SIOG 231
GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 16

Modeling in Higher Dimensions

2/29/2024

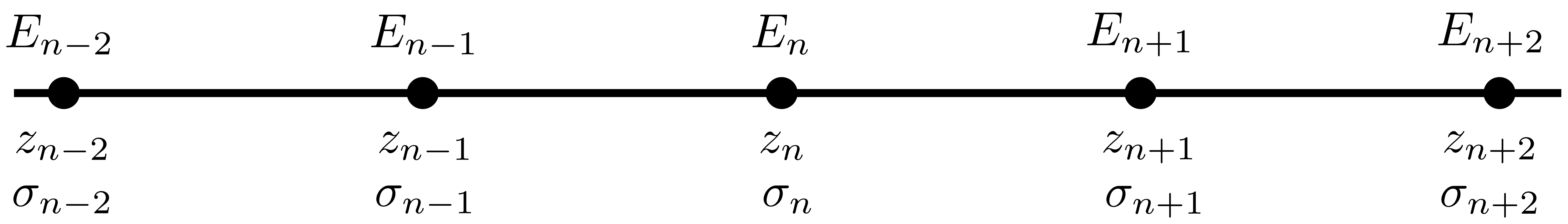
In lecture 12 we showed that the governing equation for the 1D problem was the same as for a half-space of constant conductivity:

$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

which reduces to solving

$$\text{Given} \quad \sigma(z) \quad \text{solve} \quad \frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z) \quad \text{for} \quad E(\infty) = 0 \quad \text{and} \quad E'(0) = -1$$

where *E(0)* at the surface was our admittance, c. We can solve this equation numerically using a **finite difference** approach where conductivity is defined on nodes

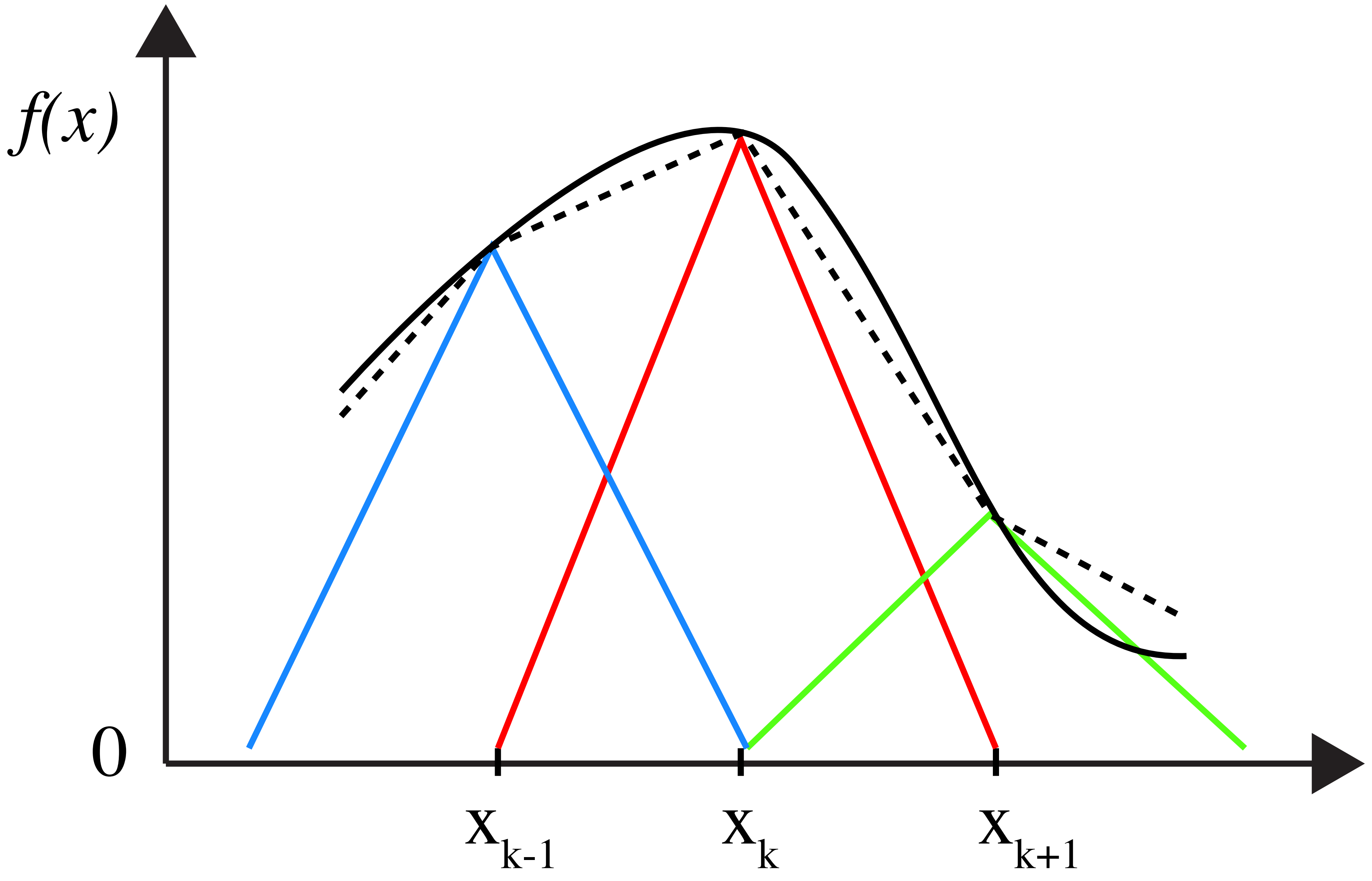


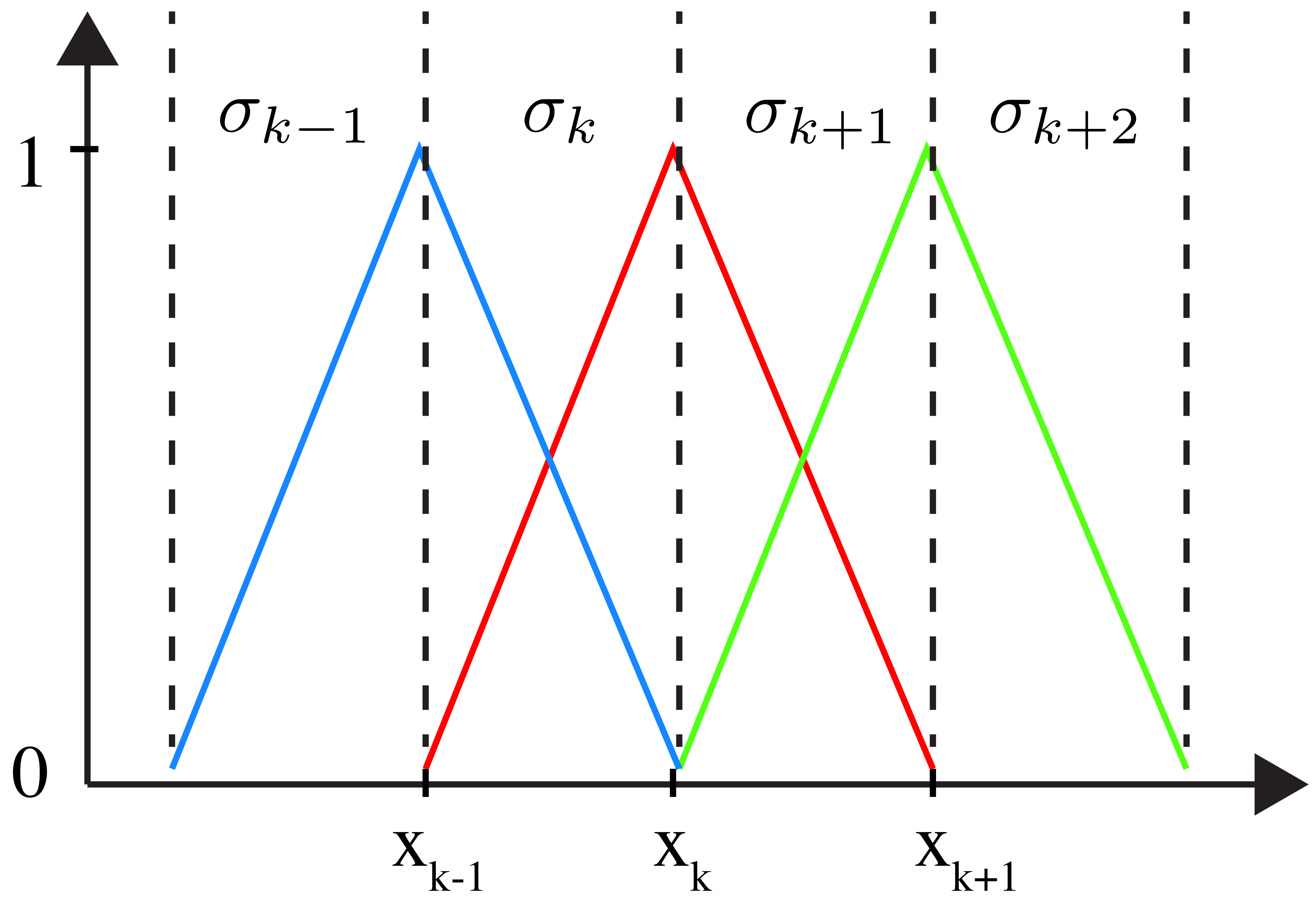
and we cast the solution as a linear system of equations

$$\frac{E_{n+1} - 2E_n + E_{n-1}}{\Delta z^2} - i\omega\mu_0\sigma_n E_n = 0 \qquad n = 2, 3, ..., N-1 \qquad \frac{E_2 - E_1}{\Delta z} = -1 \qquad E_N = 0$$

$$\mathbf{A}E = b$$

Alternatively, in the **finite element** approach, we defined conductivity on elements, and used linear basis functions to describe how *E* behaves



