SIOG 231
GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 14

The Geodynamo:

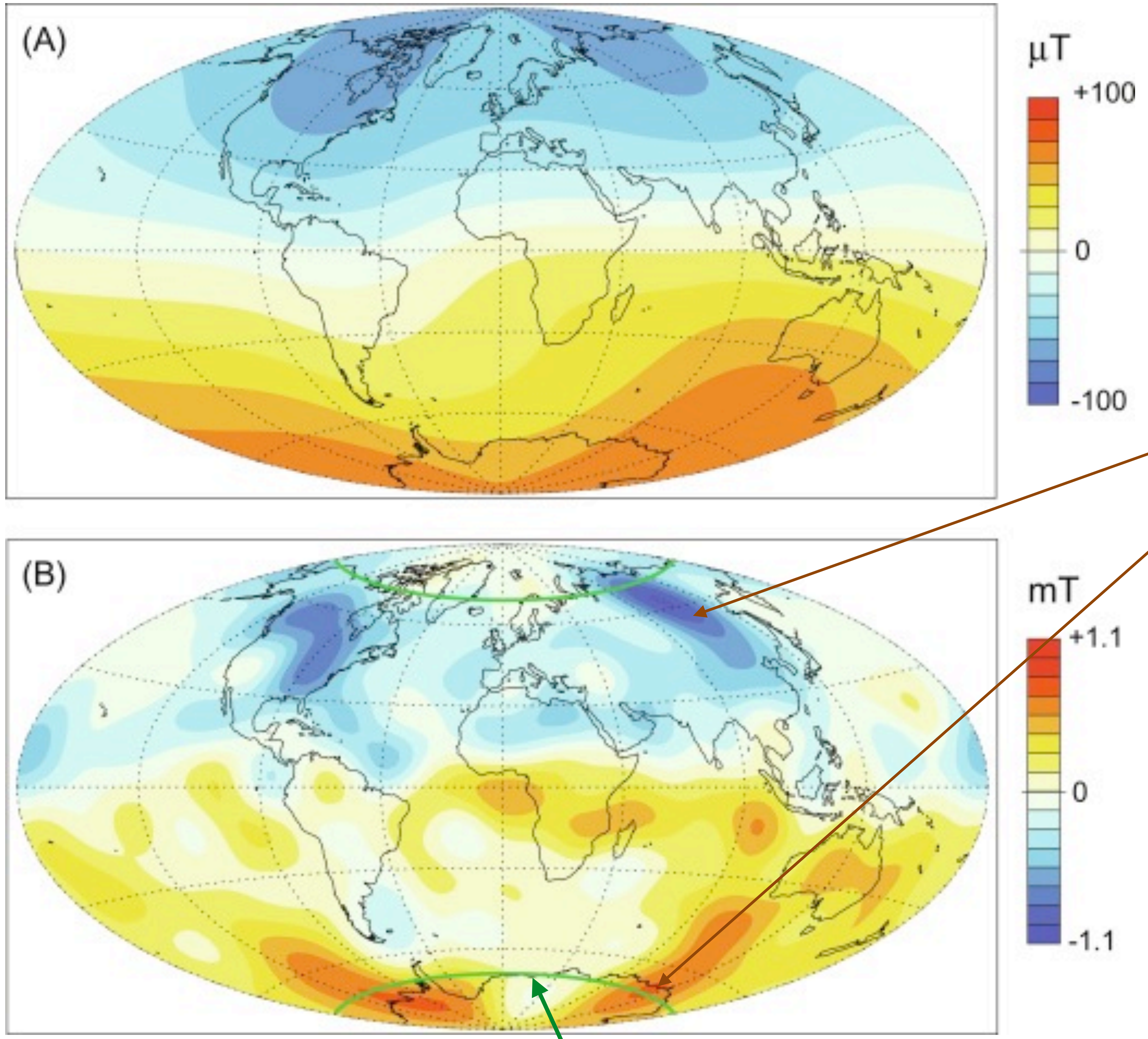
Magnetohydrodynamics in Earth's Core I

2/22/2024

Average Radial Magnetic Field from 1860 -2016 - clues to core dynamics

r=a

r=c



outline of tangent cylinder (TC)

Paired northern and southern hemisphere high intensity flux patches suggest columnar structure to magnetic field inside the core.

What is origin of reverse flux patches in both Atlantic and inside the TC near the poles?

What about Southern Hemisphere intense equatorial flux patches? Are they connected to high intensity Antarctic flux?

Today's Class

- What is a planetary dynamo?
- Moving into the dynamo source region in Earth's core: another encounter with non-uniqueness
- Reminder on important vector identities
- Toroidal and Poloidal Field decomposition of a solenoidal vector field
- α and Ω
- Induction in a moving conductor; Ohm's law in a changing reference frame
- The Magnetohydrodynamics (MHD) approximation
- The magnetic induction equation for changing magnetic field in Earth's core.
- Two end member cases - i. diffusive decay; ii. the frozen flux approximation

What is a planetary dynamo?

- A dynamo is any device or system that converts mechanical energy into electromagnetic energy - electric currents are induced by relative motion between an electrical conductor and a magnetic field.

- In the geodynamo (along with other planetary dynamos) electric and magnetic fields are induced by motions of electrically conducting fluid in the liquid outer core.

- The geodynamo is self-sustaining as long as the fluid motions continue to support electric and magnetic fields in face of inevitable Ohmic decay. No external sources of **J** or **B** are required.

Why do we need to invoke a self-sustaining geodynamo?

- Earth’s magnetic field is global scale and has a deep internal origin.
- External fields are too weak to produce the global field.
- The core is too hot to sustain remanent magnetization.
- The internal field changes continuously on a wide range of temporal and spatial scales
- The geodynamo polarity has changed many times in the past, ~200 reversals are recorded in the magnetic anomaly record which extends to about 160 Ma.
- The field has been present for most of Earth history which extends to 4.6 Ga.

Three Ingredients needed for a Dynamo

- Large volume of highly electrically conducting fluid.

