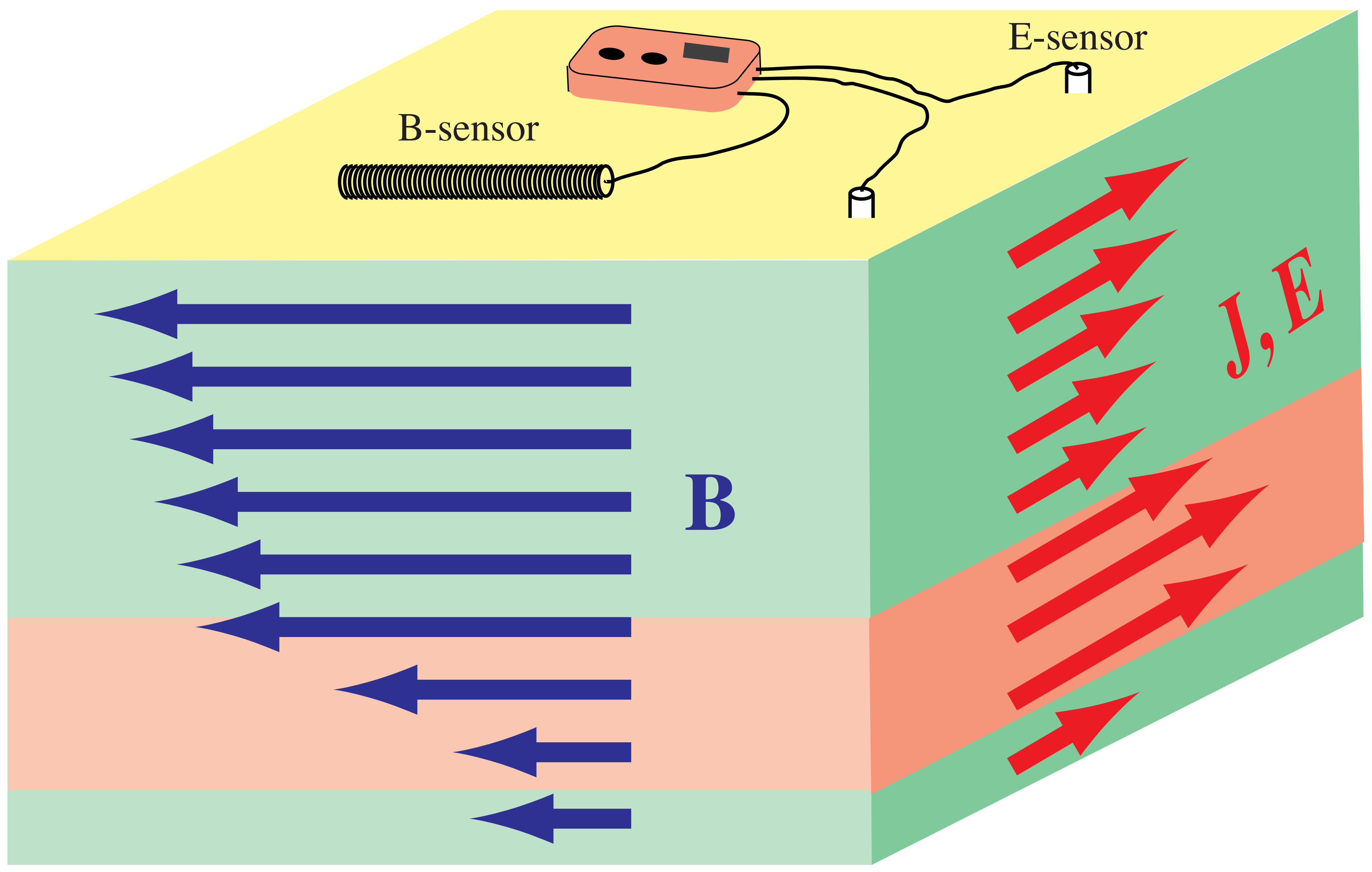


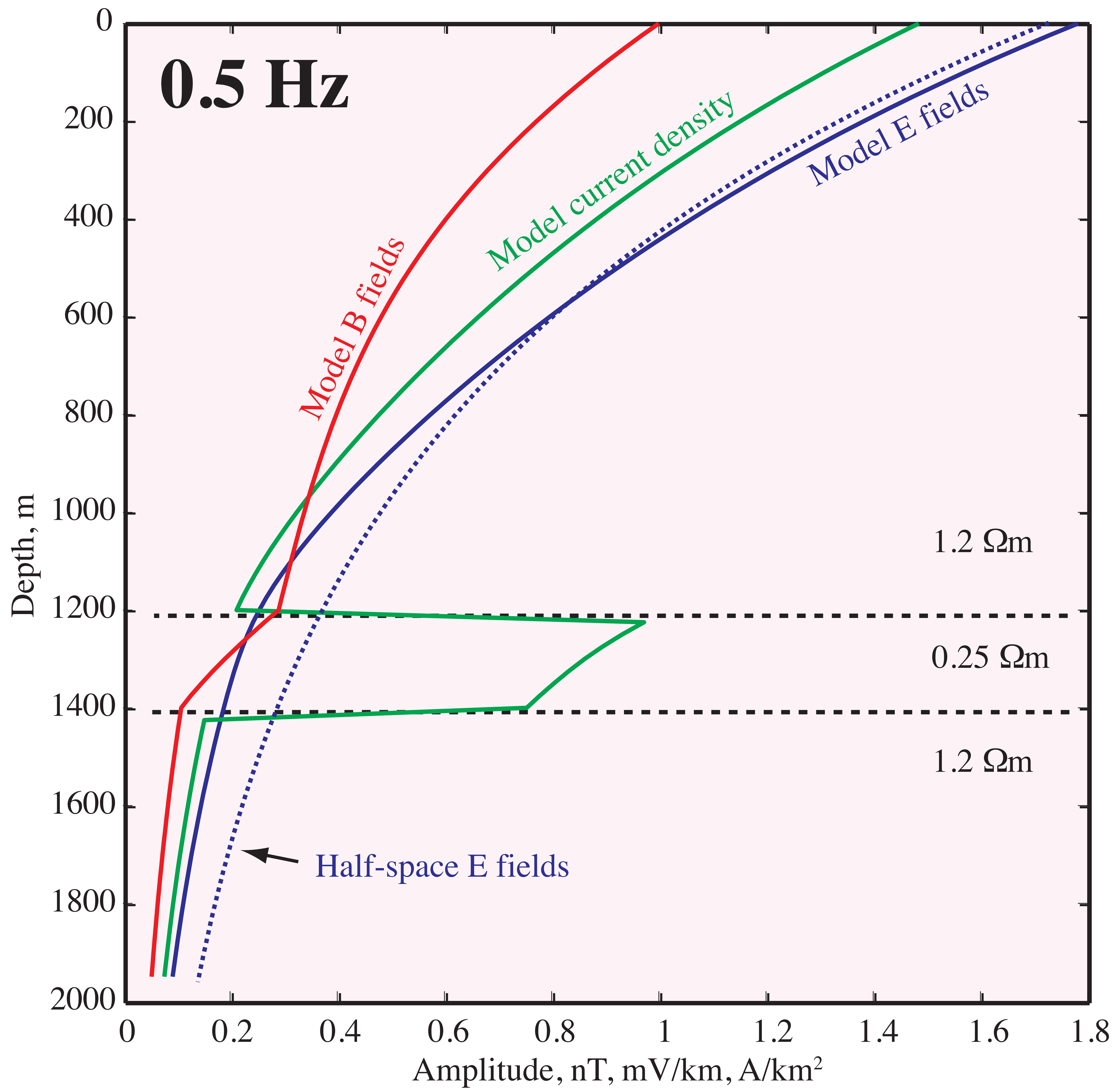



# 5 Hz fields do not reach the target layer

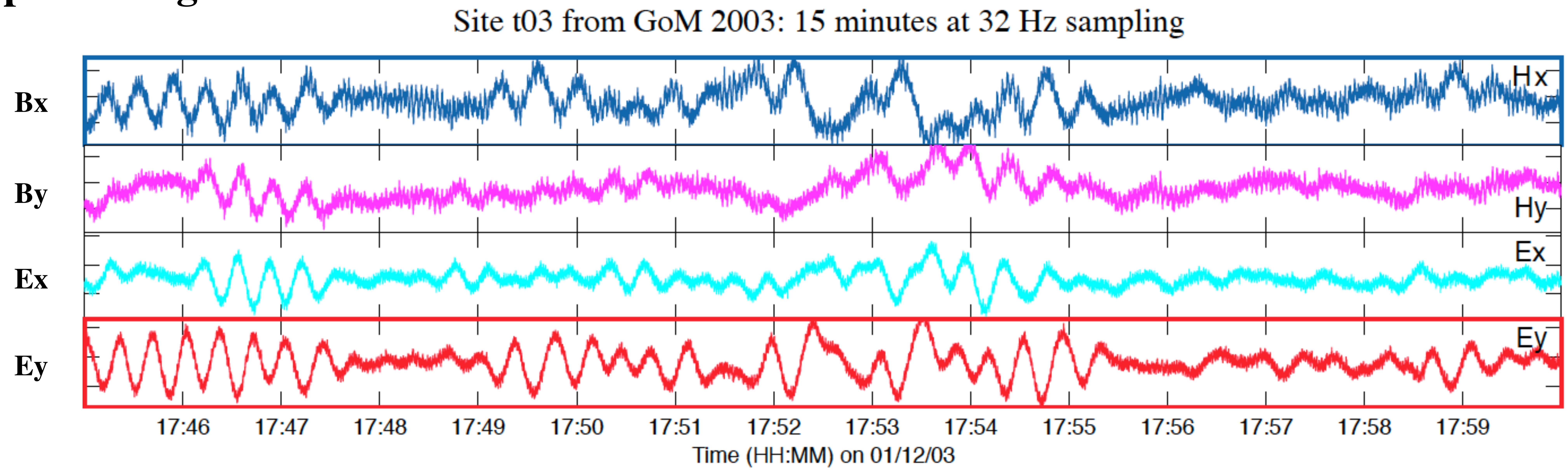


Changing deep structure alters surface electric fields



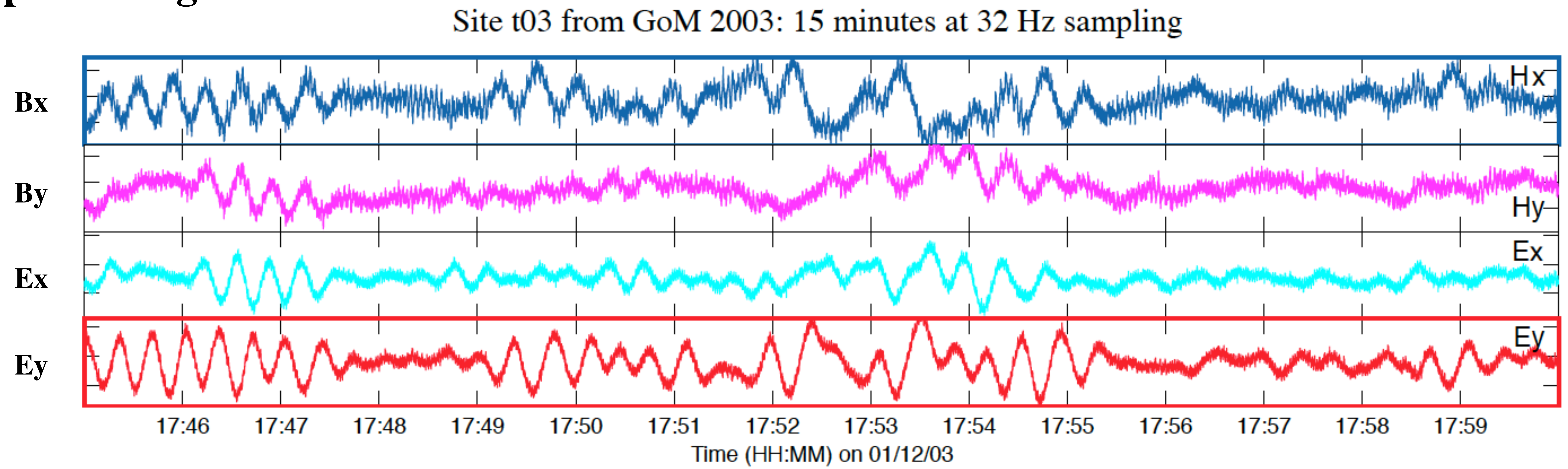



## Data processing:




We need the ratios **of the coherent parts** of the magnetic and electric field channels as a function of frequency. Because we need to separate signal from noise MT data processing is closely related to the statistics of covariance.