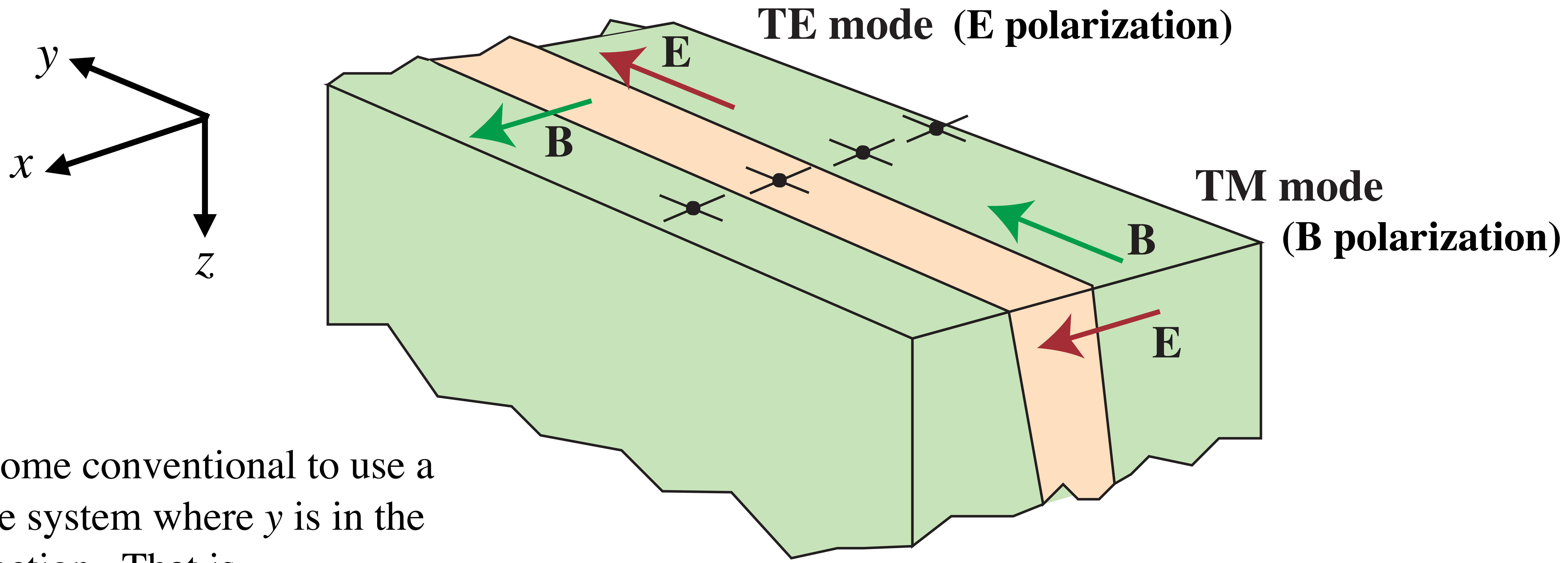


This produces a more complicated but still linear system

$$\sum_{k=1}^{n} \int_{\Omega_j} \left[v_k'(z) v_j'(z) + i\omega\mu_o \sigma(z) v_k(z) v_j(z) \right] dz E_k = 0 \quad \text{for } j = 1, ..., n$$

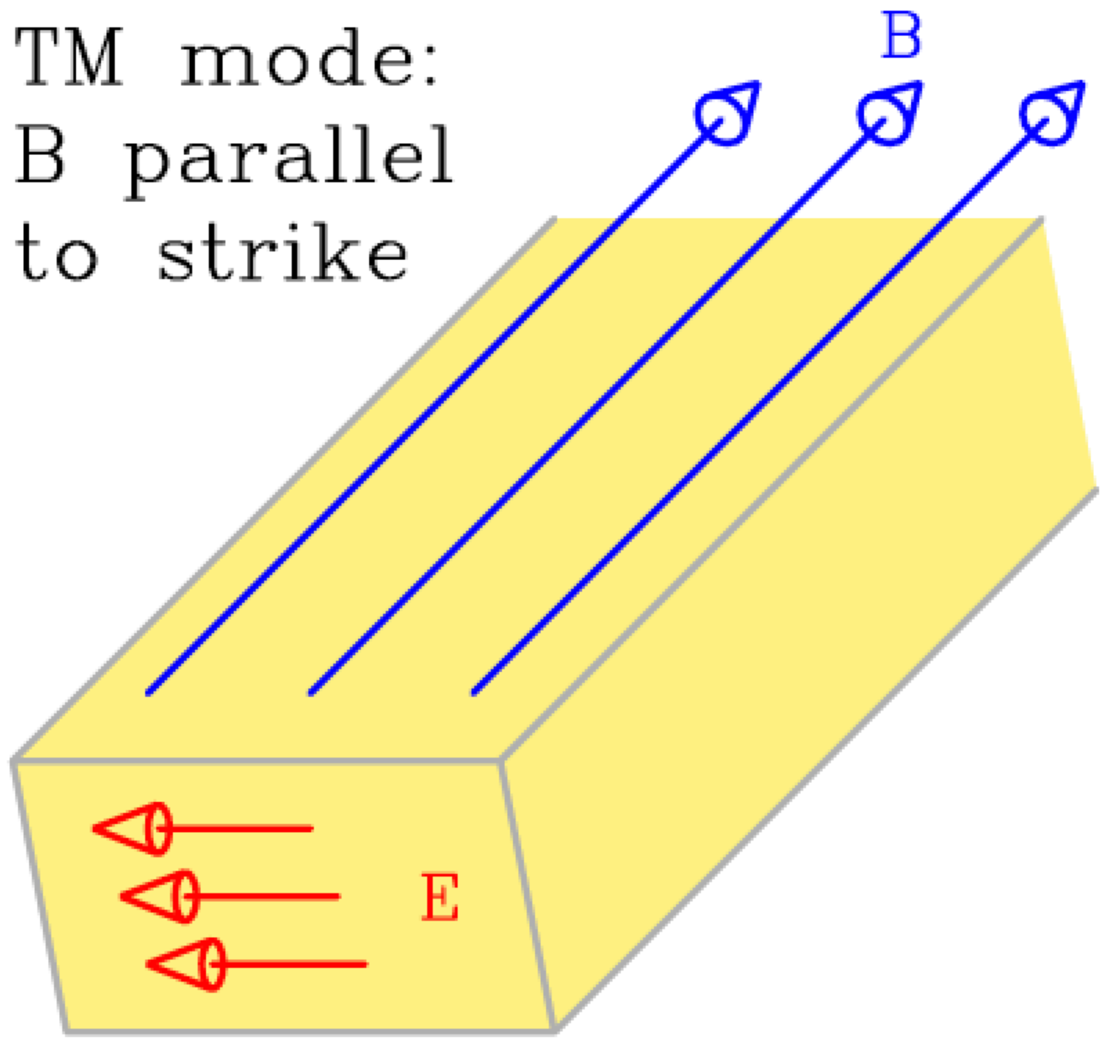
$$\mathbf{A}E = b$$

We can extend the finite difference and finite element approaches to 2D and 3D models, but the governing equations and boundary conditions are quite different. We saw that in 2D MT we could consider two independent modes, Transverse Magnetic and Transverse Electric. These two modes have different governing equations.



It has become conventional to use a coordinate system where *y* is in the strike direction. That is, conductivity only varies in *x* and *z*

Transverse Magnetic (TM) mode



The magnetic field is parallel to strike, so

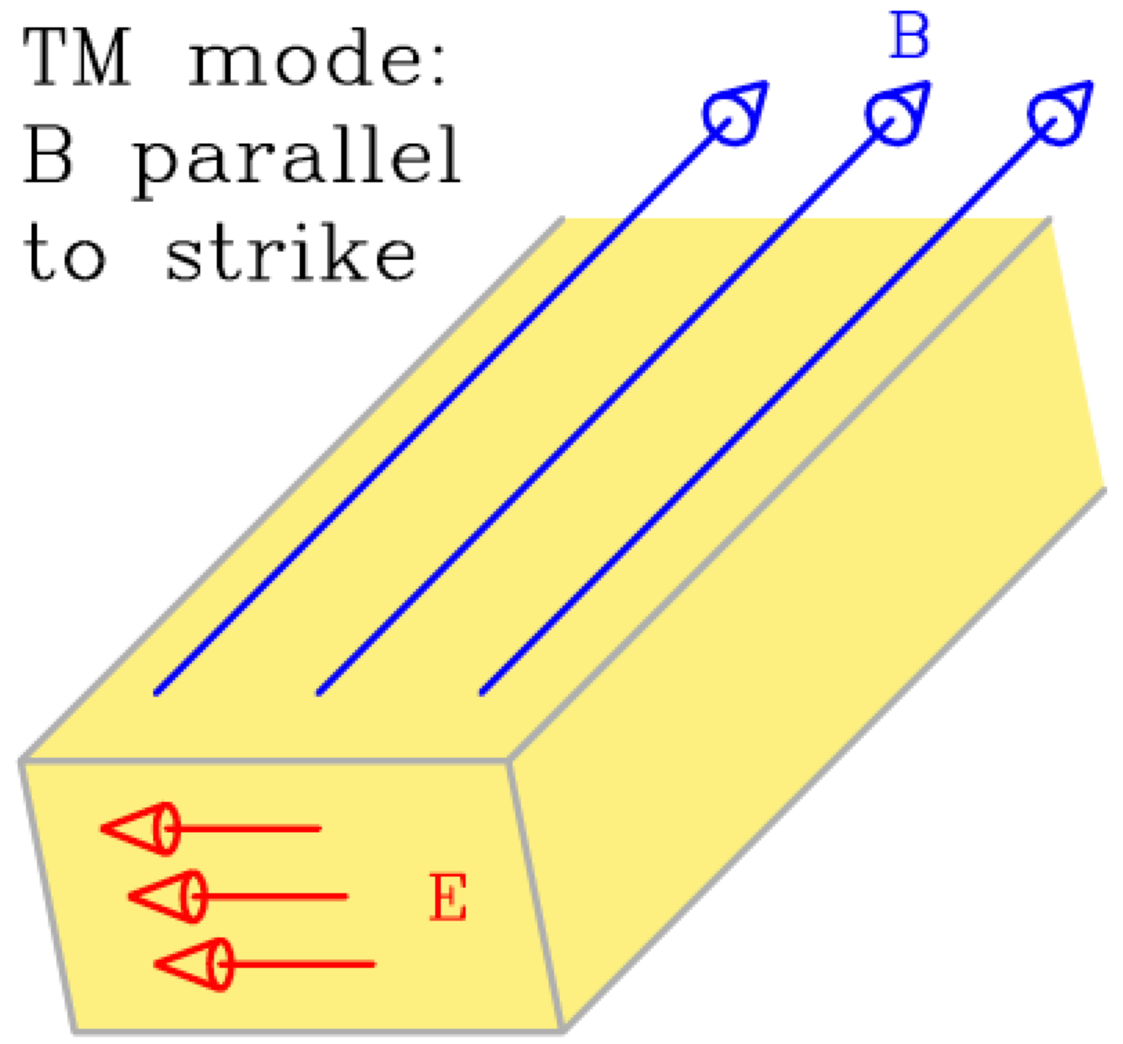
$$\mathbf{B} = B(x, y)e^{i\omega}\hat{\mathbf{y}}$$

where $B(x, y)$ is a complex, scalar function. Note that

$$\nabla \cdot \mathbf{E} \neq 0$$

so for the TM mode, we usually eliminate $\mathbf{E}$ and work in $\mathbf{B}$.