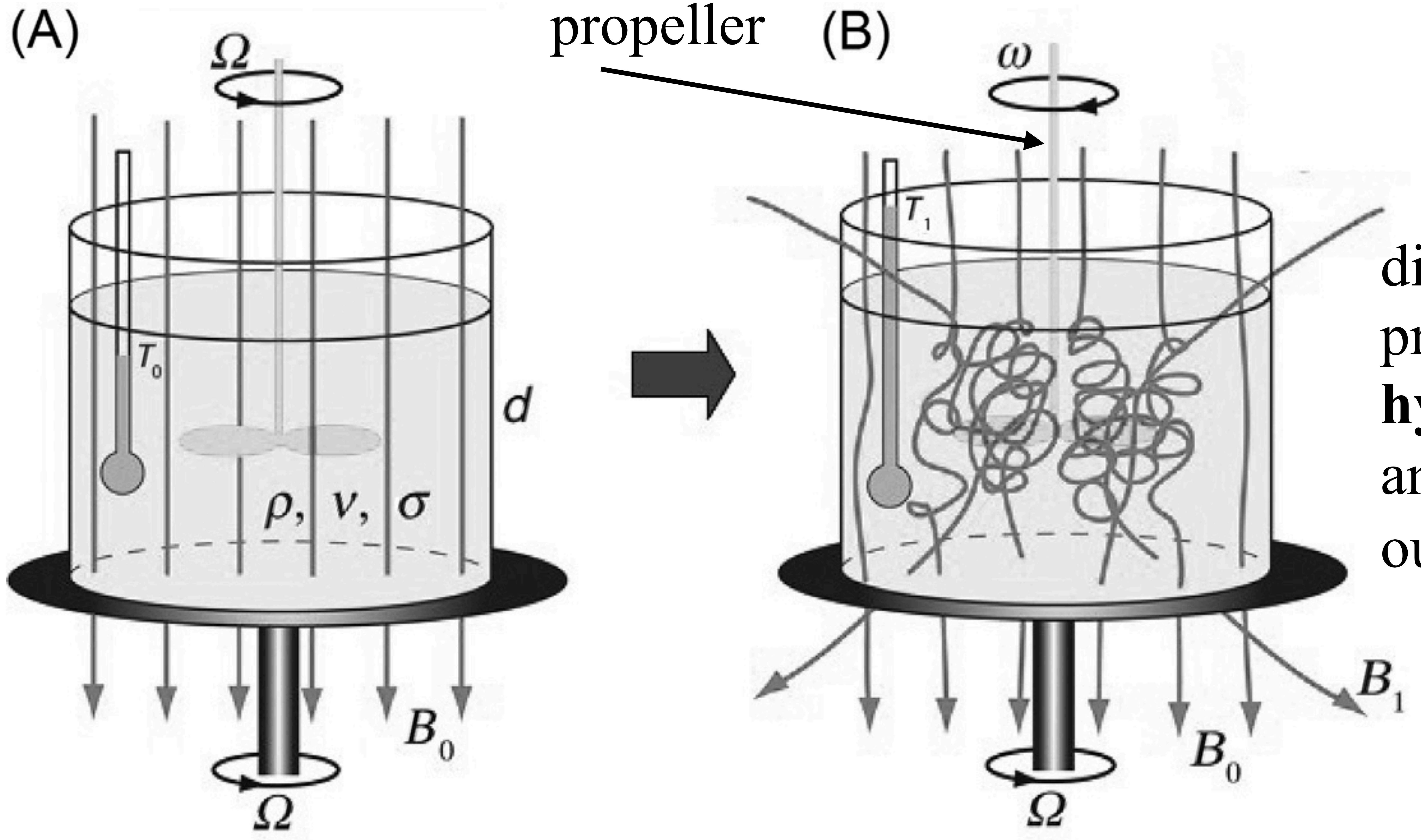
- Energy source to promote fluid motion that balances resistive decay of electric currents.

- Planetary rotation -
  i. influences interaction between **poloidal** and **toroidal** magnetic fields;
  ii. tidal forces and precession can excite fluid motions in the outer core.

A Hypothetical Dynamo

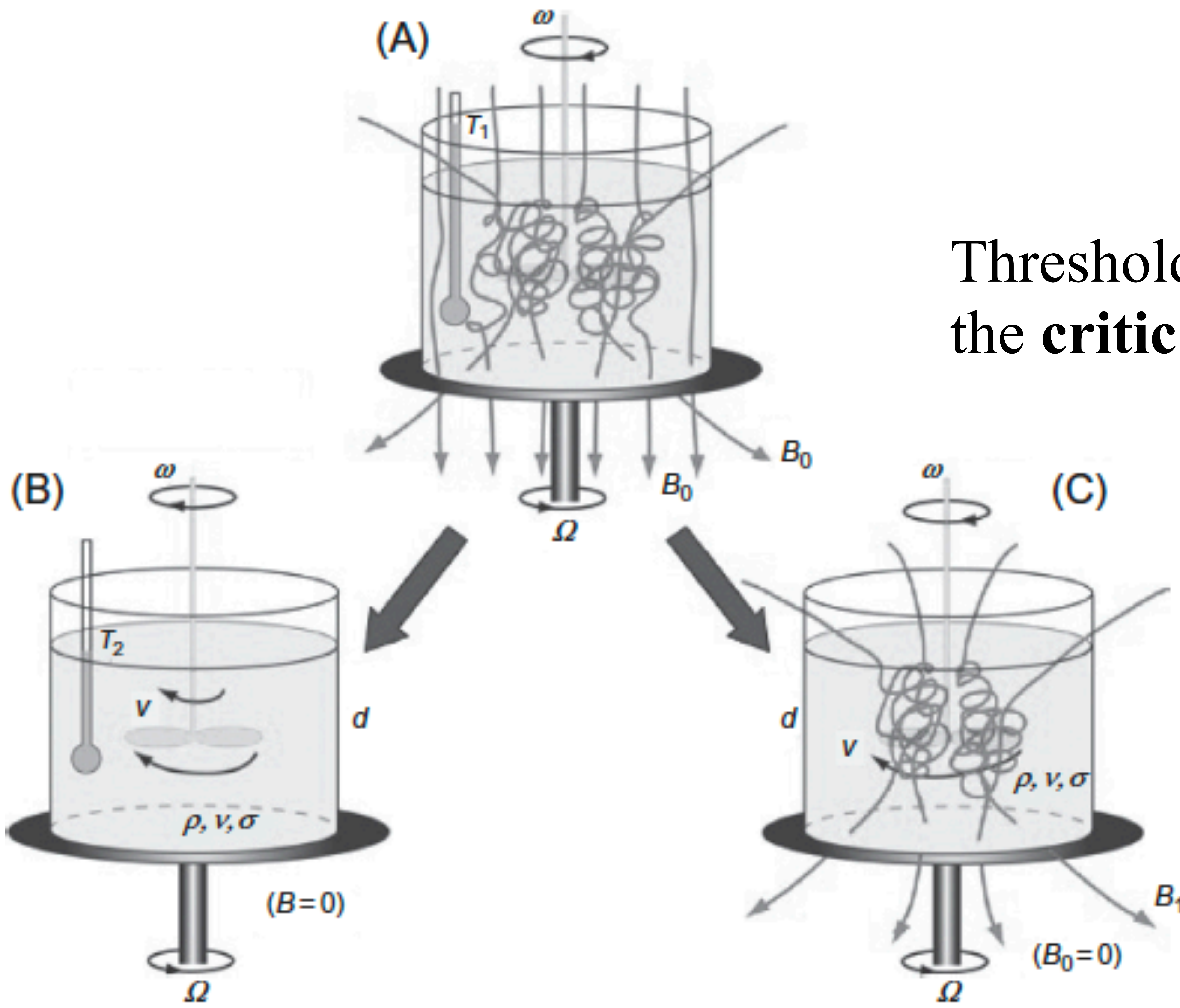
Fig 3.15, The Core

static
state



differential rotation, produces **magneto-hydrodynamic** state and induced field, B_1, outside the fluid

Step 1: Apply an external magnetic field to a rotating conducting fluid (A) and stir to induce electric currents and magnetic fields (B).



Threshold between (B) and (C) is the **critical state**

Step 2: Remove the external magnetic field in (A) and wait to see if dynamo is **subcritical** (B) or **supercritical** (C).

Critical State depends on magnetic Reynolds number

- Key dimensionless parameter $Rm = \mu_0 \sigma v d$.
- **Magnetic diffusivity** $\eta = (\mu_0 \sigma)^{-1}$, units length squared over time.
- Rm is effectively the ratio of 2 time scales: free decay time of electrical currents and magnetic fields due to electrical resistance τ_{mag}, and characteristic time for circulation of outer core fluid τ_{circ}.
- $\tau_{mag} = \dfrac{d^2}{\eta} \qquad \tau_{circ} = \dfrac{d}{v}$ so $\quad Rm = \tau_{mag}/\tau_{circ} = \dfrac{vd}{\eta}$
- Need a large Rm for a self-sustaining dynamo: i.e. $\tau_{mag} >> \tau_{circ}$
- In the core $\sigma \sim 4 - 20 \times 10^5$ S/m so $\eta \sim 0.4 - 2$ m^2/s, d is large and v smallish. Need $Rm > Rm_{crit} \sim 40$ for buoyancy driven dynamo action in core
- Lab dynamos use large v to get large Rm.