## Data processing:




Expectation: $E[X] = \int_{-\infty}^{\infty} x\Phi(x)dx$

Gaussian: $\Phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\bar{x})^2/2\sigma}$

Mean: $\bar{X} = \mathrm{E}[X]$

Variance: $\mathrm{var}[X] = \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[(X - \bar{X})^2]$

Covariance: $\mathrm{cov}[X, Y] = \mathrm{E}[(X - \bar{X})(Y - \bar{Y})]$

zero mean: $\mathrm{cov}[X, Y] = \mathrm{E}[(XY)]$

But we don't have infinite samples, so we need the sample mean etc.:

$$\bar{X} = \mathrm{E}[X] \longrightarrow \bar{x} = \frac{1}{N}\sum_{n=1}^{N} X_n$$

$$\mathrm{var}[X] = \mathrm{E}[(X - \mathrm{E}[X])^2] \longrightarrow \sigma_x^2 = \frac{1}{N}\sum_{n=1}^{N}(X_n - \bar{X})^2 = \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \bar{x})^2$$

$$\mathrm{cov}[X, Y] = \mathrm{E}[(X - \bar{X})(Y - \bar{Y})] \longrightarrow \mathrm{cov}_{xy} = \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \bar{x})(Y_n - \bar{y})$$

Covariance can be generalized to the covariance matrix for multiple random variables (here between $x_i$ and $x_k$):

$$\mathrm{cov}_{ik} = \frac{1}{N-1}\sum_{n=1}^{N}(x_{ni} - \bar{x}_i)(x_{nk} - \bar{x}_k)$$

We will assume everything is zero mean:

$$\mathrm{cov}_{ik} = \frac{1}{N}\sum_{n=1}^{N}(x_{ni})(x_{nk})$$

Back to MT: The MT impedance tensor $Z$ at a single frequency looks like

$$\begin{bmatrix} E_x(\omega) \\ E_y(\omega) \end{bmatrix} = \begin{bmatrix} Z_{xx}(\omega) & Z_{xy}(\omega) \\ Z_{yx}(\omega) & Z_{yy}(\omega) \end{bmatrix} \begin{bmatrix} H_x(\omega) \\ H_y(\omega) \end{bmatrix}$$

To convert time series to frequency domain we need the discrete Fourier transform:

$$\tilde{X}(m\Delta f) = \Delta t \sum_{n=0}^{N-1} X_{n+1} e^{-2\pi i m n/N} \quad , \qquad m = 1, 2, \ \dots \ N/2 - 1$$

complex!

Frequency bandwidth: $\Delta f = (N\Delta t)^{-1}$

Periodogram: $|\tilde{X}(m\Delta f)|^2$

Variance of periodogram is 100%, so we need to increase statistical reliability by averaging

Three approaches to averaging:

- frequency averaging
- window averaging
- multi taper averaging

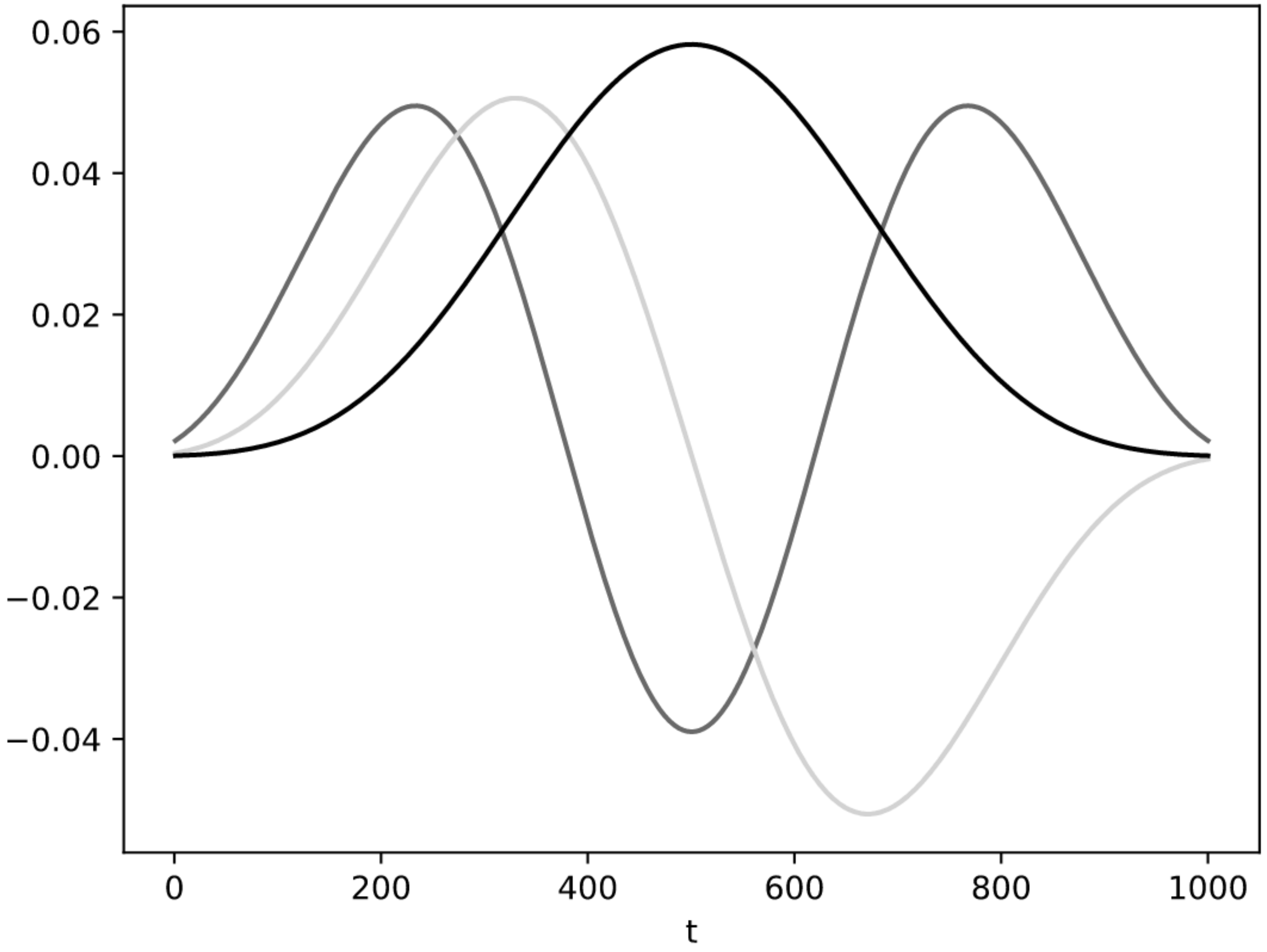
Tapering:

$$\tilde{X}(m\Delta f) = \Delta t \sum_{n=0}^{N-1} w_n X_{n+1} e^{-2\pi imn/N}$$

Here *w* is a taper - a smooth bunch of weights that usually go to zero at the ends of a series.

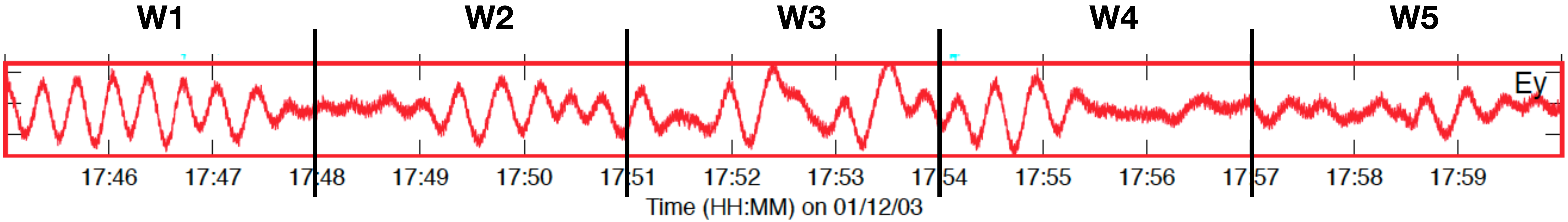
The multitaper method uses an orthogonal set of tapers and averages the resulting Fourier coefficients which should be statistically independent.

First 3 Slepian tapers




*Wikipedia*

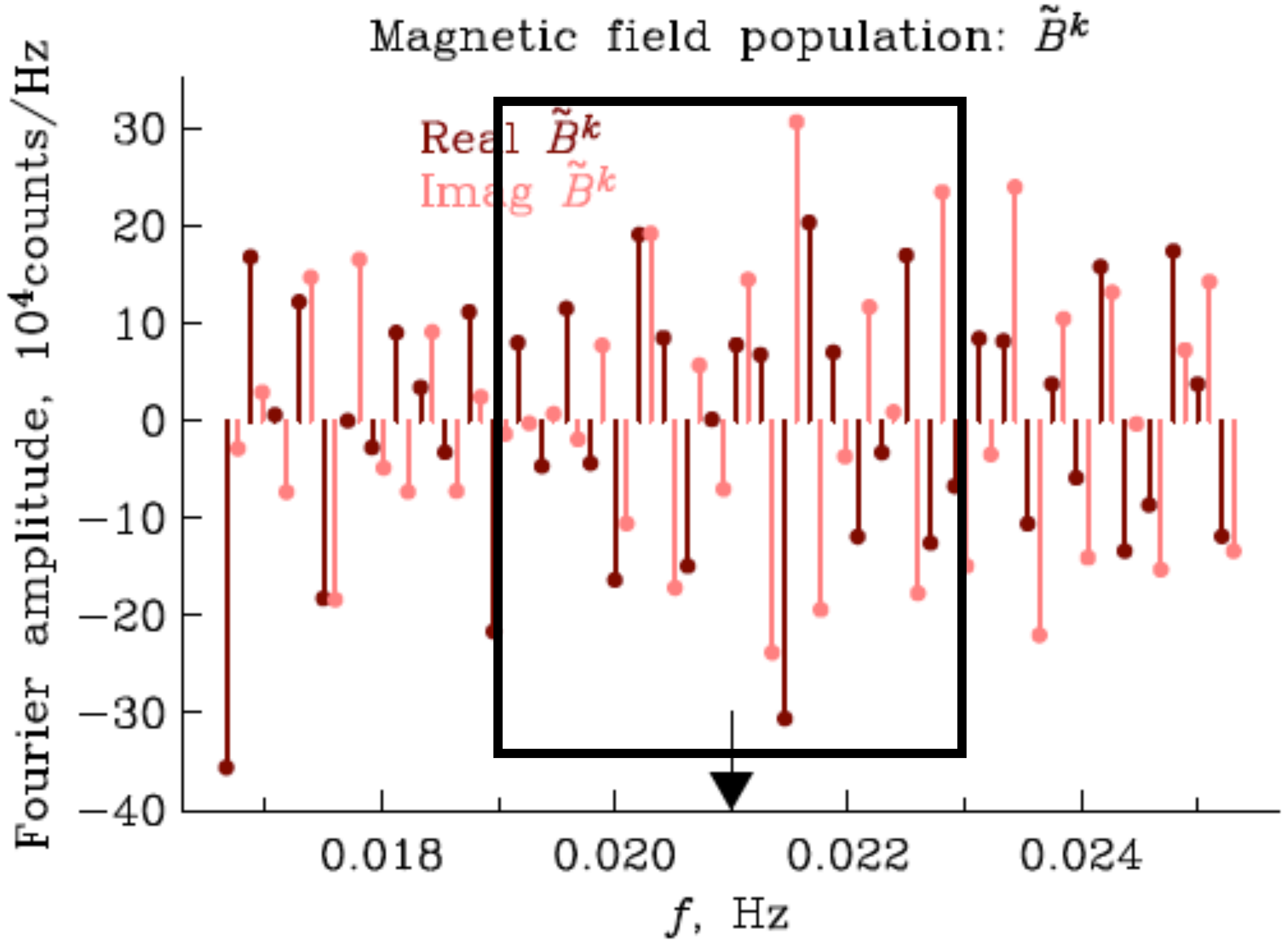
**Window averaging**: Chop the time series up into M pieces, Fourier transform each piece, and average.