Transverse Magnetic (TM) mode



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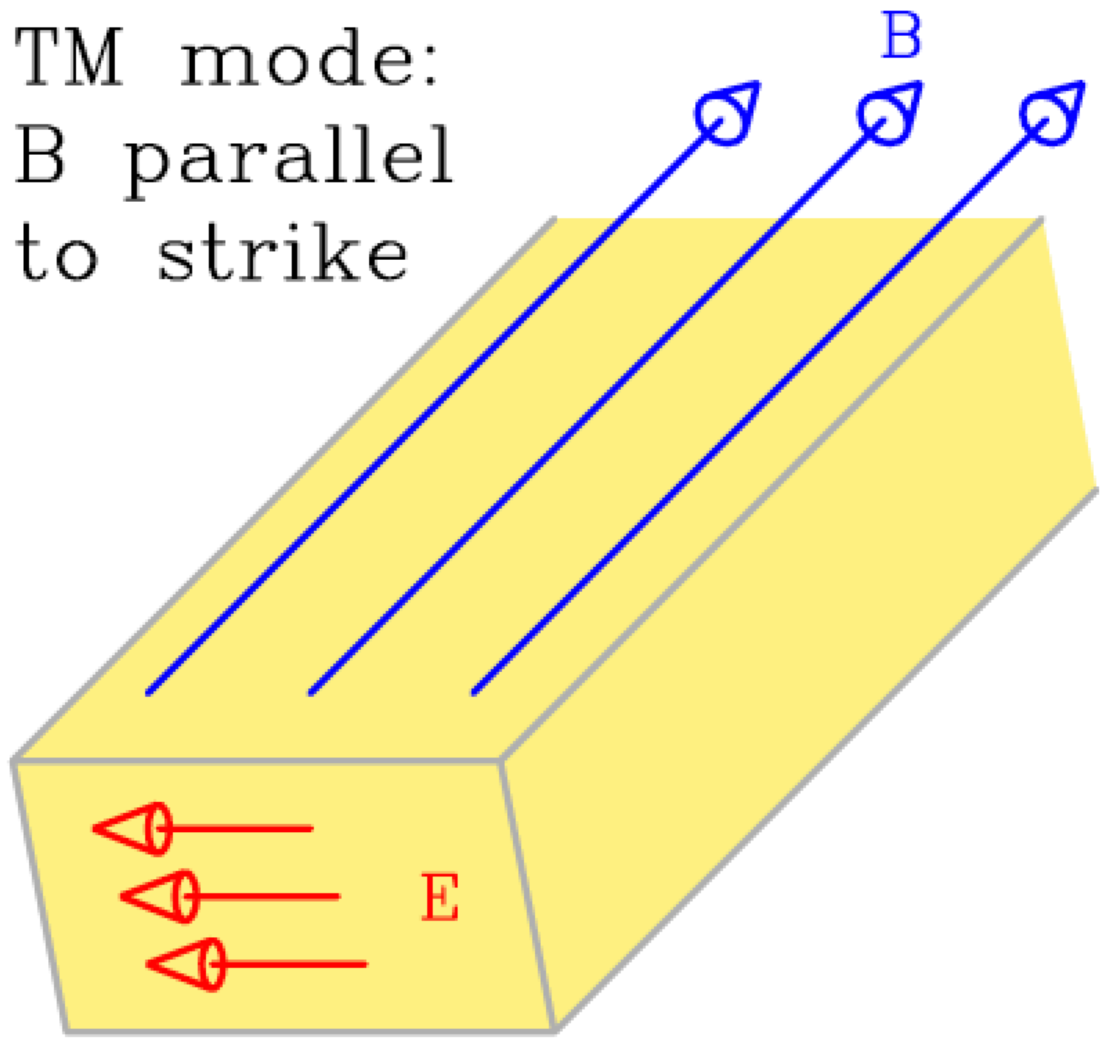
$$\mathbf{B} = B(x,y)e^{i\omega}\hat{\mathbf{y}}$$

We have our Maxwell equations for a single frequency ω and Ohm's Law

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} \qquad \nabla \times \mathbf{E} = -i\omega \mathbf{B} \qquad \mathbf{J} = \sigma \mathbf{E}$$

Transverse Magnetic (TM) mode



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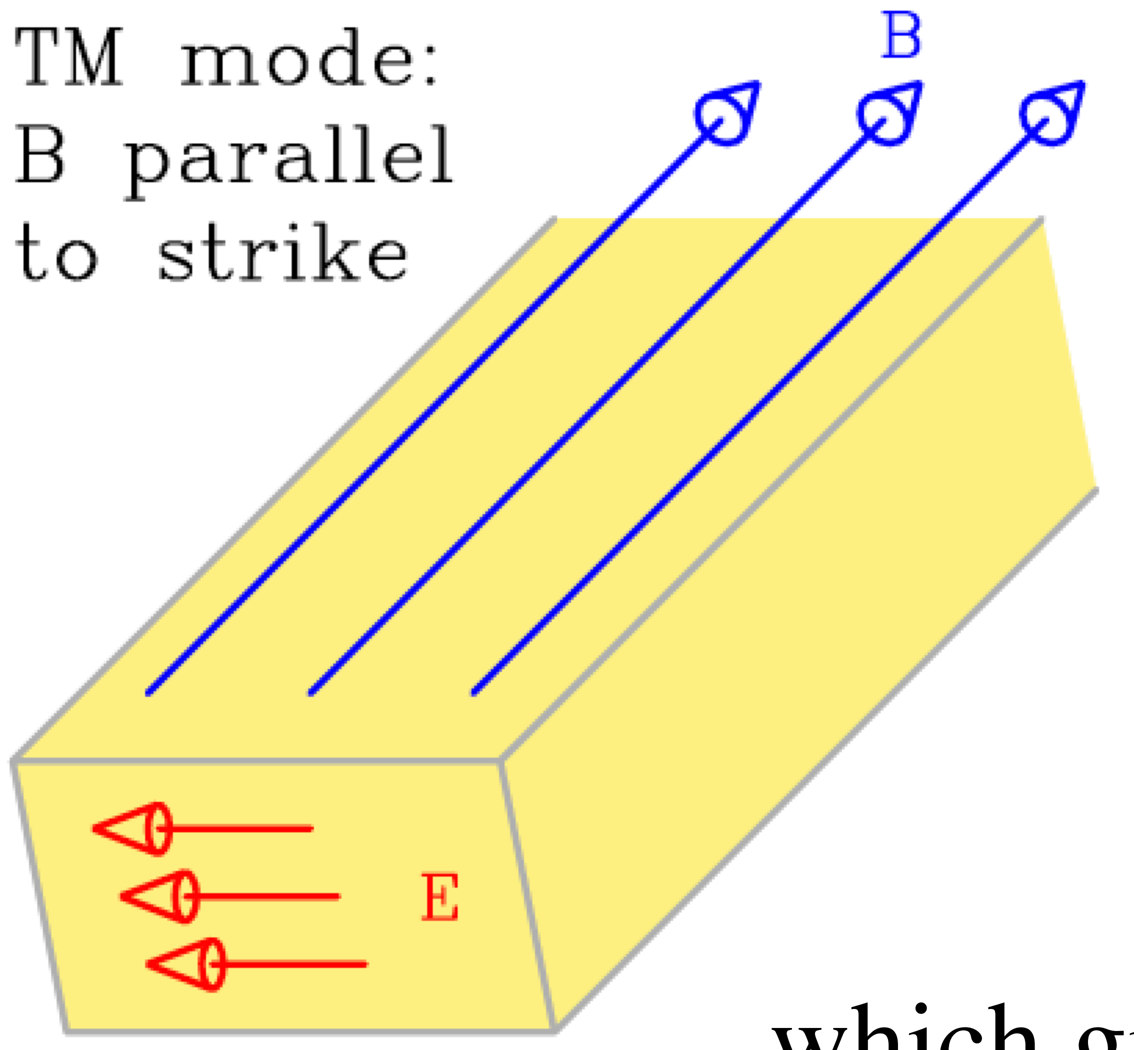
$$\rightarrow \quad \nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E} \quad \rightarrow \quad \nabla \times \left(\frac{1}{\sigma} \nabla \times \mathbf{B} \right) = \mu_o \nabla \times \mathbf{E}$$