Useful Vector Identities: A is a vector, *s*,*t* are scalars

$\nabla \cdot (\nabla \times \mathbf{A}) = 0$ Divergence of a curl of a vector is zero (I1)

$\nabla \times (\nabla s) = 0$ Curl of a scalar field is zero (I2)

$\nabla(st) = s\nabla t + t\nabla s$ Product rule for vector gradient of scalars (I3)

$\nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s\nabla \cdot \mathbf{A}$ Product rule for divergence (I4)

$\nabla \times (s\mathbf{A}) = s\nabla \times \mathbf{A} + \nabla s \times \mathbf{A}$ Product rule for curl (I5)

Dot product rule $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A}$ (I6)

Cross product rules

$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$ (I7)

$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$ (I8)

$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ Curl of curl (I9)

For more see wikipedia.org/wiki/Vector_calculus_identities

$$\mathbf{A} \times \mathbf{B} = [A_y B_z - A_z B_y \ , \ A_z B_x - A_x B_z \ , \ A_x B_y - A_y B_x]$$

Toroidal and Poloidal Magnetic Fields

Recall that since $\mathbf{B}$ is solenoidal (effectively $\nabla \cdot \mathbf{B} = 0$) we can always write

$$\mathbf{B} = \nabla \times \mathbf{A}$$

A is a vector potential

Now consider a sphere of radius c (Earth's core) surrounded by an insulator. When $\mathbf{J}$ does not vanish in $r < c$, we need two scalars, not one to describe $\mathbf{B}$ completely.
We divide the vector potential into parts parallel to and perpendicular to $\mathbf{r}$ by writing

$$\mathbf{A} = T\mathbf{r} + \nabla P \times \mathbf{r} = T\mathbf{r} + \nabla \times (P\mathbf{r}) \tag{92}$$

where T and P are scalar functions of $\mathbf{r}$, known as the defining *scalars of the toroidal and poloidal fields.*
To find $\mathbf{B}$ we take the curl:

$$\mathbf{B} = \nabla \times (T\mathbf{r}) + \nabla \times \nabla \times (P\mathbf{r}) = \mathbf{B}_T + \mathbf{B}_P \tag{93}$$

and $\mathbf{B}_T$ is called the *toroidal part* of $\mathbf{B}$, while $\mathbf{B}_P$ is the *poloidal part.* This decomposition for $\mathbf{B}$ is unique and can always be done for all solenoidal vector fields (those with $\nabla \cdot \mathbf{F} = 0$).

A Physical/ Mathematical constraint added

Conventionally, the scalars are always restricted to a class of functions whose average value over every sphere is zero, that is

$$0 = \int_{S(r)} T(r\hat{\mathbf{r}})\, d^2\hat{\mathbf{r}} = \int_{S(r)} P(r\hat{\mathbf{r}})\, d^2\hat{\mathbf{r}}$$

With this property, the scalars become unique in (92), which means that if $\mathbf{B}_T$ vanishes, then $T = 0$, and similarly for $\mathbf{B}_P$.

We can’t see the toroidal magnetic field from outside the core

- The toroidal magnetic field has no radial component:

$$\boldsymbol{B}_T = (\underbrace{\nabla \times \boldsymbol{r}}_{=0})T - \boldsymbol{r} \times \nabla T = -\boldsymbol{r} \times \nabla T \quad \longrightarrow \quad \hat{\boldsymbol{r}} \cdot \boldsymbol{B}_T = 0$$

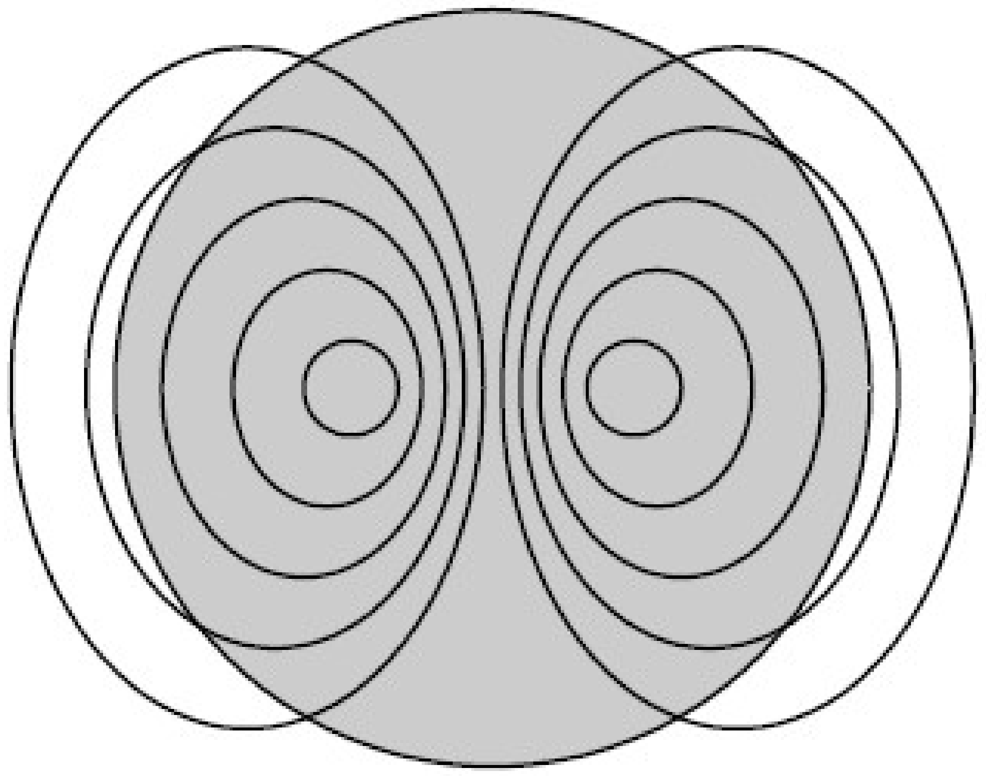
- $\boldsymbol{B}_T$ vanishes outside the conducting sphere. Hence the toroidal part of $\boldsymbol{B}$ in Earth's core is invisible outside the core and only the poloidal part, $\boldsymbol{B}_P$, has any detectable influence at the Earth's surface;

The toroidal magnetic field has no radial component. The lines of force lie on spherical surfaces and are thus confined to the interior of the conducting sphere. If we think of the sphere as Earth's core, and say that outside the core we have $\mathbf{J} = 0$ then

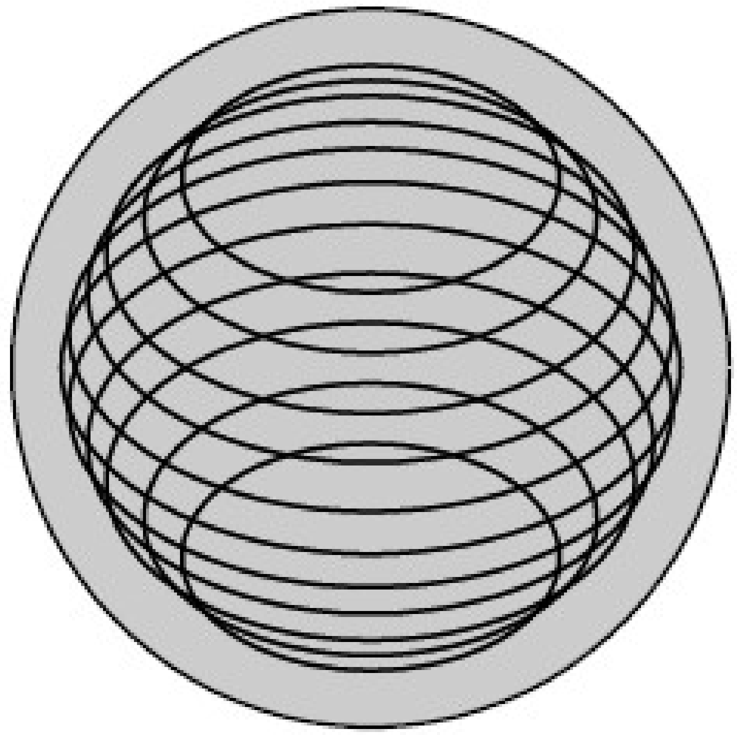
$$\mathbf{B} = -\nabla\Psi, \qquad \text{with } \nabla^2\Psi = 0. \tag{96}$$

Now $\mathbf{B}$ is continuous at $S(c)$ and since $\hat{\mathbf{r}} \cdot \mathbf{B}_T = 0$ just inside we conclude that $\hat{\mathbf{r}} \cdot \mathbf{B}_T$ is also zero just outside the core. But the equivalent source theorem tells us that any harmonic function with internal sources and vanishing radial component on $S(c)$ is identically zero outside. $\mathbf{B}_T$ vanishes outside the conducting sphere. The toroidal part of $\mathbf{B}$ in Earth's core is invisible outside the core and only the poloidal part, $\mathbf{B}_P$, has any detectable influence at the Earth's surface.