(Welch's method: taper each segment and overlap the segments.)

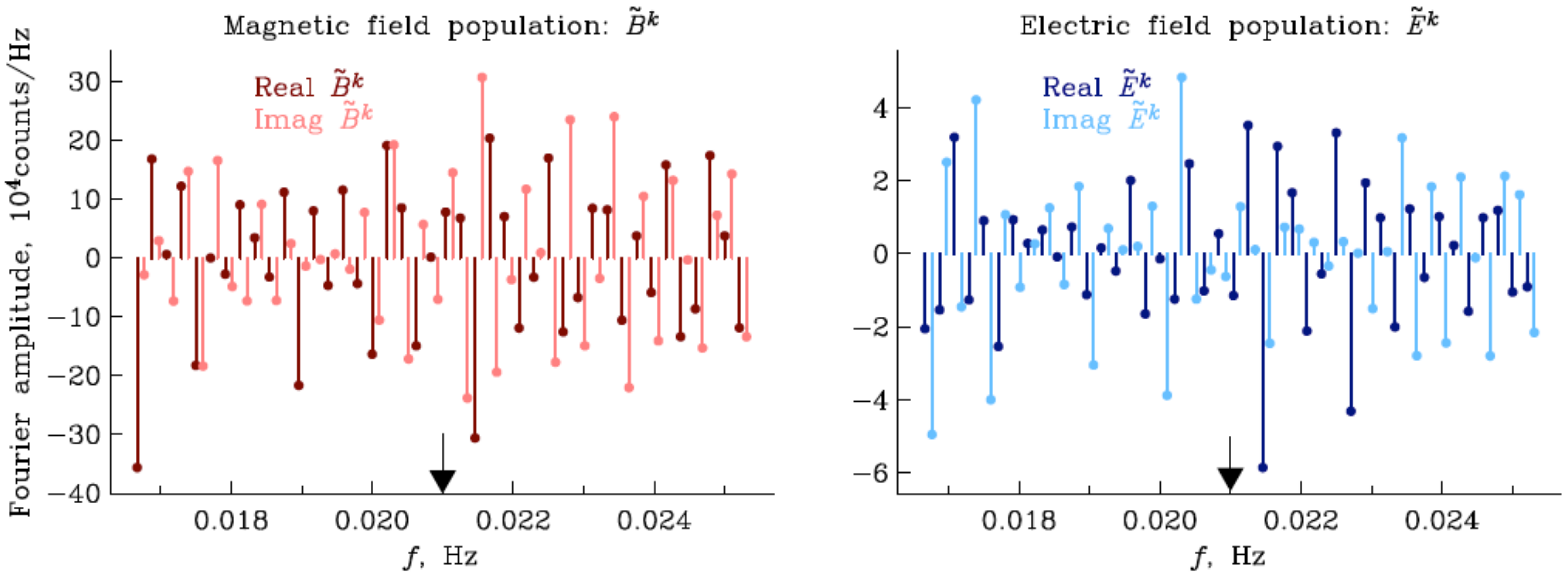
**Frequency averaging**: average over a number of adjacent frequencies.

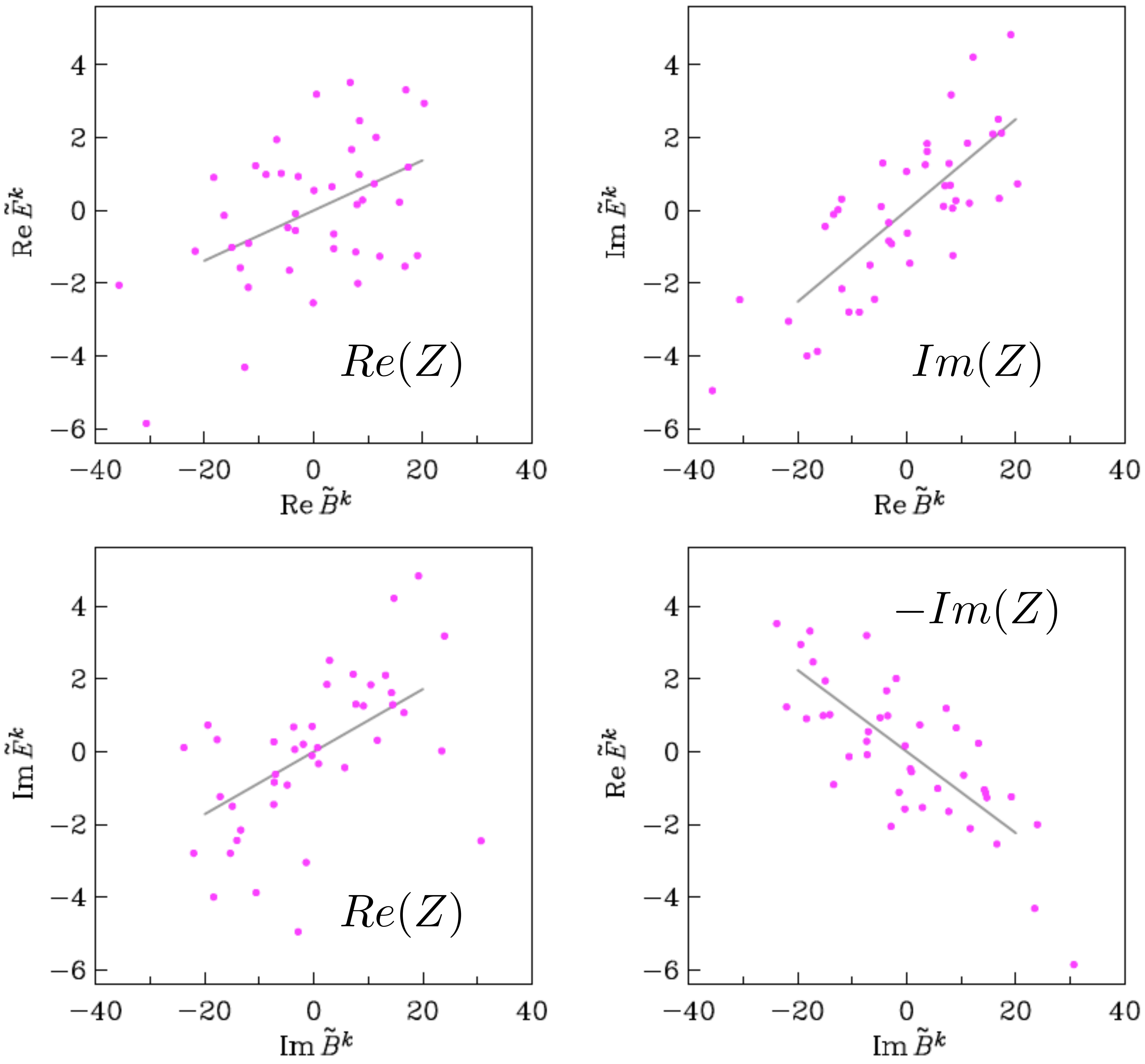
(Note: In all cases the averaging should be done on the complex components of the Fourier coefficients.)



With enough averaging one could estimate a 1D response: $$Z = \frac{\tilde{E}_y(\omega)}{\tilde{B}_x(\omega)}$$

An example from Bob Parker. He took 300,000 data sampled at 62.5 Hz from one of my seafloor instruments. Here are the Fourier coefficients sampled around 0.021 Hz. He is going to use frequency averaging over the 41 samples shown here.






$$Z = \frac{\tilde{E}_y(\omega)}{\tilde{B}_x(\omega)} \longrightarrow \tilde{E} = Z\tilde{B}$$

All terms are complex. If you multiply things out you get:

$$[Re(E) + iIm(E)]$$

$$= Re(Z)Re(B) - Im(Z)Im(B)$$

$$+iRe(Z)Im(B) + iIm(Z)Re(B)$$

(the minus sign comes from $i^2$ )

$$Re(Z) = 0.0778$$

$$Im(Z) = 0.1186$$

The full impedance matrix has 4 complex terms:

$$\begin{bmatrix}\tilde{E}_x \\ \tilde{E}_y\end{bmatrix} = \begin{bmatrix}Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy}\end{bmatrix}\begin{bmatrix}\tilde{B}_x \\ \tilde{B}_y\end{bmatrix}$$

So we need to break up our time series into *M* bits:
(this captures many source field polarizations)

(Note: I am now using B, not H, so there is an implied $\mu_O$, and I will drop the explicit frequency dependence)

And note we have to solve this for every frequency we are interested in having an MT response.

$$\begin{bmatrix}\tilde{E}_x^1 \\ \tilde{E}_x^2 \\ \vdots \\ \tilde{E}_x^M \\ \tilde{E}_y^1 \\ \tilde{E}_y^2 \\ \vdots \\ \tilde{E}_y^M\end{bmatrix} = \begin{bmatrix}Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy}\end{bmatrix}\begin{bmatrix}\tilde{B}_x^1 \\ \tilde{B}_x^2 \\ \vdots \\ \tilde{B}_x^M \\ \tilde{B}_y^1 \\ \tilde{B}_y^2 \\ \vdots \\ \tilde{B}_y^M\end{bmatrix}$$

Taking $\tilde{E}_x/\tilde{B}_y$ would work (or, $\tilde{E}_y/\tilde{B}_x$) but not very well, because there is noise in the measurements. Start with noise in $E$:

$$\tilde{E} = Z\tilde{B} + \epsilon \qquad \textbf{(1)}$$

The least squares solution minimizes

$$\min\left(\sum_{k=1}^{M} |\tilde{E}_k - Z\tilde{B}_k|^2\right)^{\frac{1}{2}}$$

and is the cross spectrum of $E$ and $B$ divided by the power spectrum in $B$

$$\hat{Z} = \frac{\sum_k \tilde{E}_k \tilde{B}_k^*}{\sum_k |\tilde{B}_k|^2}$$

Another way to look at this is to multiply both sides of (1) by $\tilde{B}^*$ and take the expectation values

$$\mathrm{E}[\tilde{E}\tilde{B}^*] = Z\mathrm{E}[\tilde{B}\tilde{B}^*] + \mathrm{E}[\epsilon\tilde{B}^*]$$

**= 0 because the noise in E is assumed to be uncorrelated with B**

$$\hat{Z} = \frac{\mathrm{E}[\tilde{E}\tilde{B}^*]}{\mathrm{E}[\tilde{B}\tilde{B}^*]} = \frac{\mathrm{cov}[\tilde{E}, \tilde{B}^*]}{\mathrm{var}[\tilde{B}]}$$

That is, $Z$ is described by two terms of a covariance matrix.