Transverse Magnetic (TM) mode



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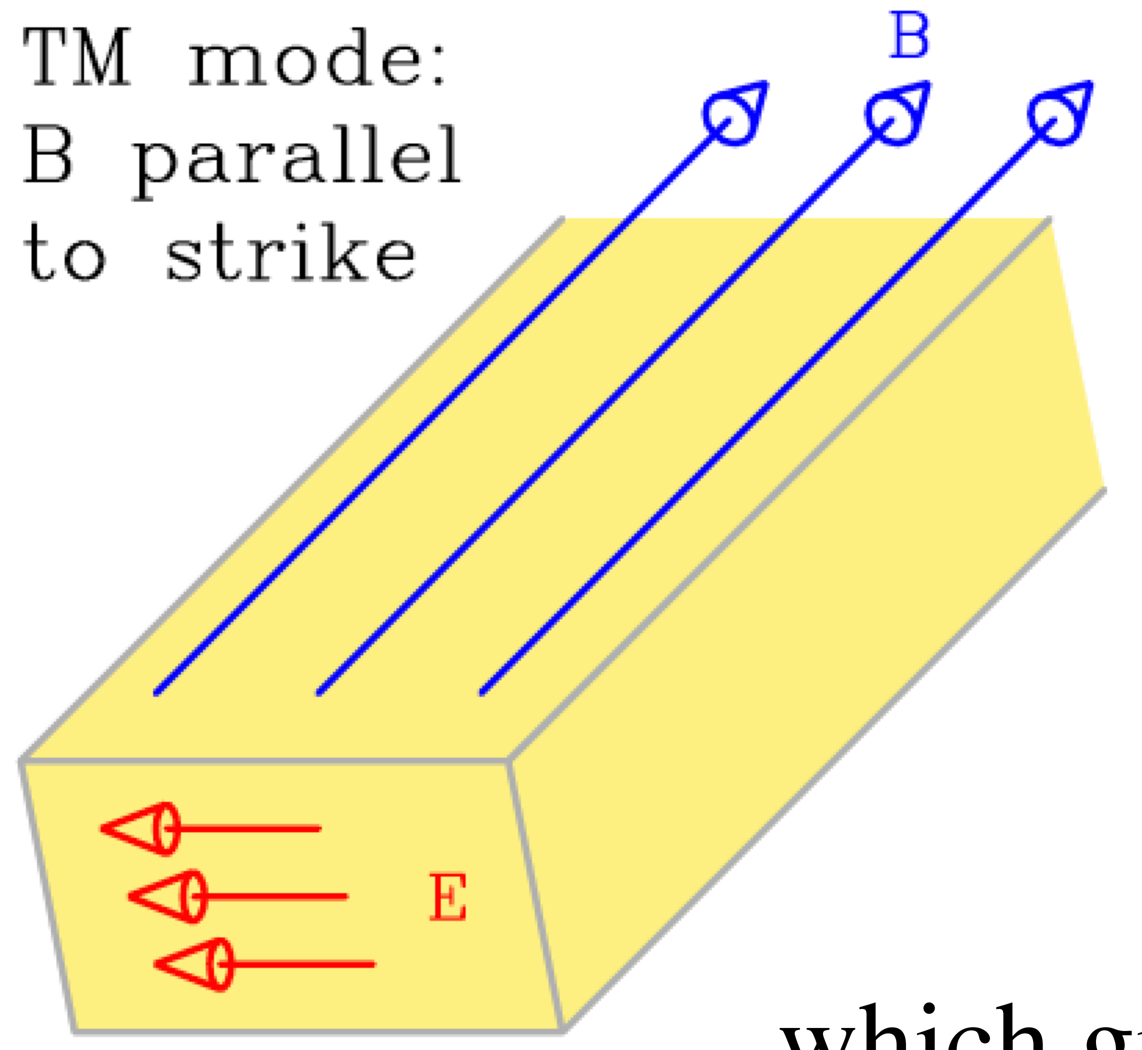
which gives a curl-curl equation in **B** $\quad \nabla \times \left(\frac{1}{\sigma}\nabla \times \mathbf{B}\right) = -i\omega\mu_o \mathbf{B}$

(Note that when σ was constant $\nabla \times \nabla \times \mathbf{B} = -i\omega\mu_o\sigma\mathbf{B}$)

Transverse Magnetic (TM) mode



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(Note that when σ was constant $\nabla \times \nabla \times \mathbf{B} = -i\omega\mu_o\sigma\mathbf{B}$)

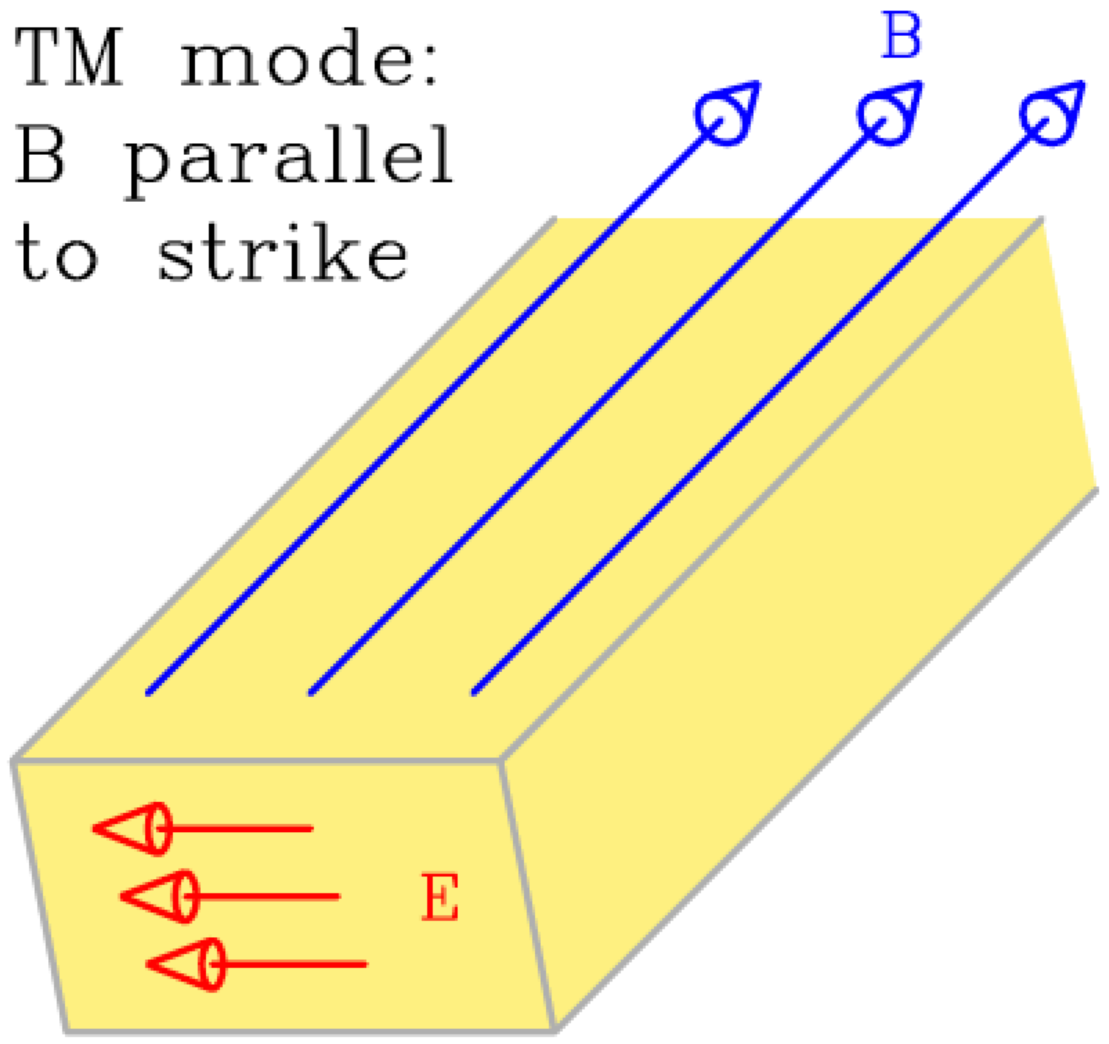
In terms of resistivity $\quad -i\omega\mu_o \mathbf{B} = \nabla \times (\rho \nabla \times \mathbf{B})$

Substituting for **B** $\quad -i\omega\mu_o B\hat{\mathbf{y}} = \nabla \times (\rho \nabla \times B\hat{\mathbf{y}}) = \nabla \times (\rho \nabla B \times \hat{\mathbf{y}})$

where we have used $\nabla \times s(\mathbf{A}) = \mathbf{s}\nabla \times \mathbf{A} + \nabla \mathbf{s} \times \mathbf{A}$ and $\nabla \times \hat{\mathbf{y}} = 0$

Transverse Magnetic (TM) mode

$$-i\omega\mu_o B\hat{\mathbf{y}} = \nabla \times (\rho\nabla B \times \hat{\mathbf{y}})$$



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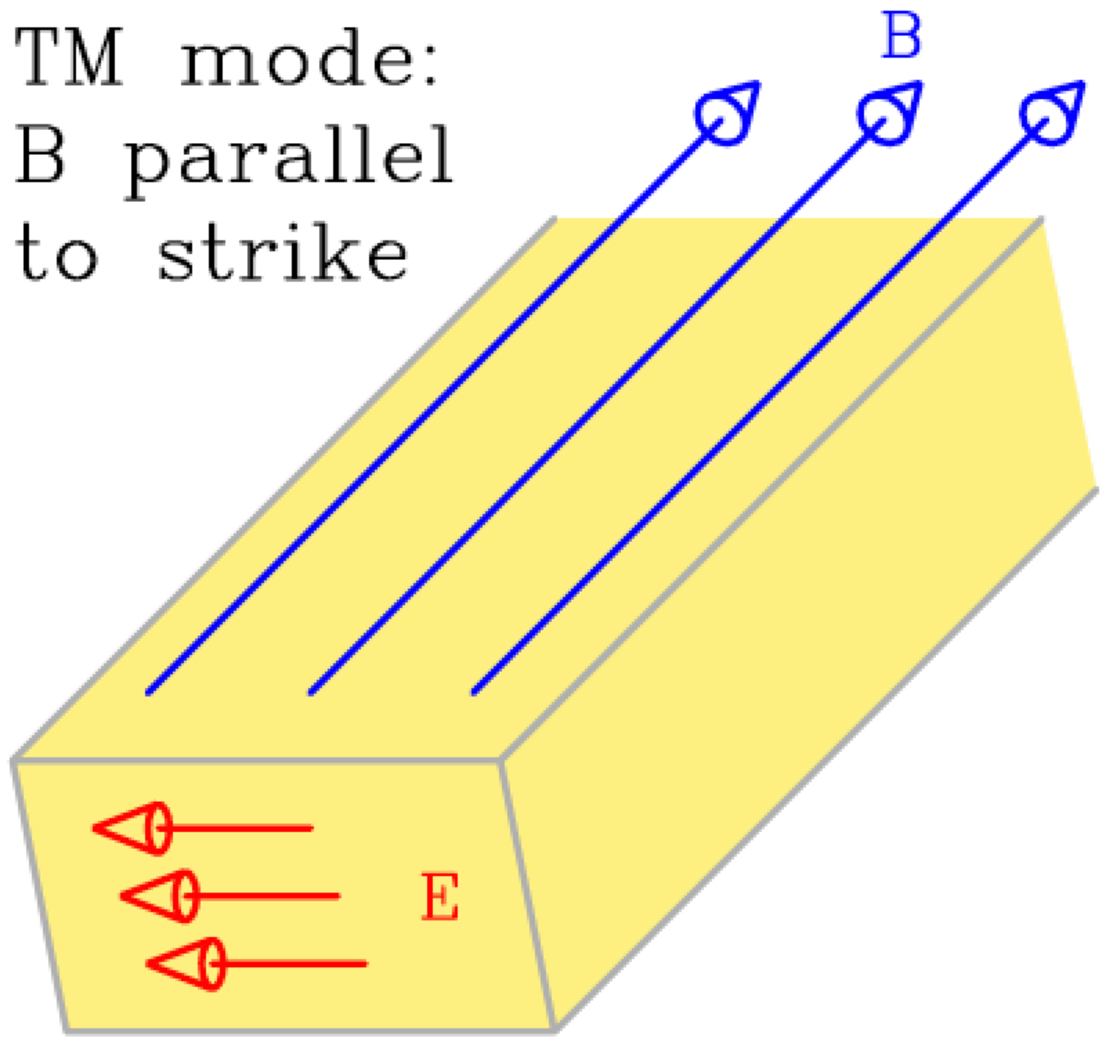