Typical B_p field lines

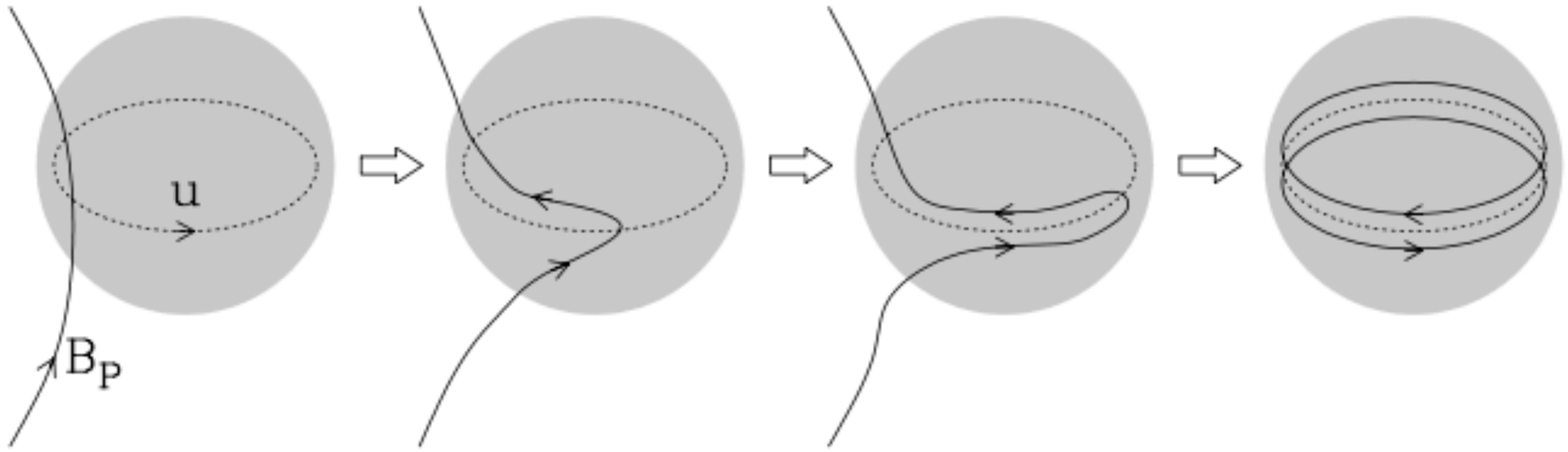


Typical B_T field lines

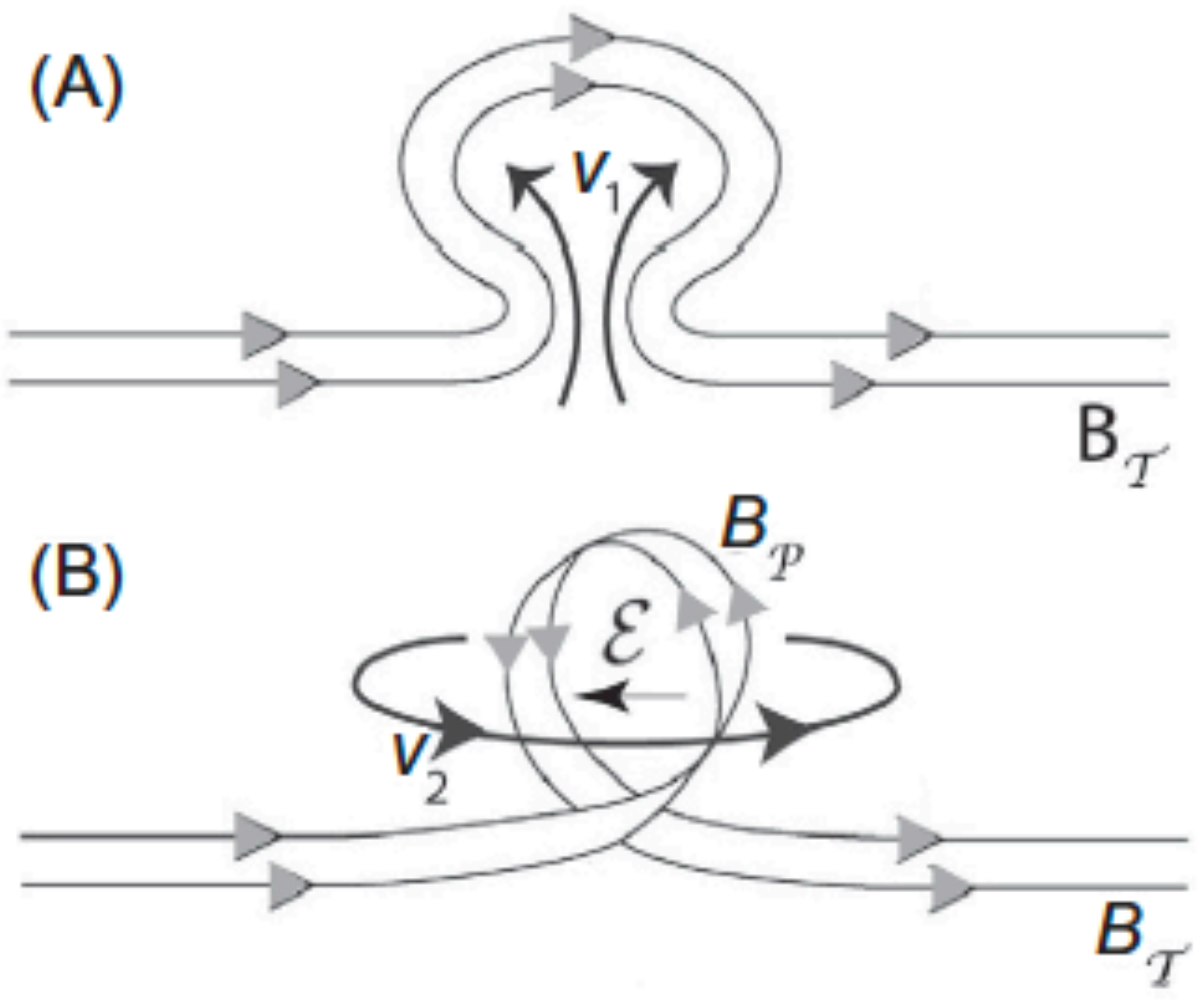


Both toroidal and poloidal magnetic fields are needed to make a self sustaining dynamo - for example

alpha-Ω model
Ω effect transforms large scale poloidal field into toroidal field



Then alpha effect drives helicity from overall rotation



3.3 The geodynamo process 99

FIG. 3.21 Production of poloidal magnetic field $B_{\mathcal{P}}$ from a toroidal magnetic field $B_{\mathcal{T}}$ by a helical convective eddy. Flow in the eddy consists of two parts, a convective upwelling $\mathbf{v}_1$ in (A) and a quasigeostrophic circulation around the upwelling $\mathbf{v}_2$ in (B) with vorticity in the same direction as the upwelling, generating positive kinematic helicity. Positive helicity induces an e.m.f. $\mathcal{E}$ antiparallel to $B_{\mathcal{T}}$, and a loop of poloidal magnetic field with the polarity indicated by the *filled arrows*.