Recap: $\tilde{E} = Z\tilde{B} + \epsilon$ $\hat{Z} = \frac{\mathrm{E}[\tilde{E}\tilde{B}^*]}{\mathrm{E}[\tilde{B}\tilde{B}^*]} = \frac{\mathrm{cov}[\tilde{E}, \tilde{B}^*]}{\mathrm{var}[\tilde{B}]}$

$Z$ is described by two terms of a covariance matrix of a vector of random variables made by

$$\mathbf{X} = \begin{bmatrix} \tilde{E} \\ \tilde{B} \end{bmatrix}$$

That is

$$C = \begin{bmatrix} \mathrm{E}[\tilde{E}\tilde{E}^*] & \mathrm{E}[\tilde{E}\tilde{B}^*] \\ \mathrm{E}[\tilde{B}\tilde{E}^*] & \mathrm{E}[\tilde{B}\tilde{B}^*] \end{bmatrix} = \begin{bmatrix} Z^2\sigma_{\tilde{b}}^2 + \sigma_\epsilon^2 & Z\sigma_{\tilde{b}}^2 \\ Z^*\sigma_{\tilde{b}}^2 & \sigma_{\tilde{b}}^2 \end{bmatrix}$$

Where

$$\mathrm{E}[\tilde{B}\tilde{B}^*] = \sigma_b^2$$

$$\mathrm{E}[\tilde{B}\tilde{E}^*] = Z^*\sigma_b^2$$

$$\mathrm{E}[\tilde{E}\tilde{E}^*] = Z^2\sigma_b^2 + \sigma_\epsilon^2$$

4 constraints on 4 unknowns: Soluble!

But not… There is noise in $B$ as well

$$\tilde{B} = \tilde{b} + \beta \qquad \tilde{E} = Z\tilde{b} + \epsilon$$

so our covariance matrix is

$$C = \begin{bmatrix} \mathrm{E}[\tilde{E}\tilde{E}^*] & \mathrm{E}[\tilde{E}\tilde{B}^*] \\ \mathrm{E}[\tilde{B}\tilde{E}^*] & \mathrm{E}[\tilde{B}\tilde{B}^*] \end{bmatrix} = \begin{bmatrix} Z^2\sigma_{\tilde{b}}^2 + \sigma_\epsilon^2 & Z\sigma_{\tilde{b}}^2 \\ Z^*\sigma_{\tilde{b}}^2 & \sigma_{\tilde{b}}^2 + \sigma_\beta^2 \end{bmatrix}$$

or 4 constraints for 5 unknowns. Our LS estimate is biased down by the unknown error in $B$

$$\hat{Z} = \frac{\mathrm{cov}[\tilde{E}, \tilde{B}^*]}{\mathrm{var}[\tilde{B}]} = \frac{Z\sigma_b^2}{\sigma_b^2 + \sigma_\beta^2} = \frac{Z}{1 + \sigma_\beta^2/\sigma_b^2}$$

or in terms of cross spectra there is noise power in the magnetic field spectrum

$$\hat{Z} = \frac{E[\tilde{E}\tilde{B}^*]}{E[\tilde{B}\tilde{B}^*]}$$

This led Gamble et al. (1979) to suggest collecting another channel of **remote reference** data, $R$:

$$\tilde{B} = \tilde{b} + \beta \qquad \tilde{E} = Z\tilde{b} + \epsilon \qquad \tilde{R} = \tilde{b} + \eta$$

If the noise in the remote is uncorrelated with the noise in the station measurement, the magnetic field cross spectrum is not biased

$$\hat{Z} = \frac{E[\tilde{E}\tilde{R}^*]}{E[\tilde{B}\tilde{R}^*]}$$

**cross spectrum between remote B (R) and site B**

or in terms of the correlation matrix

$$\mathbf{X} = \begin{bmatrix} \tilde{E} \\ \tilde{B} \\ \tilde{R} \end{bmatrix} \qquad C = \begin{bmatrix} Z^2\sigma_b^2 + \sigma_\epsilon^2 & Z\sigma_b^2 & Z\sigma_b^2 \\ Z^*\sigma_b^2 & \sigma_b^2 + \sigma_\beta^2 & \sigma_b^2 \\ Z^*\sigma_b^2 & \sigma_b^2 & \sigma_b^2 + \sigma_\eta^2 \end{bmatrix}$$

we have 9 constraints on 6 unknowns. Indeed, one could introduce a new impedance, $Y$, for the remote

$$\tilde{R} = Y\tilde{b} + \eta$$

and still solve the problem.

We noted that adding channels (n) increased the number of constraints ($n^2$) faster than the number of unknowns (~n). Gary Egbert (1997) took this to extreme with his "**multivariate errors in variables**" approach.

$$\mathbf{X}_i = \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{E}_{2i} \\ \tilde{B}_{3i} \\ \tilde{E}_{4i} \\ \tilde{B}_{5i} \\ \vdots \\ \tilde{E}_{Ki} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ e_{21} & e_{22} \\ b_{31} & b_{32} \\ e_{41} & e_{42} \\ b_{51} & b_{52} \\ \vdots & \\ e_{K1} & e_{K2} \end{bmatrix} \begin{bmatrix} c_{1i} \\ c_{2i} \end{bmatrix} + \begin{bmatrix} \epsilon_{1i} \\ \epsilon_{2i} \\ \epsilon_{3i} \\ \epsilon_{4i} \\ \epsilon_{5i} \\ \vdots \\ \epsilon_{Ki} \end{bmatrix} = \mathbf{U}\mathbf{c}_i + \epsilon_i$$

We have *K* channels of data divided into *M* segments (here we show the *i*th segment), **U** are the ideal NS and EW polarized magnetic fields, **c** are the polarizations of the field for the *i*th data segment, and *e* is the noise. This is our forward model. Everything on the right is unknown. And all this is only for a single frequency.

$$\mathbf{X}_i = \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{E}_{2i} \\ \tilde{B}_{3i} \\ \tilde{E}_{4i} \\ \tilde{B}_{5i} \\ \vdots \\ \tilde{E}_{Ki} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ e_{21} & e_{22} \\ b_{31} & b_{32} \\ e_{41} & e_{42} \\ b_{51} & b_{52} \\ \vdots & \\ e_{K1} & e_{K2} \end{bmatrix} \begin{bmatrix} c_{1i} \\ c_{2i} \end{bmatrix} + \begin{bmatrix} \epsilon_{1i} \\ \epsilon_{2i} \\ \epsilon_{3i} \\ \epsilon_{4i} \\ \epsilon_{5i} \\ \vdots \\ \epsilon_{Ki} \end{bmatrix} = \mathbf{U}\mathbf{c}_i + \epsilon_i$$

If we can find $\mathbf{U}$, the MT impedance for the $j$ th site is given by the product of two subsets of $\mathbf{U}$:

$$\mathbf{Z}_j = \begin{bmatrix} ex_{j1} & ex_{j2} \\ ey_{j1} & ey_{j2} \end{bmatrix} \begin{bmatrix} bx_{j1} & bx_{j2} \\ by_{j1} & by_{j2} \end{bmatrix}^{-1} = \mathbf{U}_1 \mathbf{U}_2^{-1}$$

Enter our friend the covariance matrix (here also a **spectral density matrix**):

$$\mathbf{S} = E[\mathbf{X}\mathbf{X}^*]$$

**S** is $\underline{K}$x$K$, and we increase reliability by averaging over the *M* estimates of the data sample. If we know the covariance matrix of noise $\Sigma_{\mathrm{N}}$ then an unbiased and maximum-likelihood (for Gaussian noise) is obtained by solving the eigenvalue problem.

$$\mathbf{S}\mathbf{u} = \lambda \Sigma_{\mathrm{N}} \mathbf{u}$$

But we don't know $\Sigma_{\mathrm{N}}$. Instead we make an estimate of uncorrelated noise

$$\Sigma_{\mathrm{N}} = \mathrm{diag}(\epsilon_1^2\ \epsilon_2^2\ ...\ \epsilon_K^2)$$

and fold the correlated noise into the model

$$\mathbf{X}_i = \mathbf{U}\mathbf{c}_i + \mathbf{V}\mathbf{b}_i + \epsilon_i$$

where **V** is a *K*x*L* matrix describing *L* sources of correlated noise with polarizations **b**. The first step of the algorithm is an iterative method to estimate $\Sigma_{\mathrm{N}}$ .

The second step is to compute an eigenvalue decomposition of the spectral density matrix scaled by the noise model

$$\mathbf{S}' = \Sigma_{\mathrm{N}}^{-\frac{1}{2}} \mathbf{S} \Sigma_{\mathrm{N}}^{-\frac{1}{2}} = \mathbf{W} \Lambda \mathbf{W}^*$$

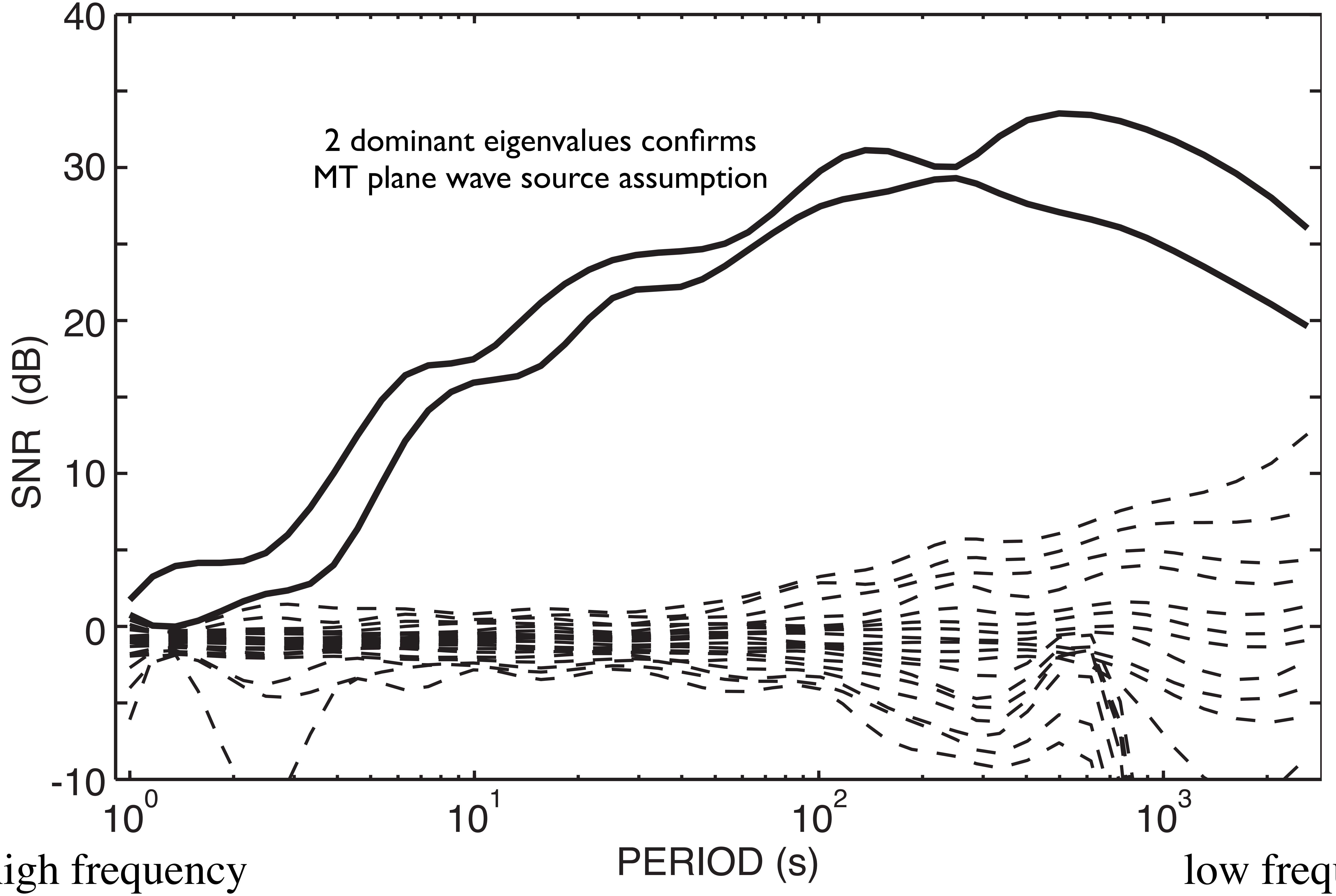
which gives a diagonal matrix of eigenvalues

$$\Lambda = \mathrm{diag}(\lambda_1 ... \lambda_K)$$

and a matrix **W** whose columns are the eigenvectors. If all goes well, the first two eigenvectors will describe the two MT source field polarizations, and the rest will be estimates of the correlated noise (or more complicated source field terms).