Using $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

$$-i\omega\mu_o B\hat{\mathbf{y}} = \rho\nabla B \nabla \cdot \hat{\mathbf{y}} - \hat{\mathbf{y}}\nabla \cdot (\rho\nabla B) + (\hat{\mathbf{y}} \cdot \nabla)(\rho\nabla B) - (\rho\nabla B \cdot \nabla)\hat{\mathbf{y}}$$

Transverse Magnetic (TM) mode

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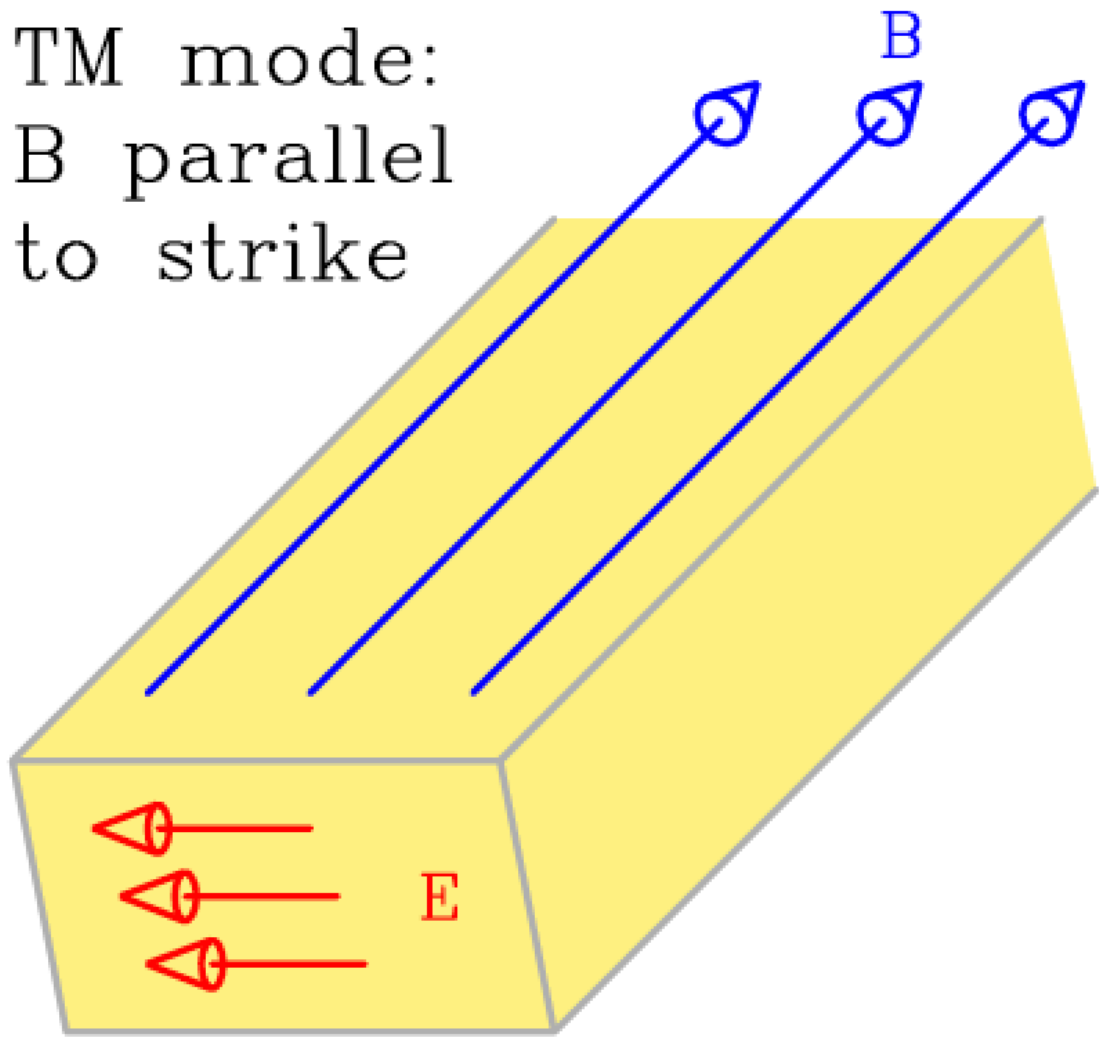

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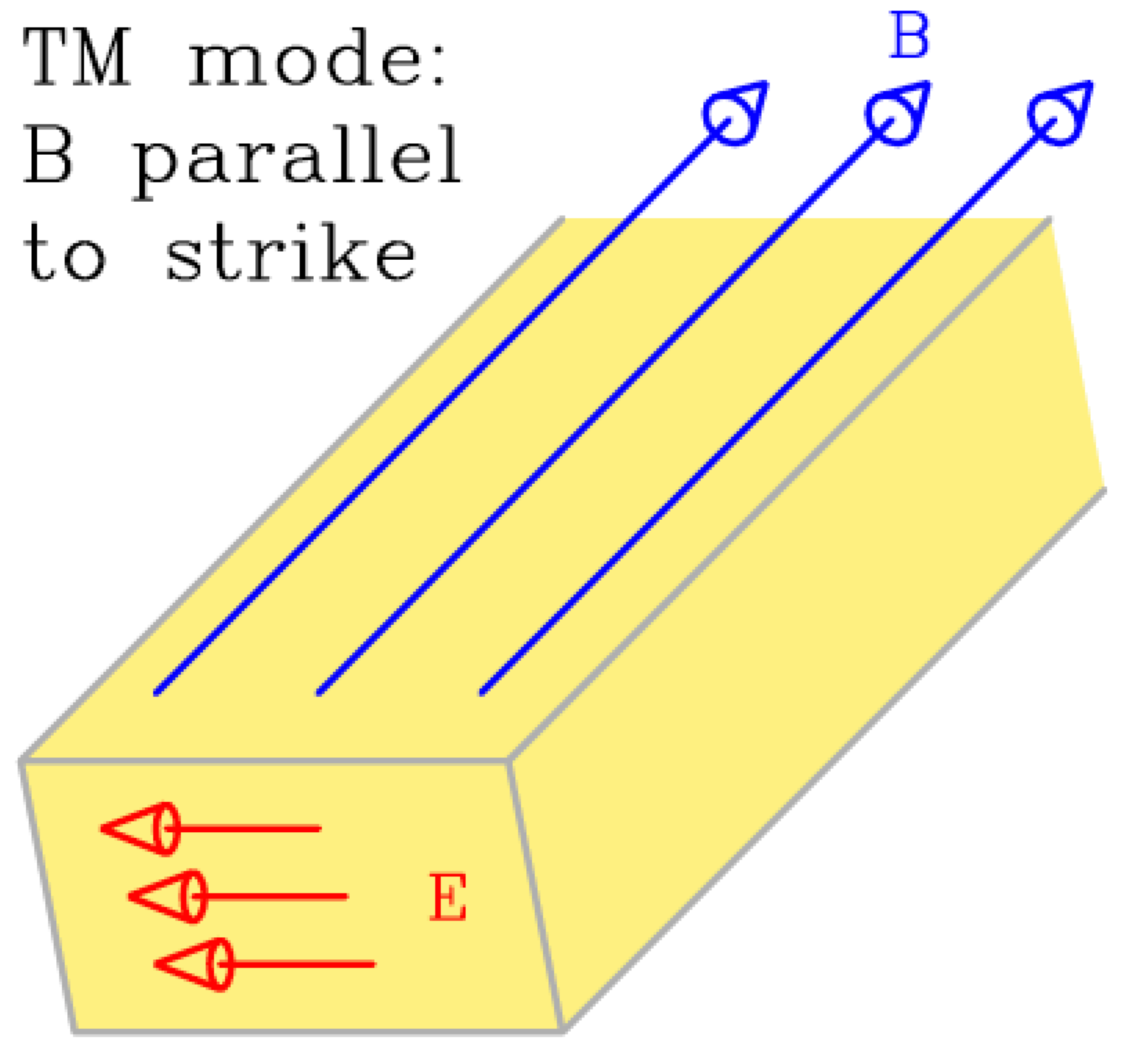
$$-i\omega\mu_o B\hat{\mathbf{y}} = \rho\nabla B \nabla \cdot \hat{\mathbf{y}} - \hat{\mathbf{y}}\nabla \cdot (\rho \nabla B) + (\hat{\mathbf{y}} \cdot \nabla)(\rho\nabla B) - (\rho\nabla B \cdot \nabla)\hat{\mathbf{y}}$$

$\hat{\mathbf{y}}$ = const. $\quad = \frac{\partial}{\partial y} = 0 \quad$ $\hat{\mathbf{y}}$ = const.

so $$i\omega\mu_o B = \nabla \cdot (\rho \nabla B)$$

This is our governing equation for B in the TM mode.