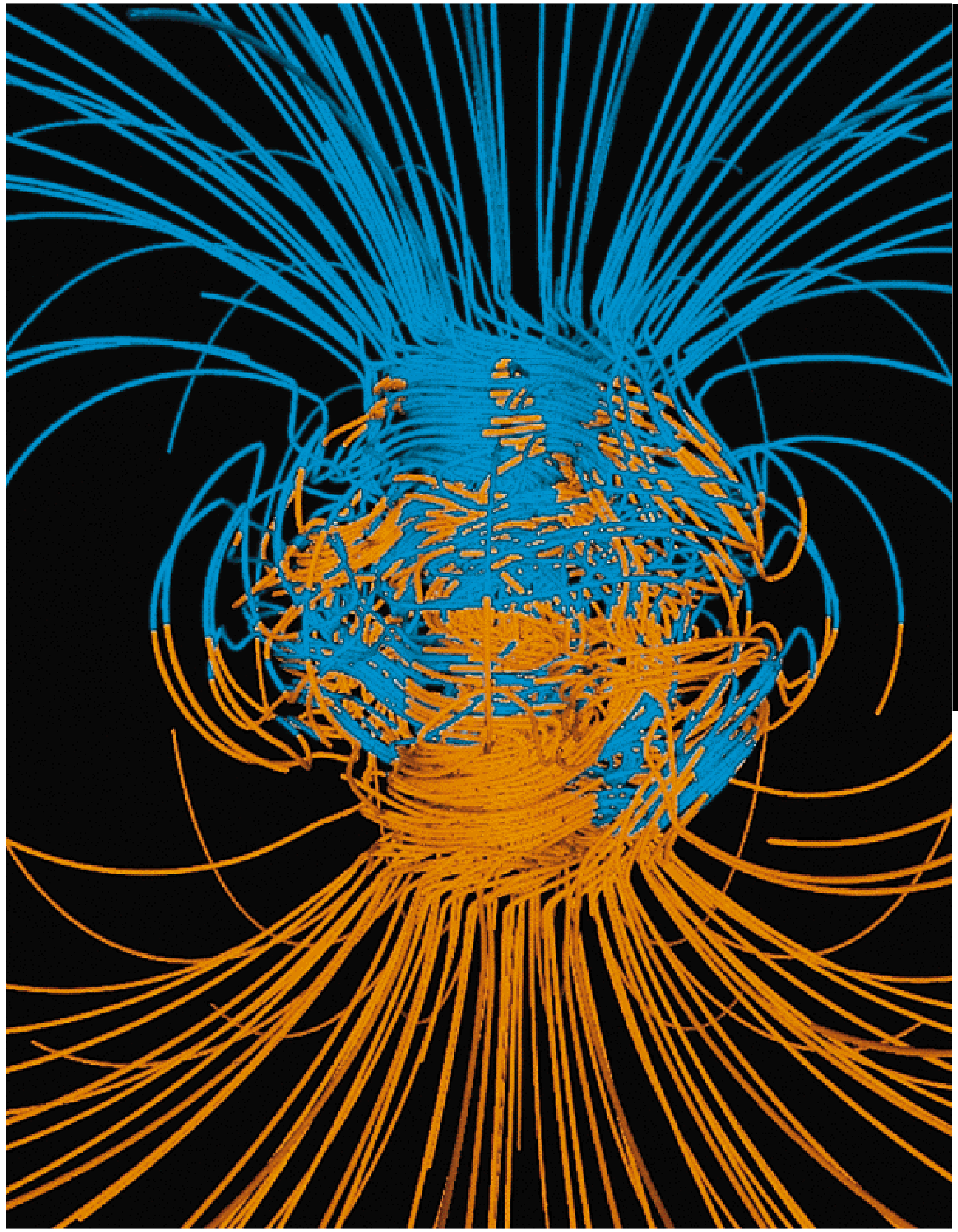
Helicity is defined as the correlation between the fluid velocity, v, and its vorticity, $\nabla \times v$:

$$H = v \cdot \nabla \times v$$

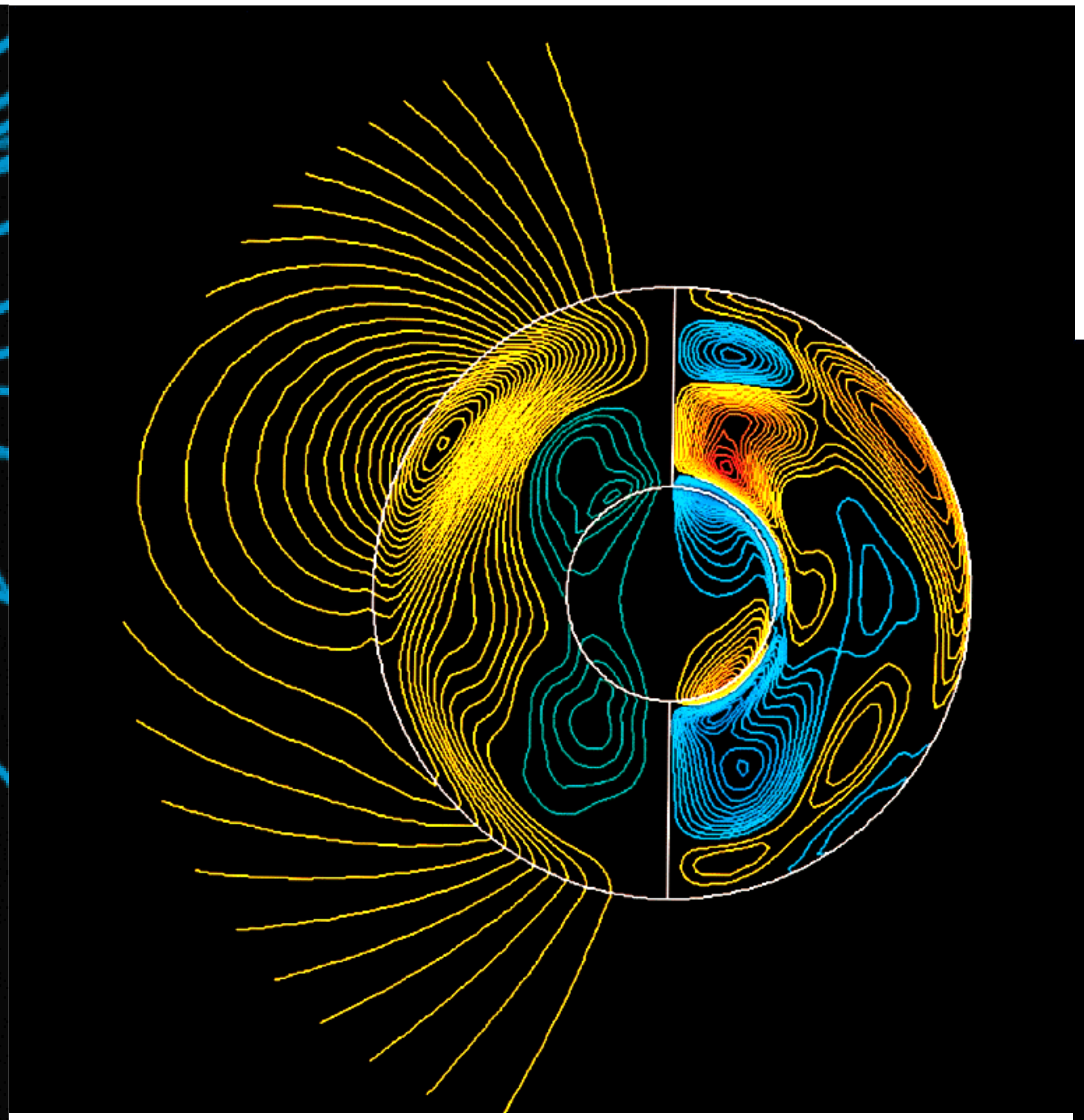
Reversals and excursions are the most extreme variations in the geomagnetic field, but it is hard to get detailed records

We use both direct and paleomagnetic observations of the field and physics based numerical simulations to infer core fields and their changes - the latter are illustrated here.