Eigenvalue spectrum indicates frequencies with good signal to noise ratio and MT source.

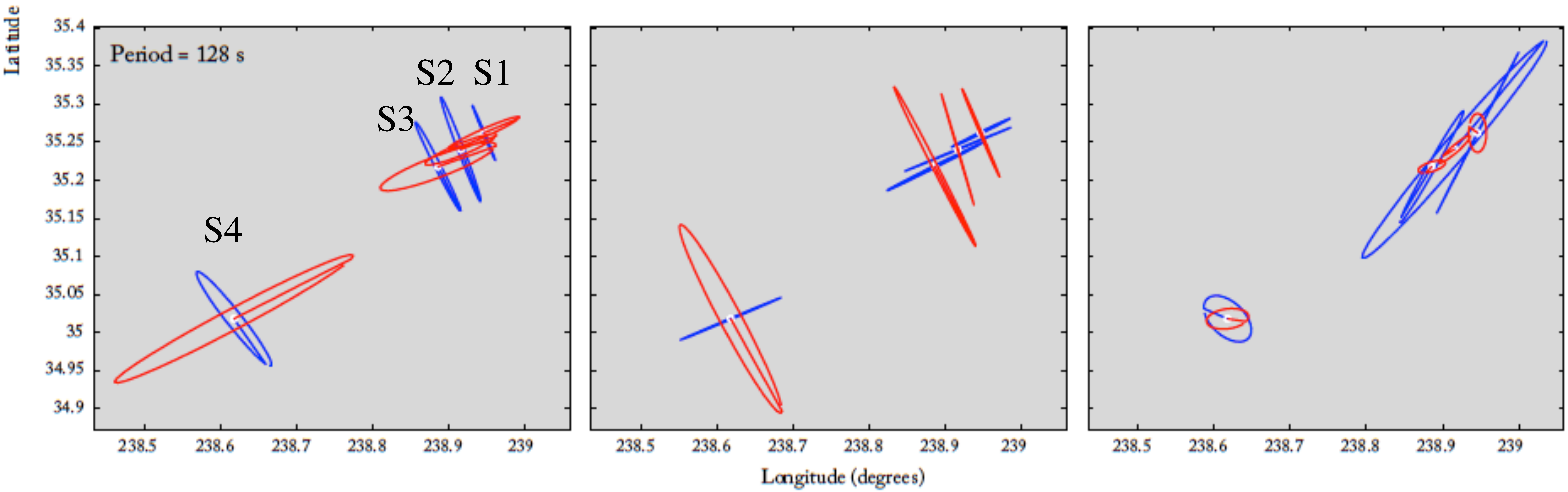
signal to noise




high frequency

low frequency

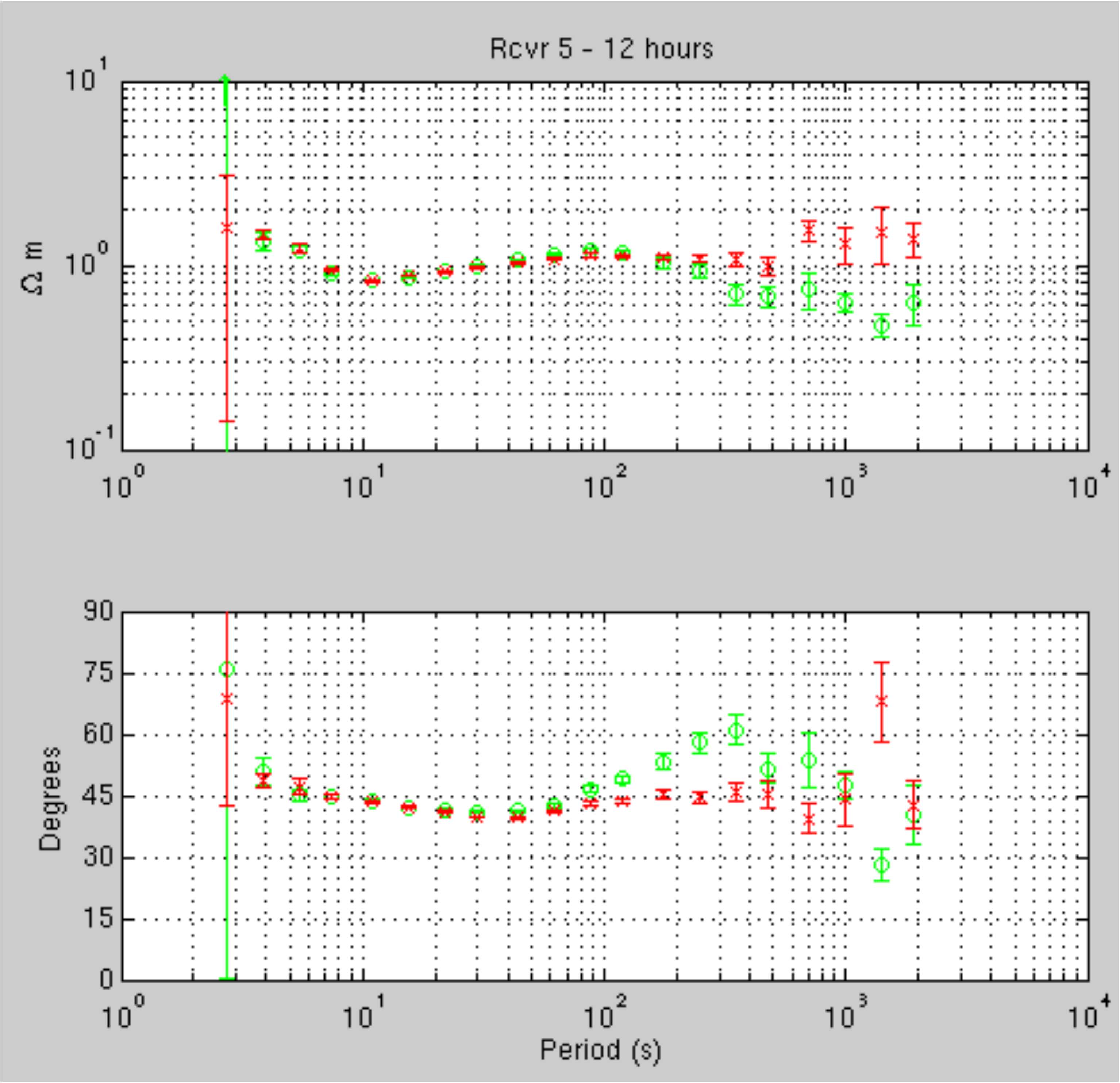
Here is what the first three eigenvectors look like for one frequency at 4 sites.