Transverse Magnetic (TM) mode



Our governing equation for B in the TM mode:

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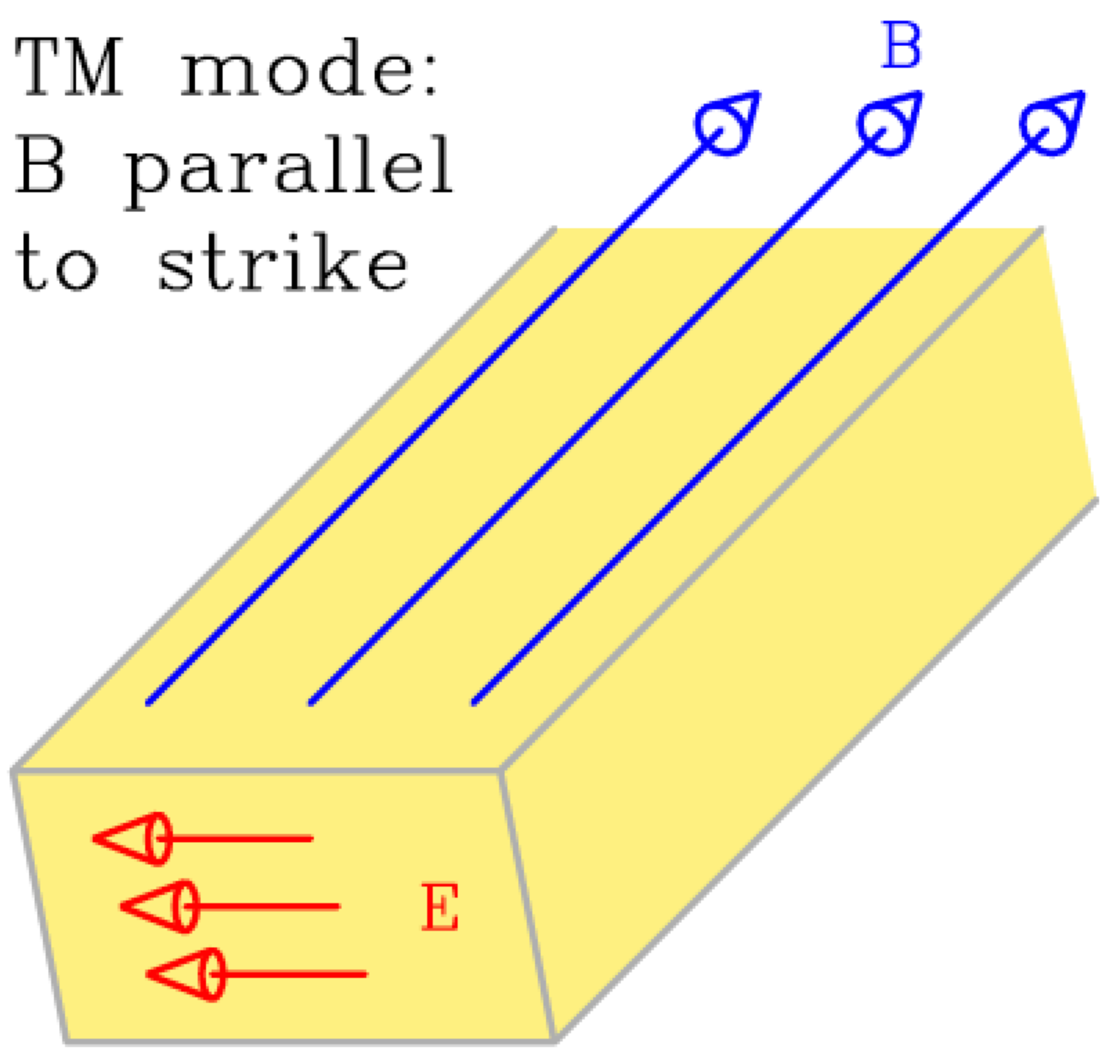
We just need some boundary conditions.

Transverse Magnetic (TM) mode



Our governing equation for B in the TM mode:

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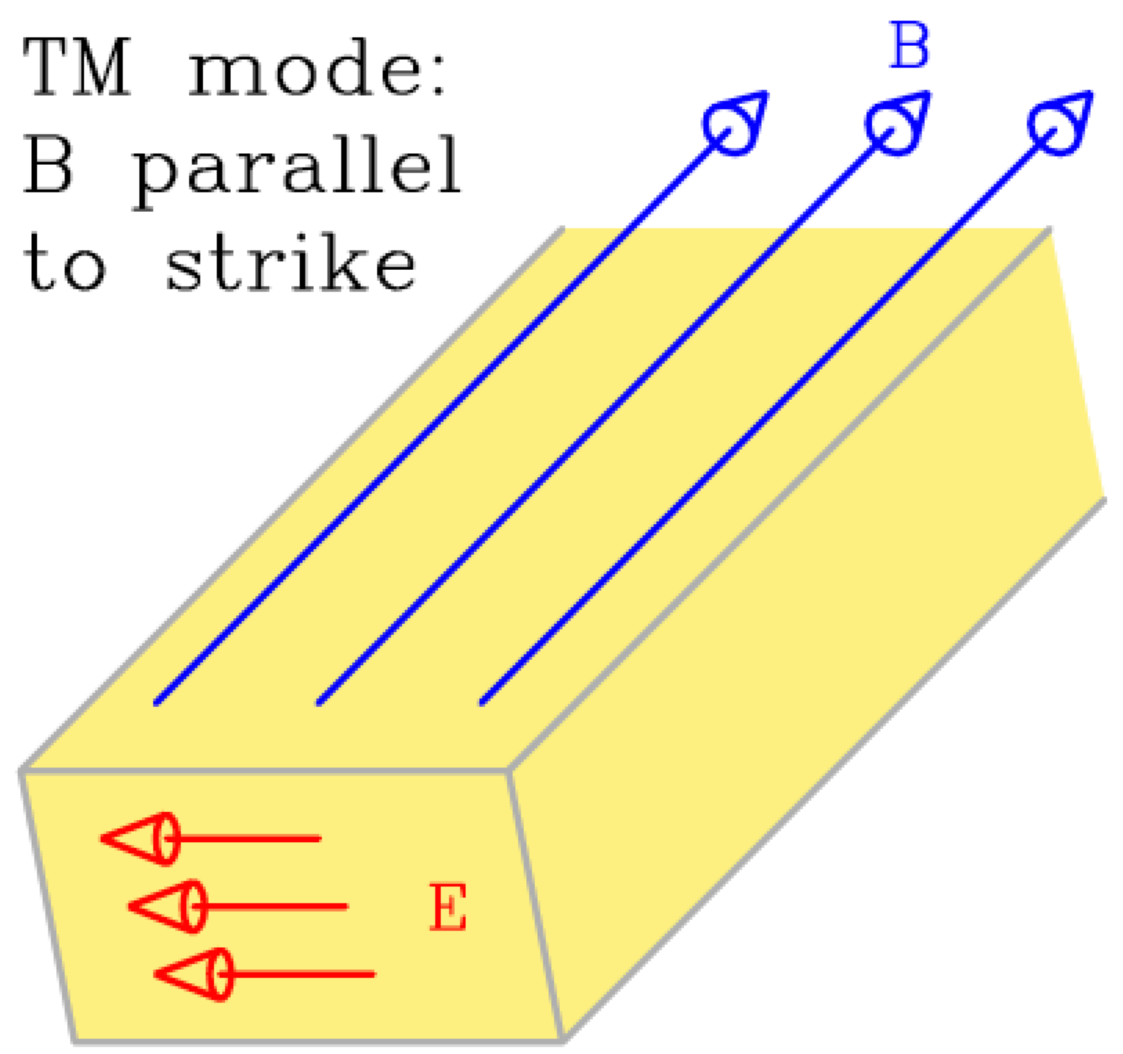
At some great depth, H, we can assume conductivity is infinite. Then E goes to zero, but we need a condition on B. Because Ex is continuous, it must be zero just above in the finite conductor, so current too must be zero, and from the x component of $\nabla \times \mathbf{B} = \mu_o \mathbf{J}$

we have that $$\frac{\partial B}{\partial z} = 0 \quad \text{on} \quad z = H$$

A condition on a normal derivative is called a **Neumann boundary condition**.

Transverse Magnetic (TM) mode



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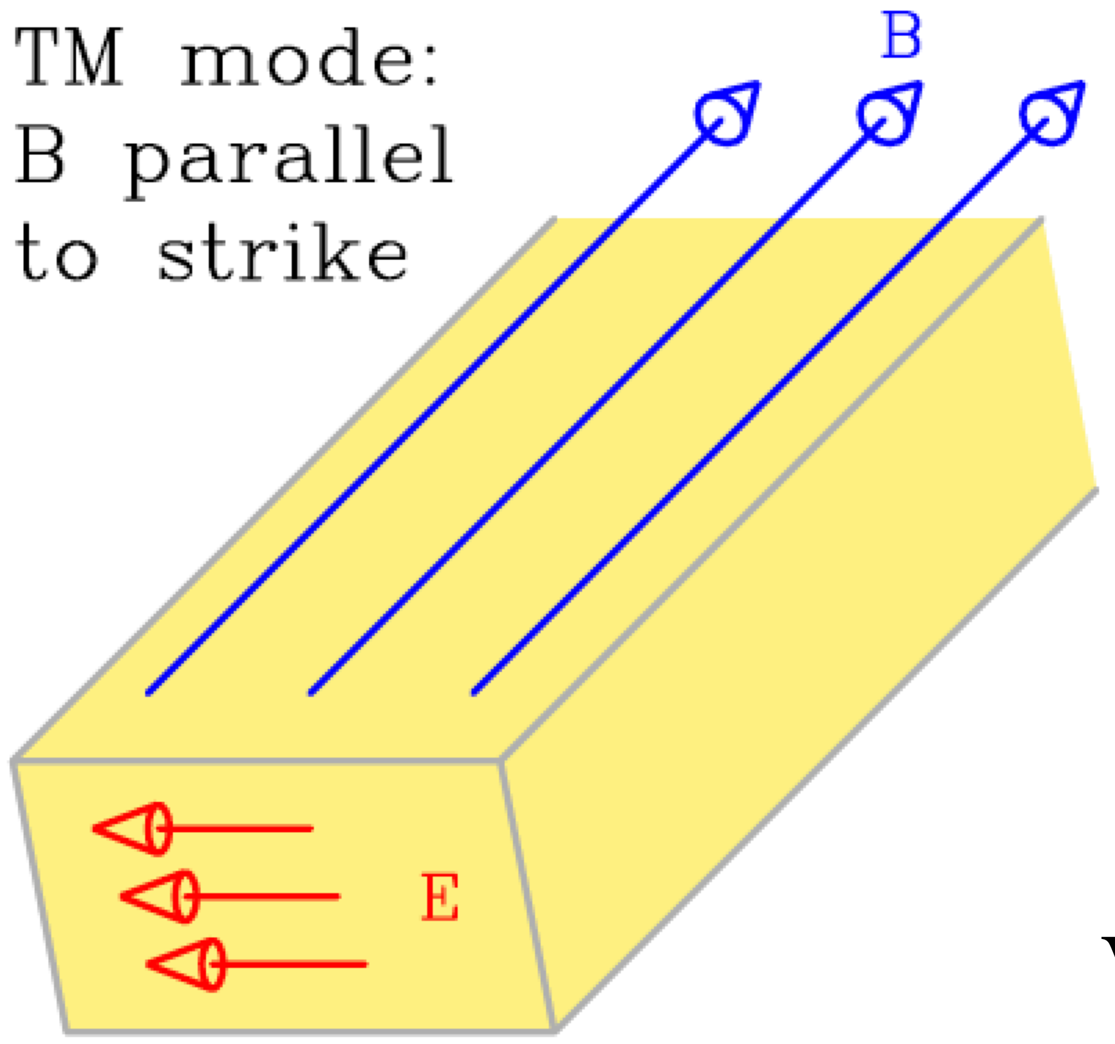
A condition on a normal derivative is called a **Neumann boundary condition**.