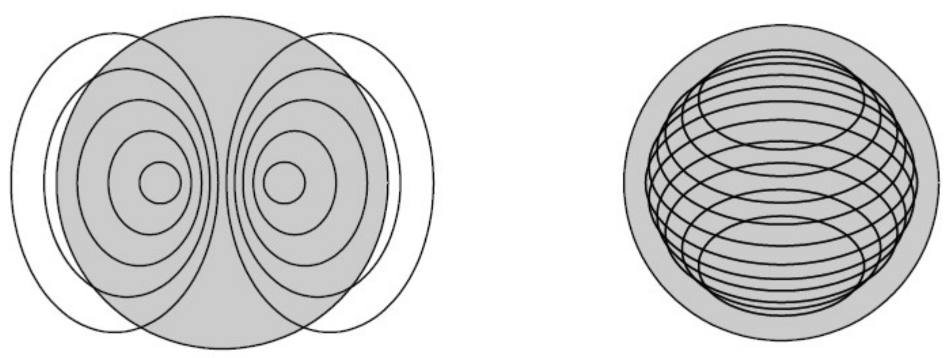
Left hemispheres are poloidal, right are toroidal, all are longitudinal averages

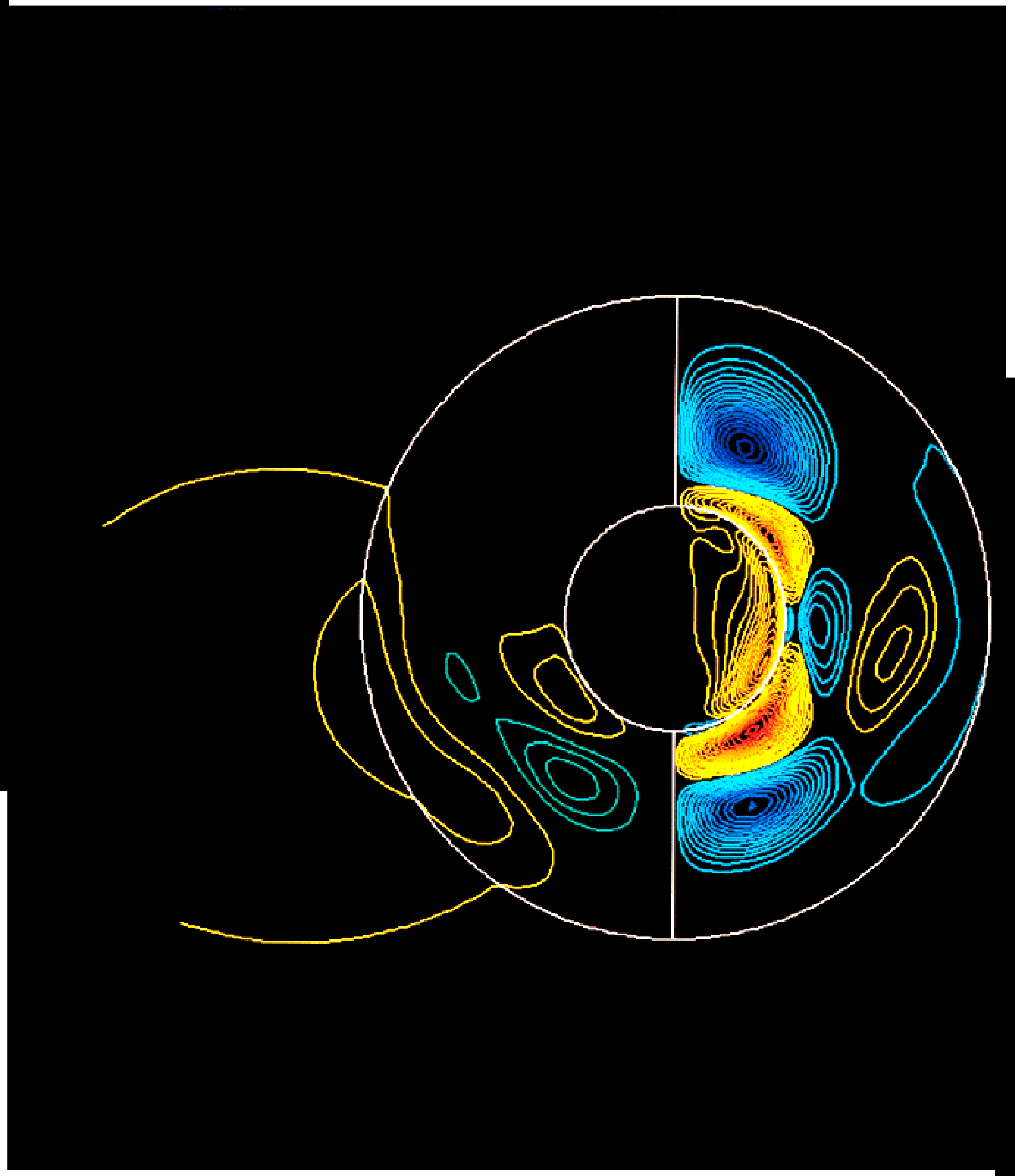


Glatzmaier & Roberts, 1995,
doi: 10.1038/377203a0

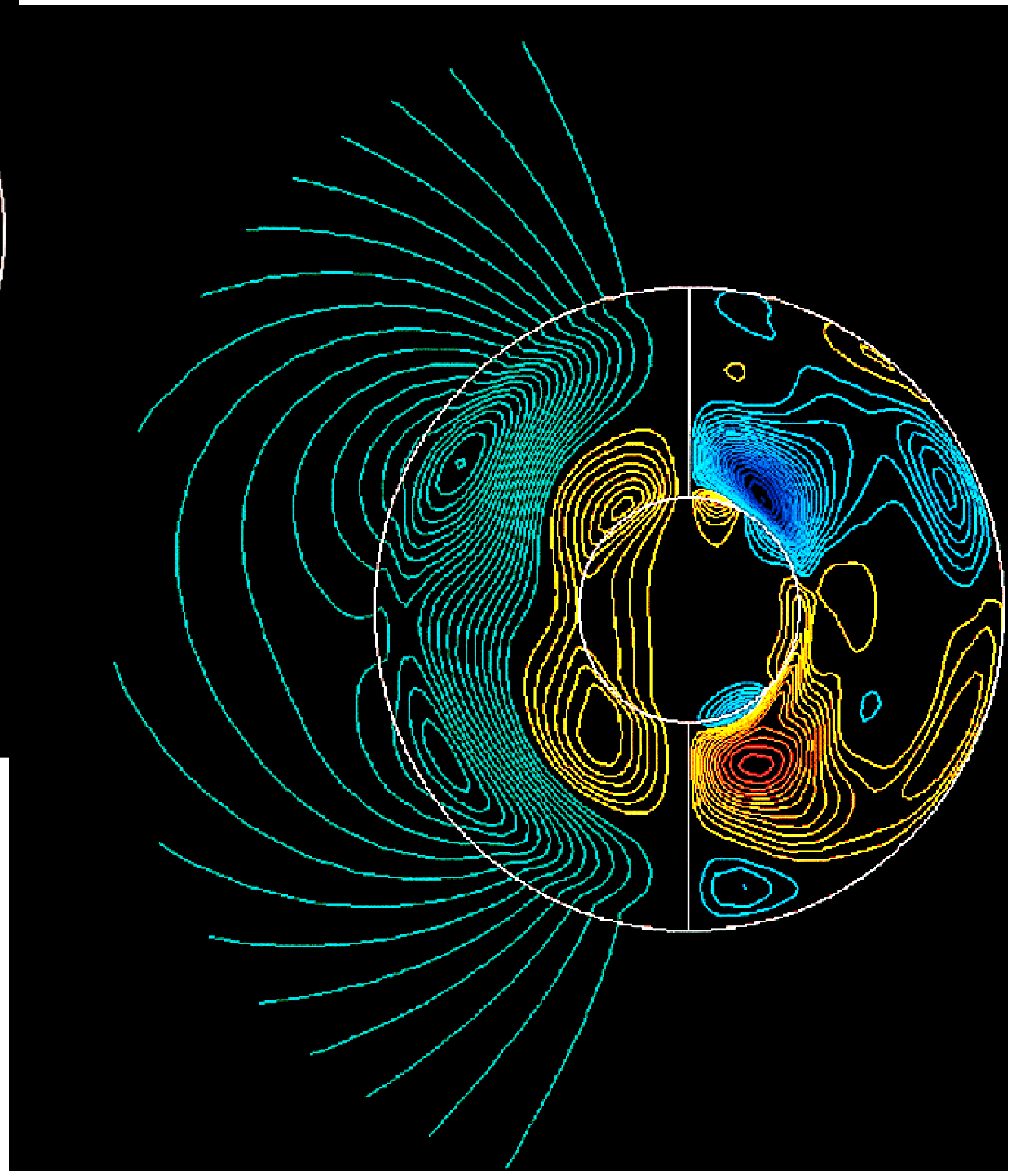


Before





During



After

Toroidal and poloidal fields

- We can make a similar decomposition for the current flow in core:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot \boldsymbol{J} = 0$$

$$\boldsymbol{J} = \boldsymbol{J}_T + \boldsymbol{J}_P$$

- $\boldsymbol{J_T}$ and $\boldsymbol{J_P}$ are the toroidal and poloidal current flow in core;
- Toroidal fields ($\boldsymbol{B}_T$) are generated by poloidal currents ($\mathbf{J}_P$);
- Poloidal fields ($\boldsymbol{B}_P$) are generated by toroidal currents ($\mathbf{J}_T$).

$$\mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B} = \nabla \times \nabla \times (T \boldsymbol{r}) + \nabla \times \nabla \times \nabla \times (P \boldsymbol{r}) = \mu_0 (\boldsymbol{J}_P + \boldsymbol{J}_T)$$

Another physical norm that can be used for regularization

Ohmic Heating Norm