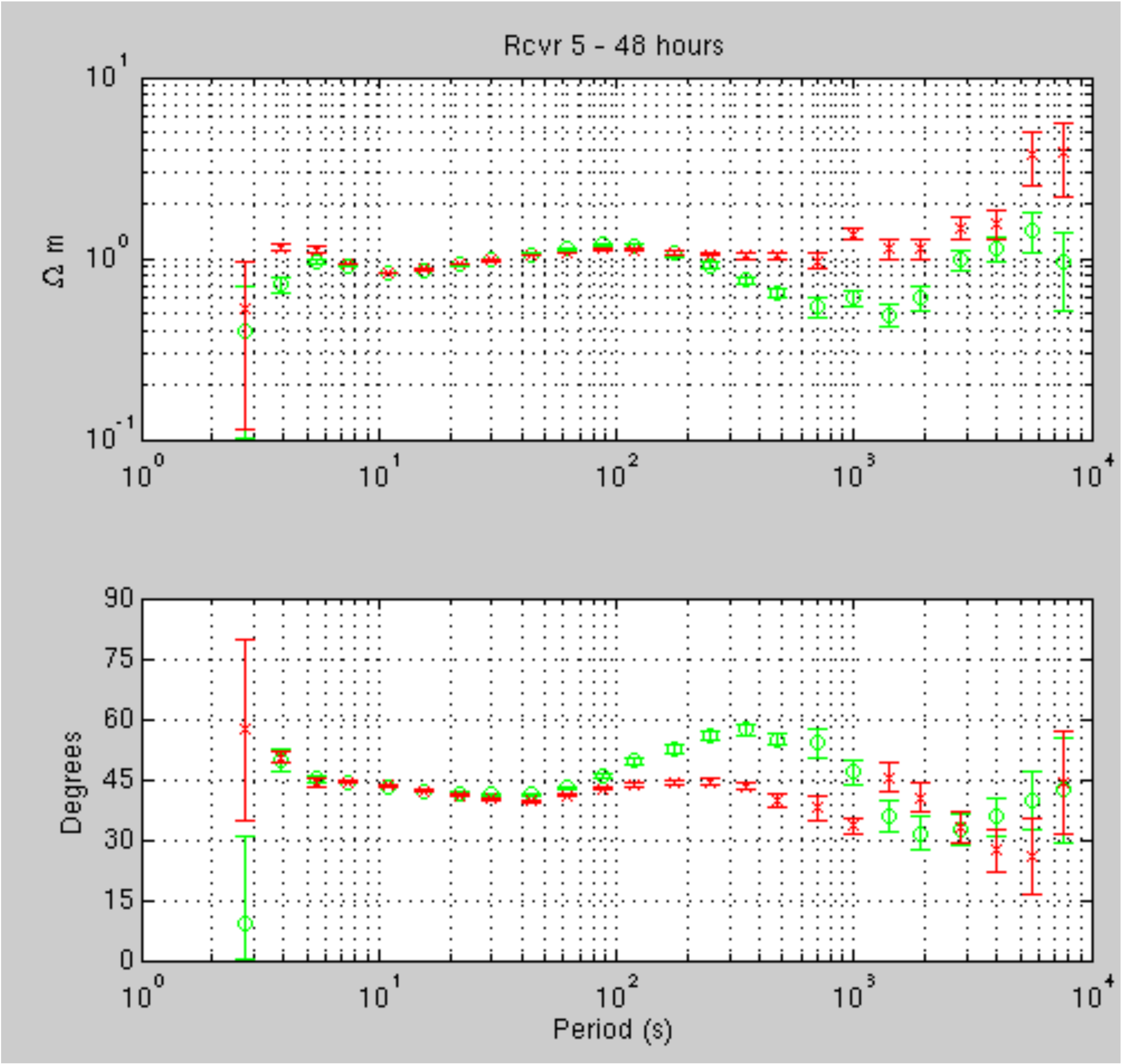

Blue - magnetic    Red - electric

# Effect of acquisition time on quality of the MT response:

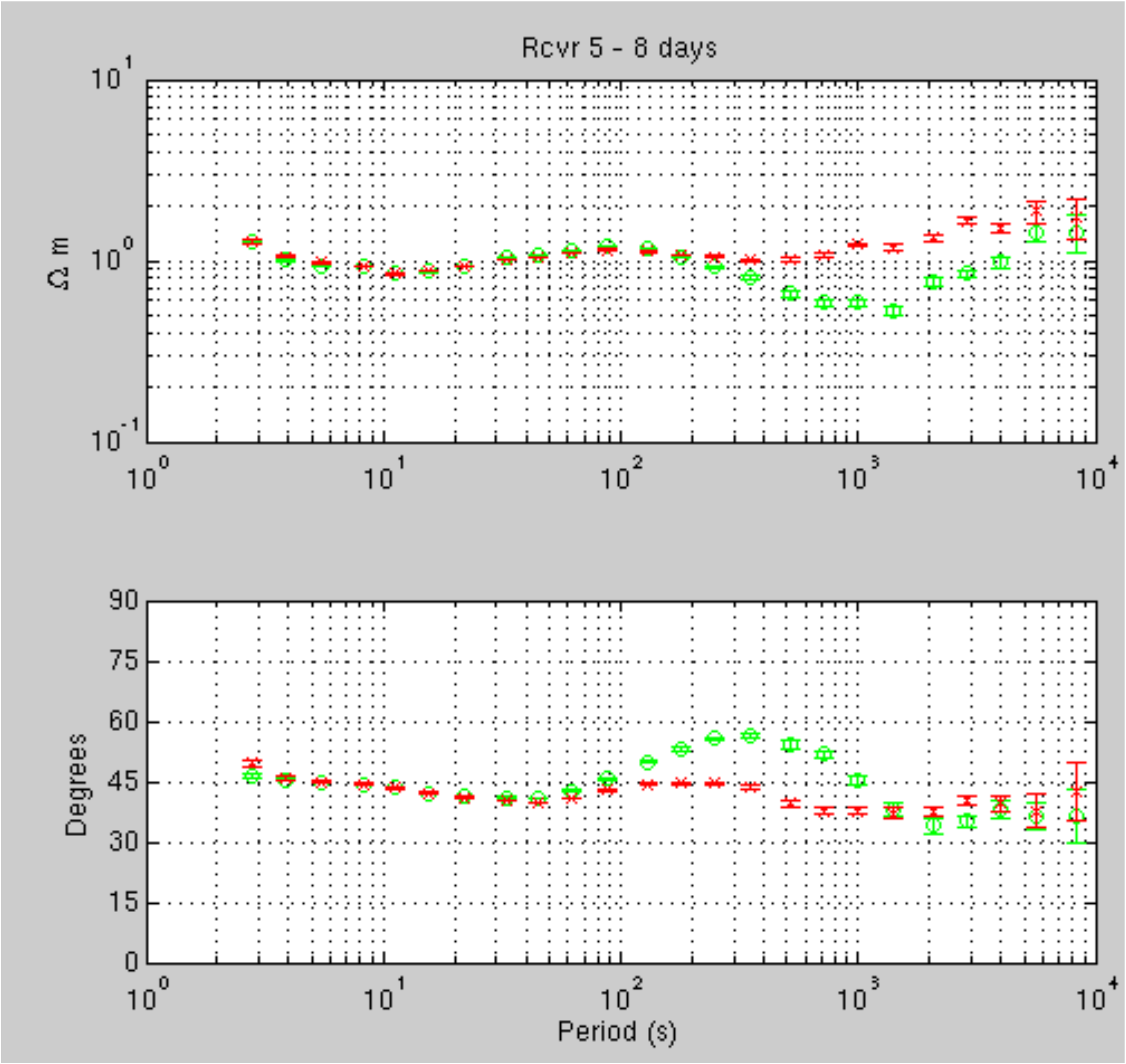
## 12 hours




## 2 days




## 8 days



**1D:** Diagonals of impedance matrix are equal and opposite (captures the difference in phase shift). Off-diagonals are zero.

$$Z = \begin{bmatrix} 0 & Z_{xy} \\ -Z_{xy} & 0 \end{bmatrix}$$

**2D:** Off-diagonals are still zero when aligned to strike, but diagonals are different.

$$Z = \begin{bmatrix} 0 & Z_{xy} \\ Z_{yx} & 0 \end{bmatrix}$$

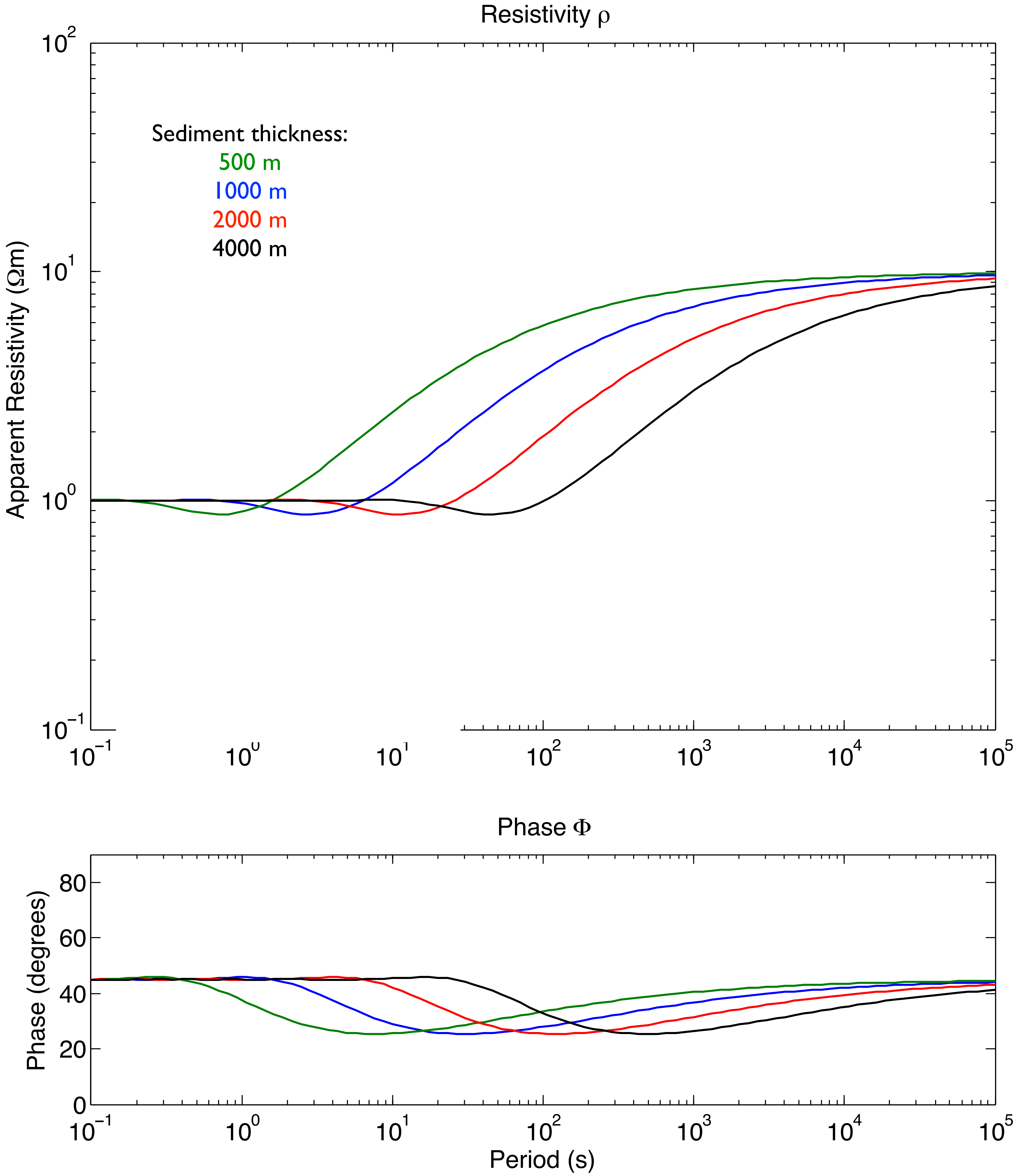
**3D:** All elements are non-zero and unique.

$$Z = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}$$

# A Simple 1D MT Example

Model

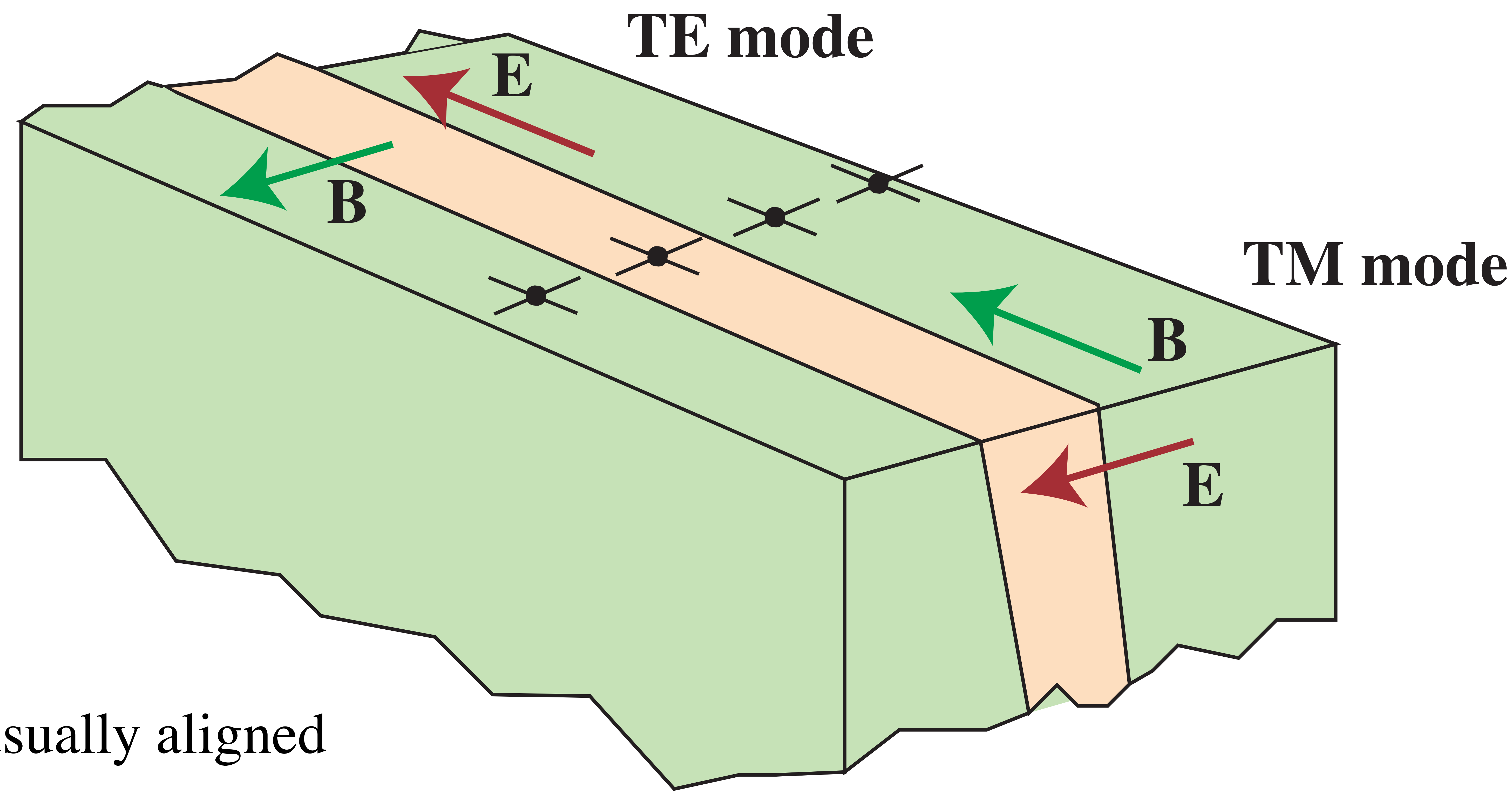

MT Response




Marine MT is good for mapping the depth to basement

**2D:** The direction with E along strike is call the Transverse Electric mode, sometimes called E-polarization.

The other direction, with E across strike, is called the Transverse Magnetic mode, sometimes called B-polarization.