At the surface the vertical component of J must be zero because no current flows in the air, and from the z component of Ampere's Law $\partial B/\partial x = 0$, so By must be constant, or

$$B = \text{constant on } z = 0$$

This is called a **Dirichlet boundary condition**, where the value of the solution is specified.

Transverse Magnetic (TM) mode



Since

$$B = \text{constant on } z = 0$$

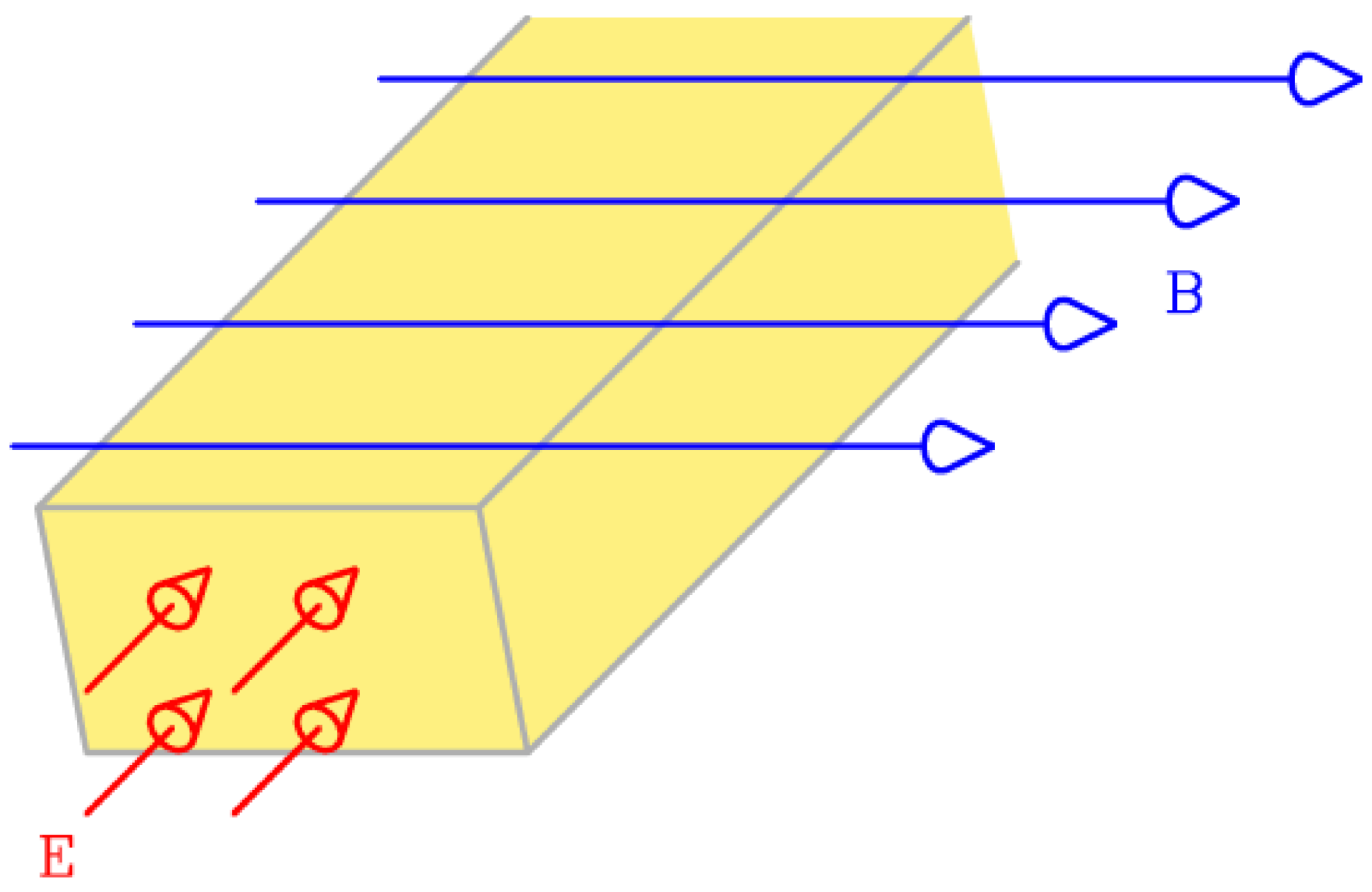
the constant field at the surface must be the source field B_o. That is, the currents induced in the TM mode do not produce fields above the conductor. The fields within the conductor are *toroidal fields*. For this reason we don't need to include the air in the model.

We could put perfect insulators or conductors at the sides of the model, but it is more efficient to assume the conductivity is constant and use the half-space solution, or become 1D and use the layered solution we derived previously.

Finally, we note that to predict our MT observations, we need the *x-y* component of the impedance tensor

$$Z_{xy} = \frac{E_x}{B_y} = -\frac{1}{\mu_o \sigma B}\frac{\partial B}{\partial z}$$

Transverse Electric (TE) mode



The magnetic field is perpendicular to strike, so

$$\mathbf{B} = B_o e^{i\omega t}\hat{\mathbf{x}}$$

and the electric field and currents flow in the y direction:

$$\mathbf{E} = E(x,y)e^{i\omega t}\hat{\mathbf{y}}$$

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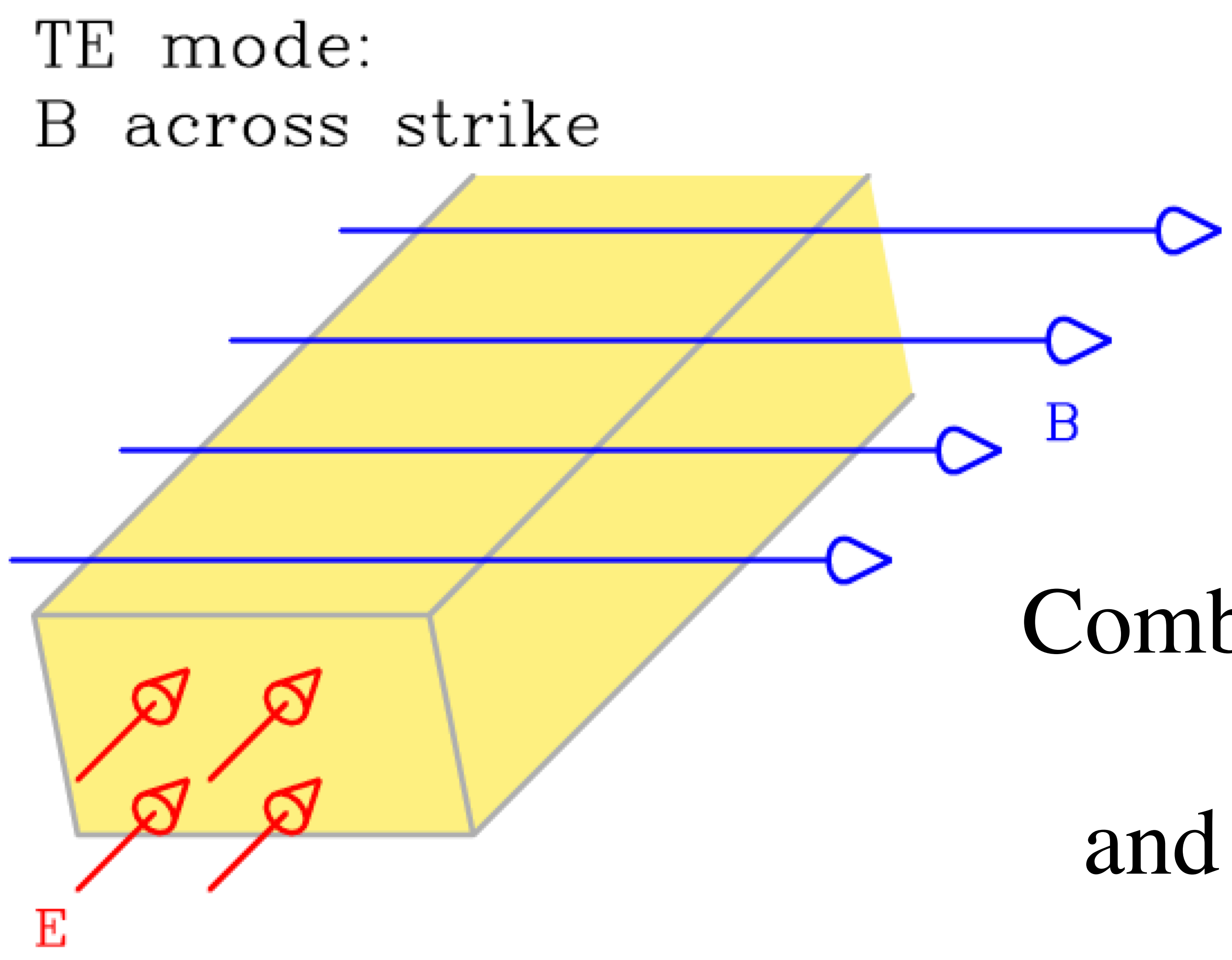