$$\mu_0^2 \int_{r<c} \boldsymbol{J}_T \cdot \boldsymbol{J}_T \, d^3\boldsymbol{r} \geq c \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{(l+1)(2l+1)^2(2l+3)}{l} |b_{lm}|^2$$

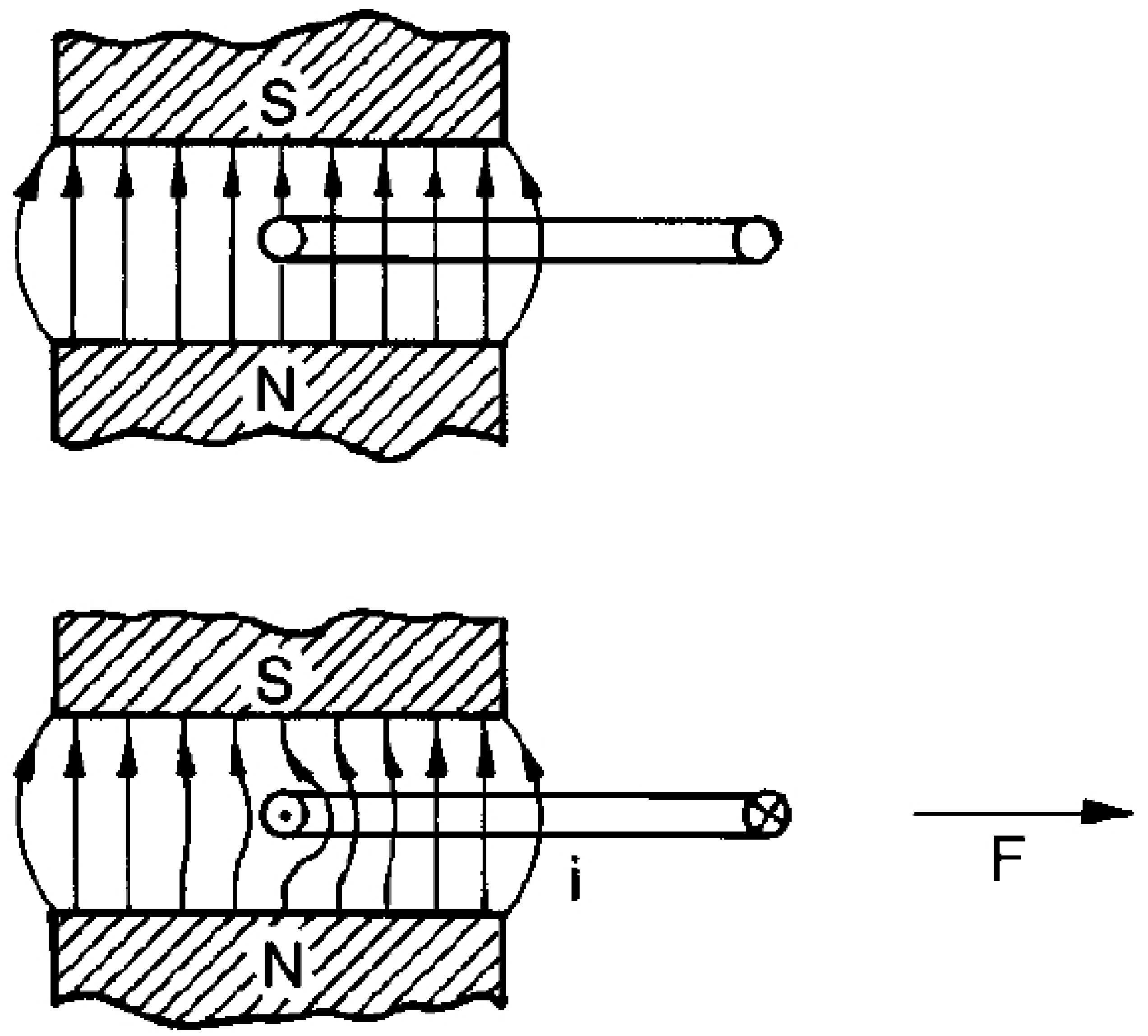
$$Q = \int_{r<c} \frac{\boldsymbol{J}_T \cdot \boldsymbol{J}_T}{\sigma} d^3\boldsymbol{r} \geq \frac{c}{\mu_0^2 \sigma} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{(l+1)(2l+1)^2(2l+3)}{l} |b_{lm}|^2$$

Getting into the Physics of the Core

Fundamentals of Electromagnetism and the Induction Equation

• Induction by a moving conductor:

To begin, let us consider what happens when an electrical conductor is pulled through a magnetic field:



Interaction of a magnetic field and a moving wire loop.

3 effects result:

(1) An electrical current is induced in the conductor.
(2) This current causes a magnetic field that adds to the original field, such that the conductor appears to drag the field along with it.
(3) The combined magnetic field interacts with the current resulting in a Lorentz force that acts on conductor, opposing its motion.

(Davidson, 2001)

Ohm's law in a moving reference frame

• Ohm's law is an empirical law stating the experimentally observed relation between the electric fields and electric current density.