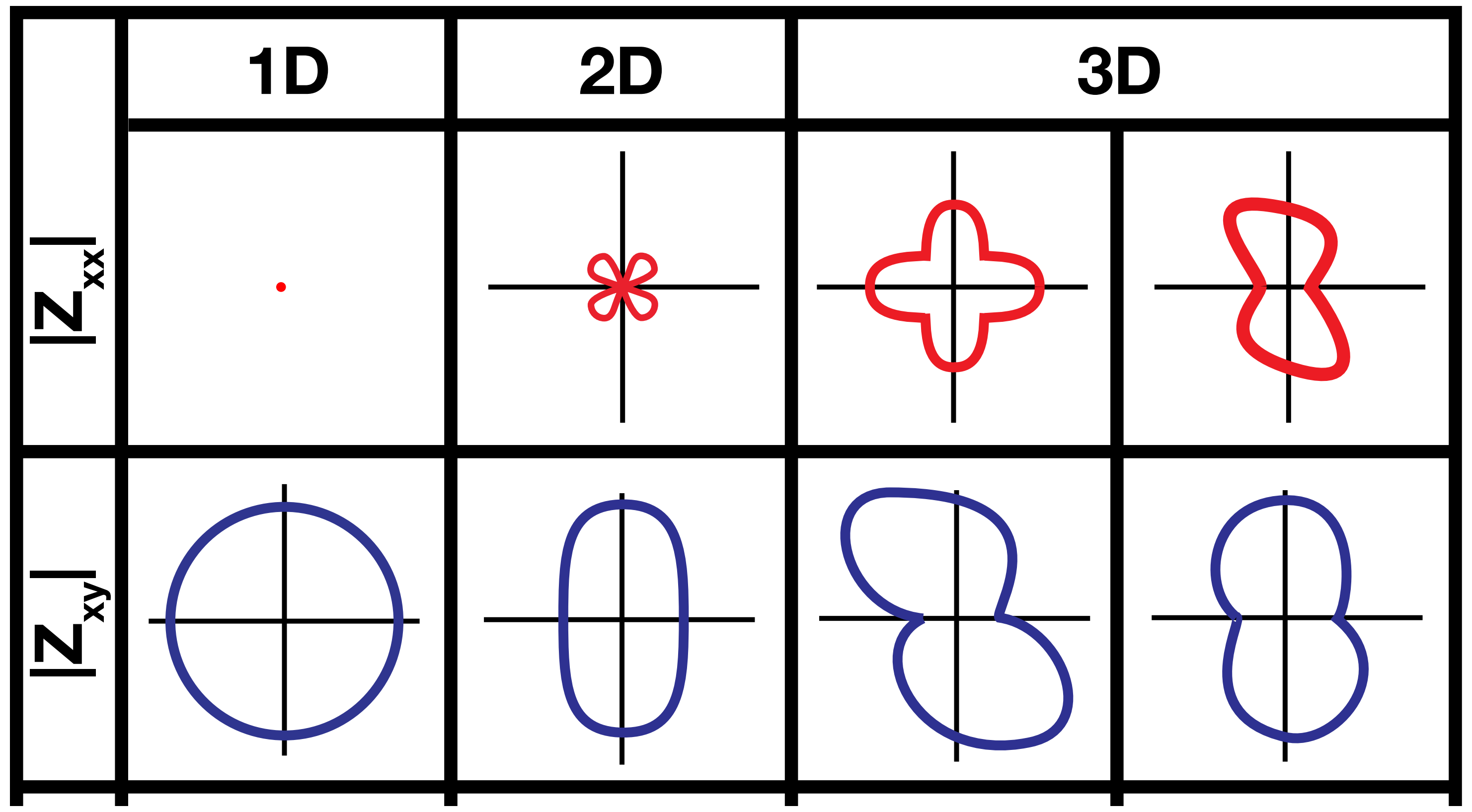

When the strike is known, instruments are usually aligned appropriately. If not, they can be rotated:

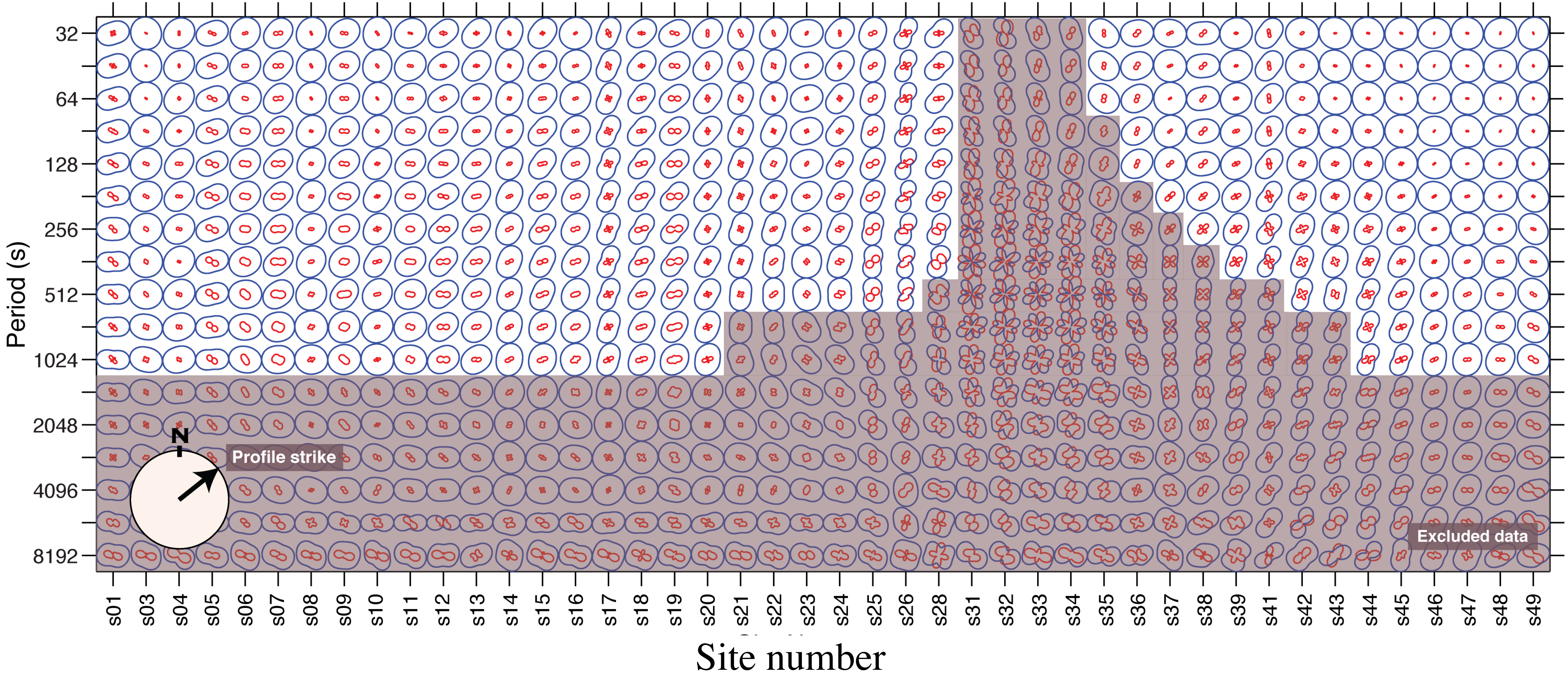
$$Z' = U_\theta Z U_\theta^T = \begin{bmatrix} (Z_{xy} + Z_{yx}) \sin\theta \cos\theta & Z_{xy} \cos^2\theta - Z_{yx} \sin^2\theta \\ -Z_{xy} \sin^2\theta + Z_{yz} \cos^2\theta & -(Z_{xy} + Z_{yx}) \sin\theta \cos\theta \end{bmatrix}$$

If the strike direction is unknown, $Z$ can be rotated to minimize the diagonals.

**Polar diagrams:** Show normalized diagonal and off diagonal components during rotation.

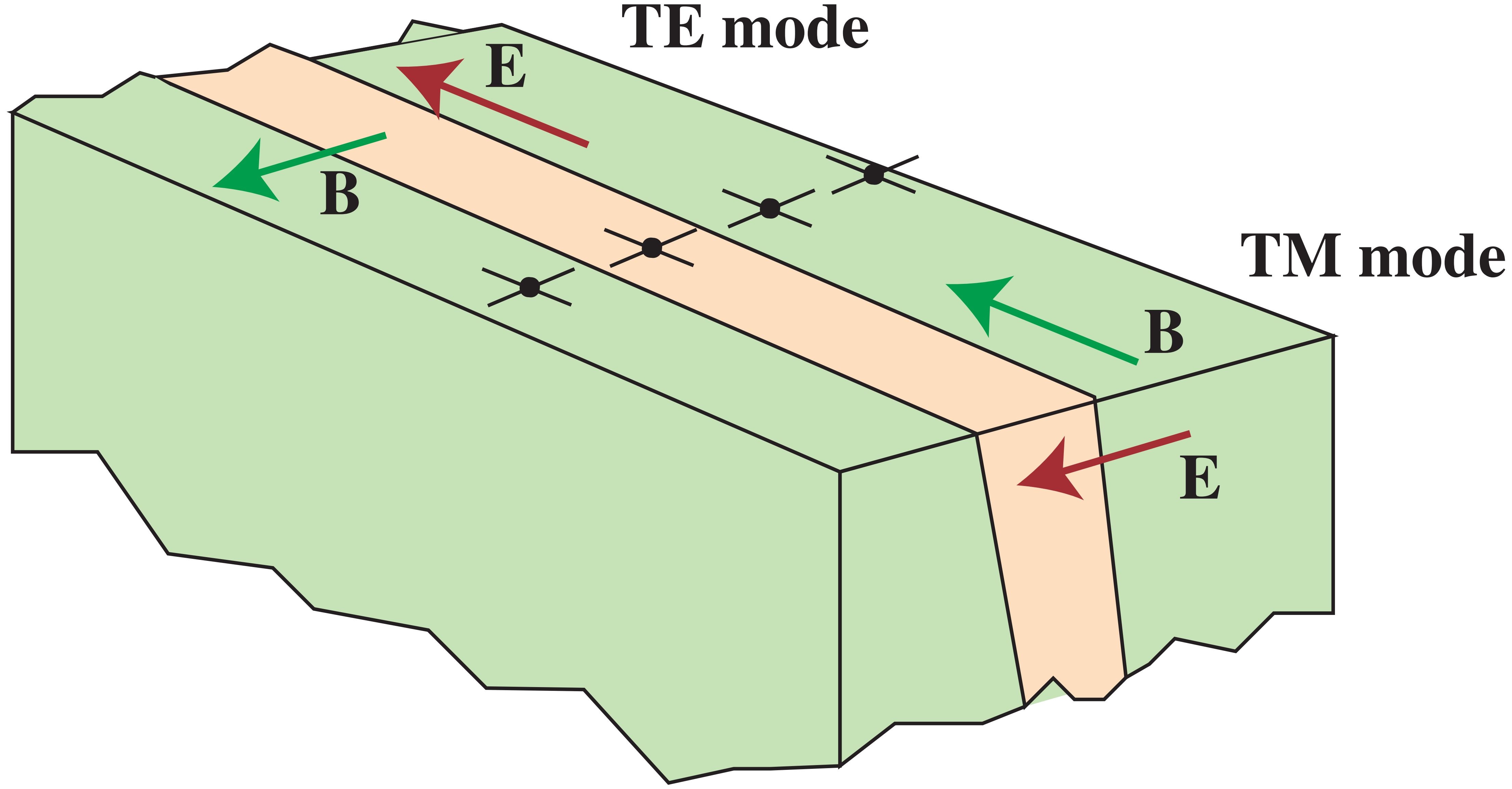
$$Z = \begin{bmatrix} \boxed{Z_{xx}} & Z_{xy} \\ \boxed{Z_{yx}} & Z_{yy} \end{bmatrix}$$

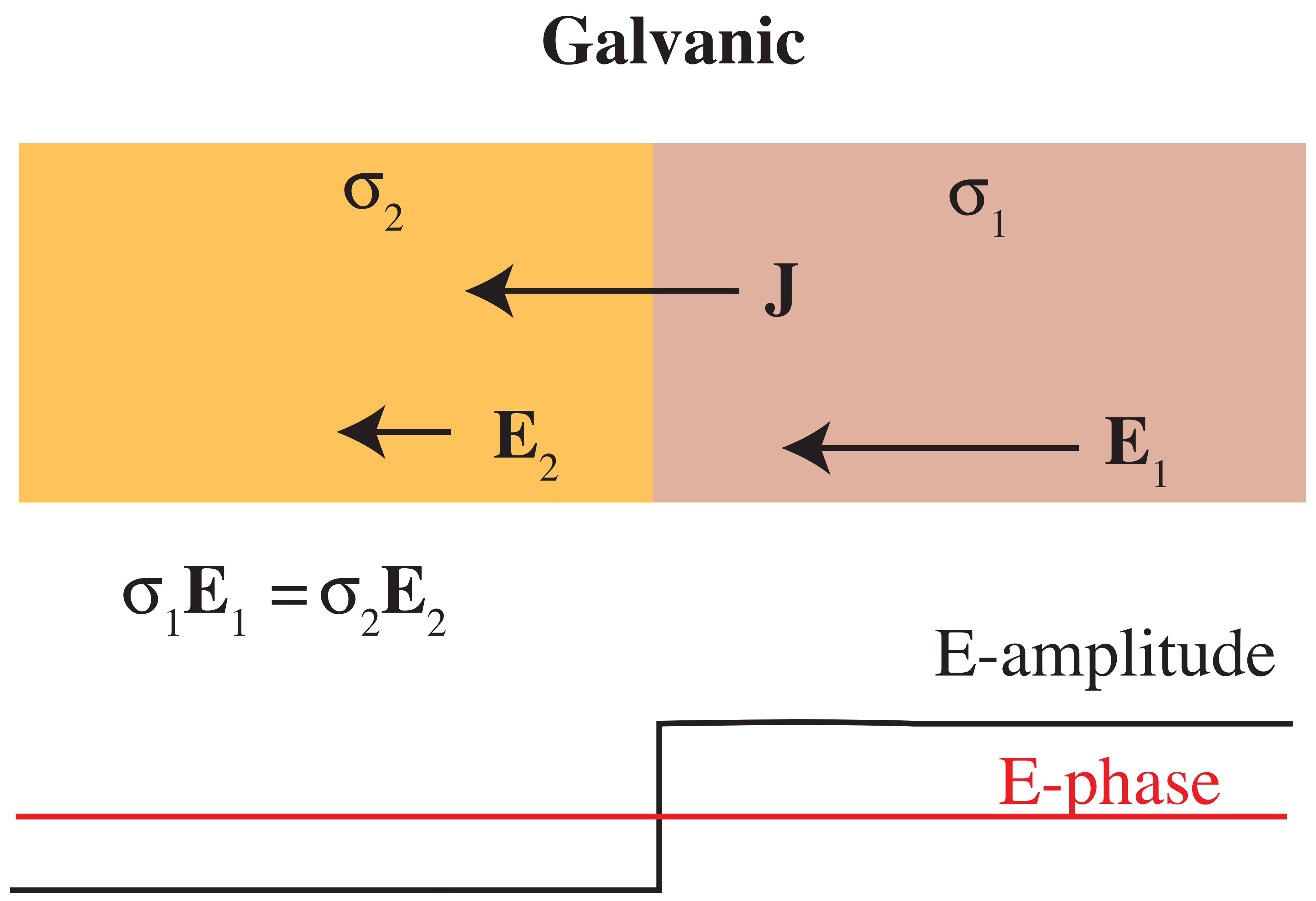


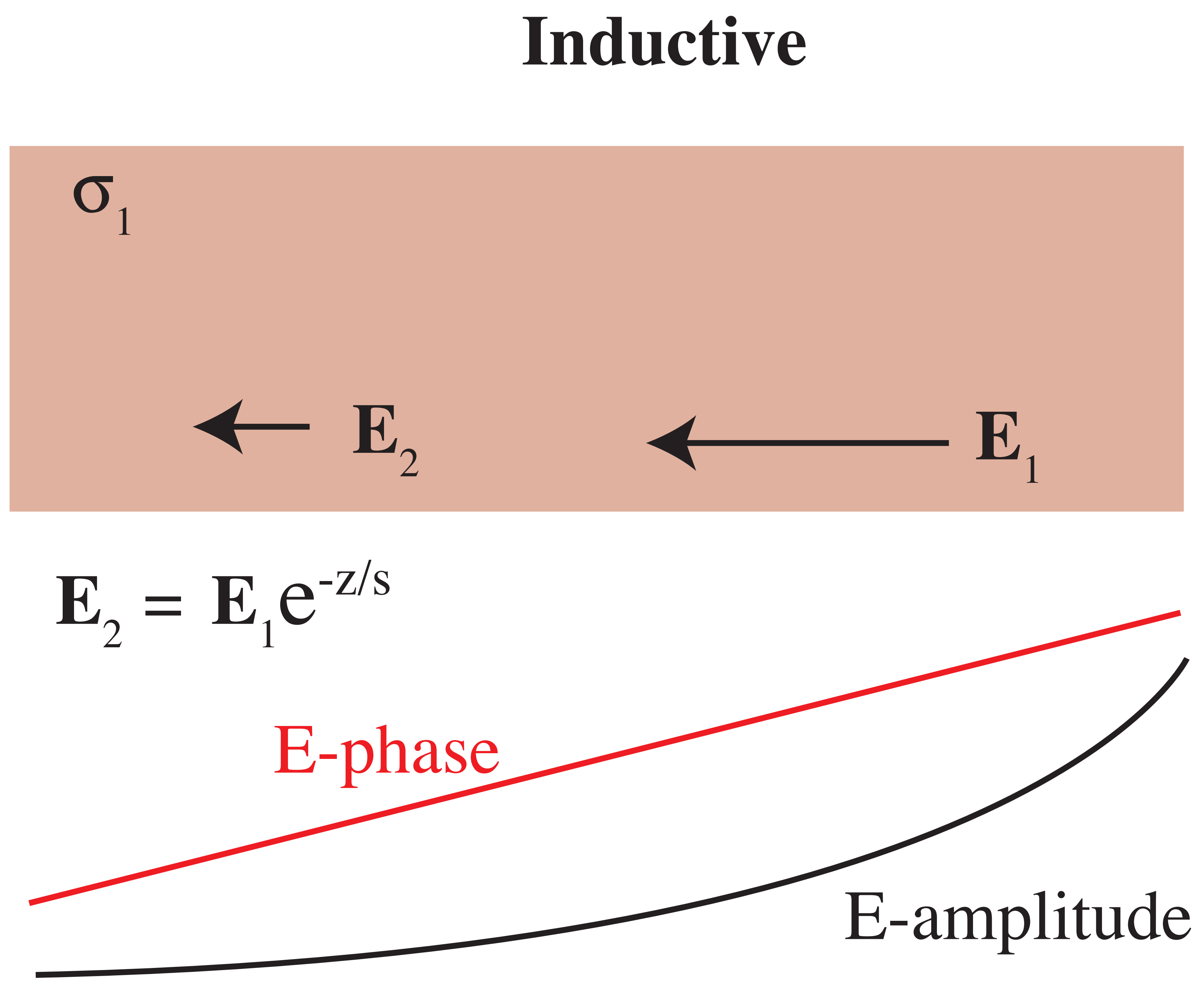



**Galvanic vs Inductive:** Currents flowing across conductivity contrasts require changes in electric field but not phase. TM mode can have galvanic effects but TE mode is purely inductive.

Near-surface changes in conductivity can produce “static shifts" in MT resistivity.



**Static shift:** Apparent resistivities are shifted vertically on a log plot, while phases are unaltered.

A simple remedy is to multiply resistivities by a constant to correct for the shift.

This and more complicated "distortion corrections" were common to invert such data in 1D and 2D.

3D inversions can include surface contrasts that create galvanic effects in both directions, and so distortion correction is no longer common.

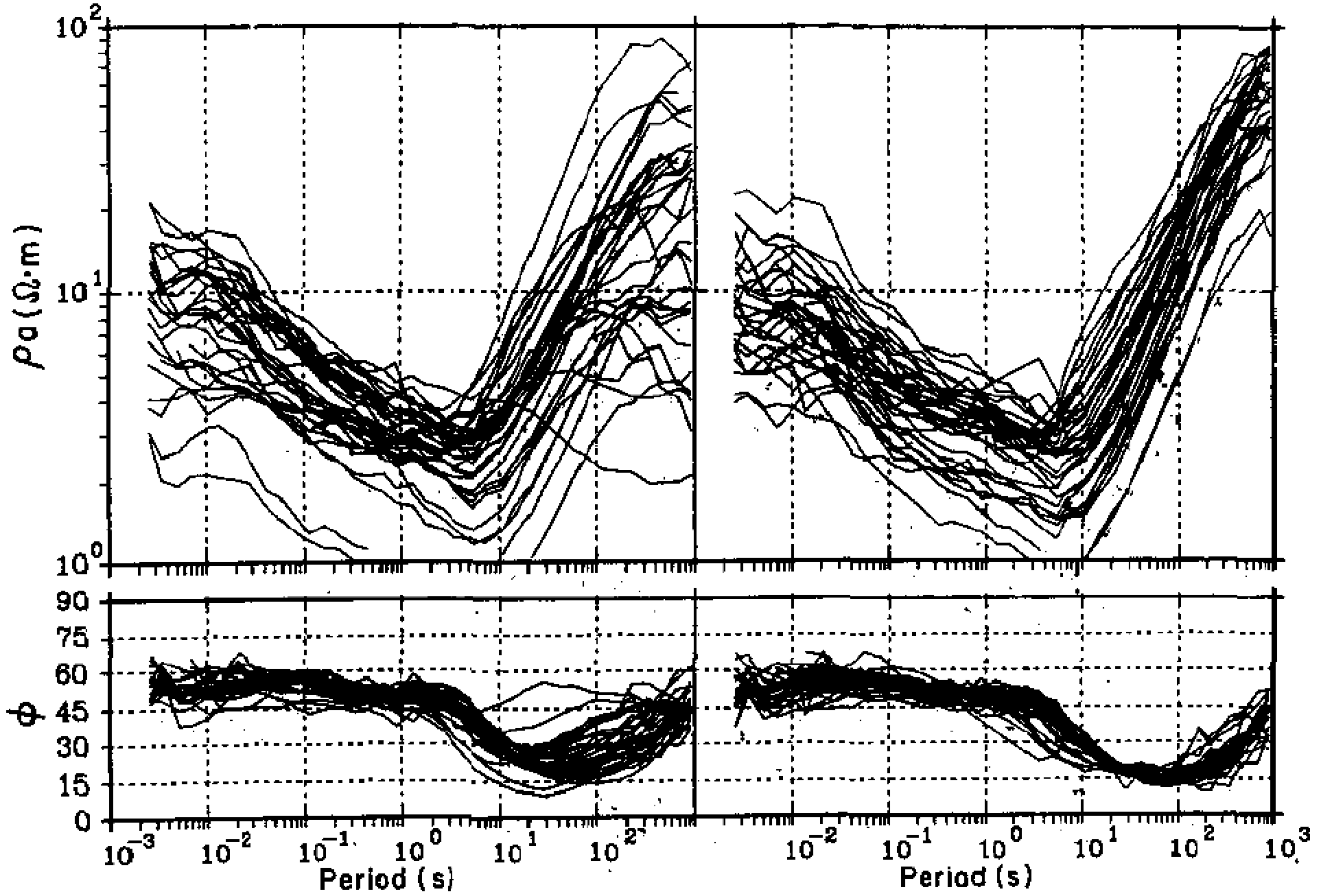



FIG. 9. MT data from all 35 locations along the Williston Basin profile. (a) are the *E*-polarization data, and (b) are the *B*-polarization data. Note the static shift corruption of the apparent resistivity data.

*Jones, Geophysics, 1988*