• For stationary conductors it takes the form:

$\boldsymbol{J} = \sigma \boldsymbol{E}$ where σ is the electrical conductivity $\left(\Omega^{-1} m^{-1}\right)$

• When considering a moving electrical conductor, the effective electric field in the frame moving with the conductor must be used:

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \boldsymbol{E}'\right) = \sigma\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right)$$

• For Earth's core (predominantly liquid Fe at high P,T) the electrical conductivity is thought to be large $\sim 0.5 \times 10^6\, \Omega^{-1} m^{-1}$

The Magnetohydrodynamic (MHD) Approximation of Electrodynamics

- For moving conductors (where $u^2 \ll c^2$) and considering slow changes in the EM fields $\left((l/\tau)^2/c^2 \ll 1\right)$ then the evolution of the magnetic fields and electric currents are specified by:

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right)$$

Note 1: Only time-derivative of the magnetic field remains.
Note 2: Neglect of the displacement current means we no longer need to consider the Gauss's electrostatic equation.

Often referred to as:
"the magnetohydrodynamic (MHD) approximation of electrodynamics"

The Magnetic Induction Equation

- Recall the equations governing electrodynamics under the MHD approximation:

$$\nabla \cdot \boldsymbol{B} = 0 \qquad (1)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (2)$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \qquad (3)$$

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \qquad (4)$$

- Substituting from (4) into (3) gives

$$\frac{1}{\mu_0 \sigma}(\nabla \times \boldsymbol{B}) = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} \qquad (5)$$

- Take the curl of this and using the magnetic diffusivity $\eta = 1/\mu_0 \sigma$

$$\nabla \times (\eta \nabla \times \boldsymbol{B}) = \nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \qquad (6)$$

The Magnetic Induction Equation

- Next, substituting from (2) into (6) and rearranging,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}) \qquad (7)$$

- If η = constant then, we can use a standard vector calculus identity together with (1) to re-write the last term as

$$\nabla \times (\eta \nabla \times \boldsymbol{B}) = \eta \nabla \times (\nabla \times \boldsymbol{B}) = \eta (\nabla (\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}) = -\eta \nabla^2 \boldsymbol{B}$$

- Substituting this into (7) we arrive at the Magnetic Induction equation:

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}} \qquad (8)$$

- Thus, under the MHD approximation, if we know the motion of the conductor and the present magnetic field, we can calculate how the field evolves in time.

Magnetic Reynolds Number

- Assume that the velocity field has a characteristic magnitude U
- Assume that the magnetic field has a characteristic magnitude B
- Assume that the lengthscale over which both fields change is L
- Then the ratio of the magnitudes of the terms on the RHS will be:

$$\frac{|\nabla\times(\boldsymbol{u}\times\boldsymbol{B})|}{\eta\nabla^2\boldsymbol{B}}=\frac{U\,B/L}{\eta B/L^2}=\frac{U\,L}{\eta}=\mu_0\sigma U\,L=R_m$$

- R_m is known as the magnetic Reynolds number.

- For global motion in Earth's core (L is the core diameter and $U \approx 2mm/s$)

$$R_m\approx 2\times 10^{-3}\times 7\times 10^6/1.6=8800$$

Diffusion of the Magnetic Field

Suppose that R_m is small and neglect the advection term, then

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B}$$

$$\boldsymbol{B}(x,t) = \boldsymbol{B}_0 e^{2\pi i x} e^{-t/t_0}$$

$$\int_{|R^3} d^3 \boldsymbol{s} \, |\nabla B_j|^2 \geqslant (\pi/c)^2 \int_{|R^3} d^3 \boldsymbol{s} \, B_j^2$$

- The longest characteristic decay time:

$$c^2/\pi^2 \eta \; = \; 7.7 \times 10^{11} s \; = 55{,}000 \text{ years}$$

Really do need a mechanism to continually re-generate the geomagnetic field

Perfect Conductivity: Frozen Flux

- Consider the case in which the second term on the right hand side is negligible (i.e. perfect conductivity), then

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

Alfvén's theorem:
Magnetic field lines move with a perfectly conducting fluid as though frozen to it.

Using the standard vector calculus relation,

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} \quad \text{since} \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{B} = 0$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} \quad \text{or} \quad \frac{D\boldsymbol{B}}{Dt} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$$

Advection of magnetic field along with flow

Stretching of magnetic field by shear of flow

Now consider a material patch of outer core fluid with surface area A oriented by a normal unit vector $\hat{\mathbf{n}}$.

The total magnetic flux Φ must obey the important condition

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int (\mathbf{B} \, . \, \hat{\mathbf{n}}) dA = 0 \quad \text{Alfvén's Theorem}$$

Total magnetic flux within the path is conserved in time. Magnetic flux moves as though frozen into the conductor.

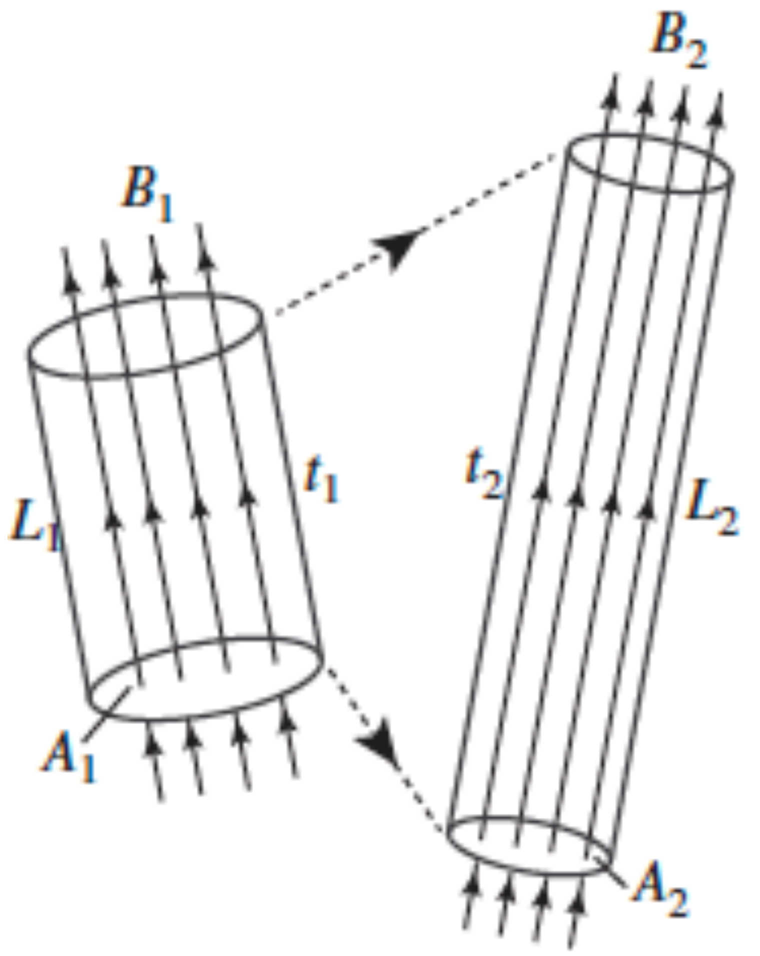


FIG. 3.17 Illustration of magnetic field intensification by fluid motion in the frozen flux (perfect conductor) limit. Stretching of an incompressible perfectly conducting cylinder containing magnetic flux $\Phi = A_1 B_1$ at t_1 and conserves Φ at t_2 but intensifies the magnetic field according to $B_2/B_1 = L_2/L_1 = A_1/A_2$.

Shape of the contour evolves with time but the total flux is invariant.

Perfect Conductivity: Frozen Flux

$$\frac{d}{dt}\int_S \boldsymbol{B}\cdot d\boldsymbol{S} = \int_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot d\boldsymbol{S} + \oint_C \boldsymbol{B}\cdot \boldsymbol{u}\times d\boldsymbol{l} =$$

$$= \int_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot d\boldsymbol{S} - \oint_C \boldsymbol{u}\times \boldsymbol{B}\cdot d\boldsymbol{l}$$

$$= \int_S \left(\frac{\partial \boldsymbol{B}}{\partial t} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B})\right)\cdot d\boldsymbol{S} = 0$$

The total flux enclosed by a material surface cannot change with time even if the shape of the contour evolves : FLUX IS FROZEN