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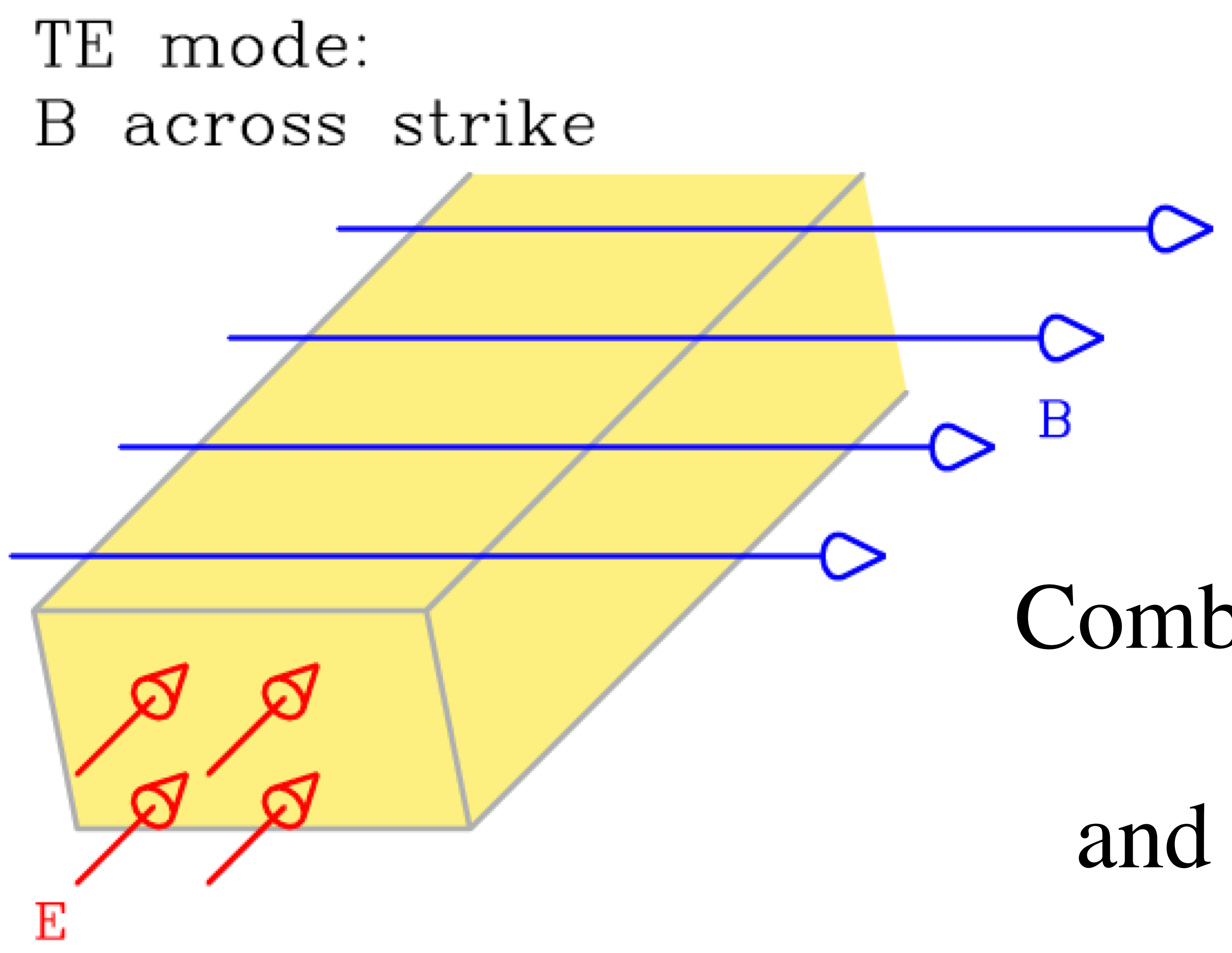
Combine Ampere's and Ohm's Laws $\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E}$

and take the divergence $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_o \nabla \cdot \sigma(x, z)\mathbf{E}$

Transverse Electric (TE) mode



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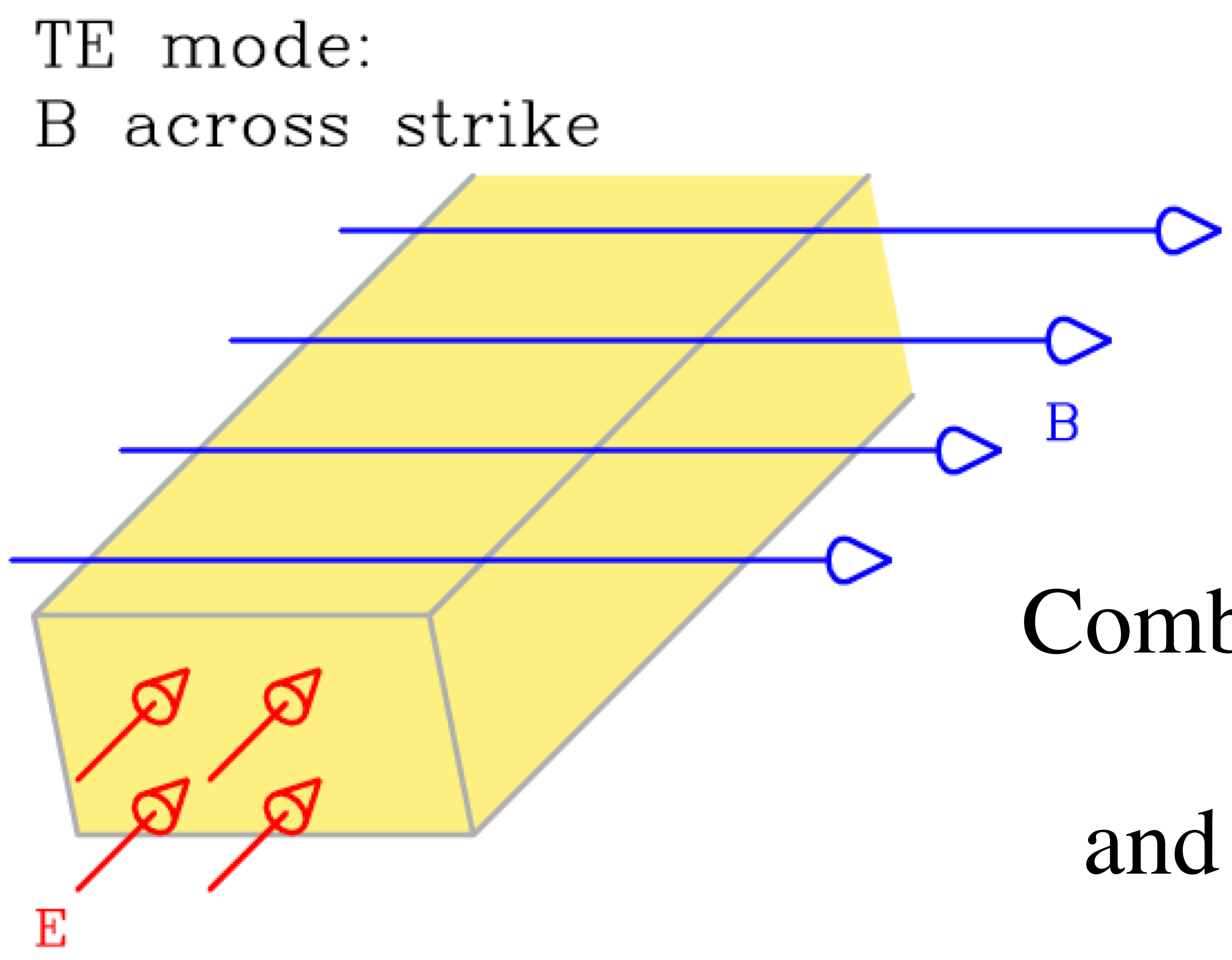
use a couple of vector identities $\quad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s\nabla \cdot \mathbf{A}$

and we have $\quad 0 = \sigma(x,z)\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla\sigma(x,z)$

Transverse Electric (TE) mode



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and we have $0 = \sigma(x,z)\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \sigma(x,z)$

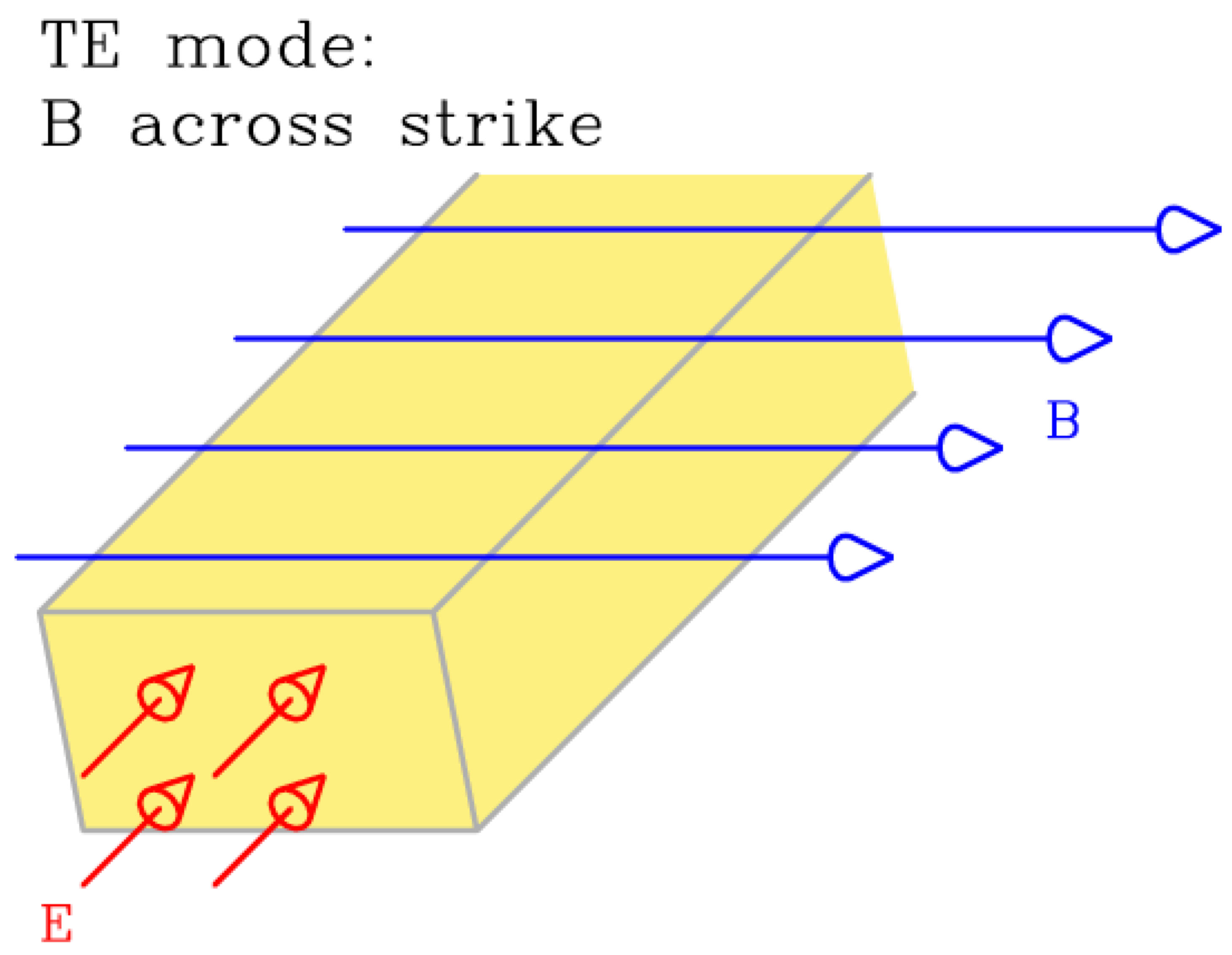
= 0 because E is only in *y* direction

We thus show that $\nabla \cdot \mathbf{E} = 0$

and our diffusion equation still holds

$$\nabla^2 E = i\omega\mu_0\sigma E$$

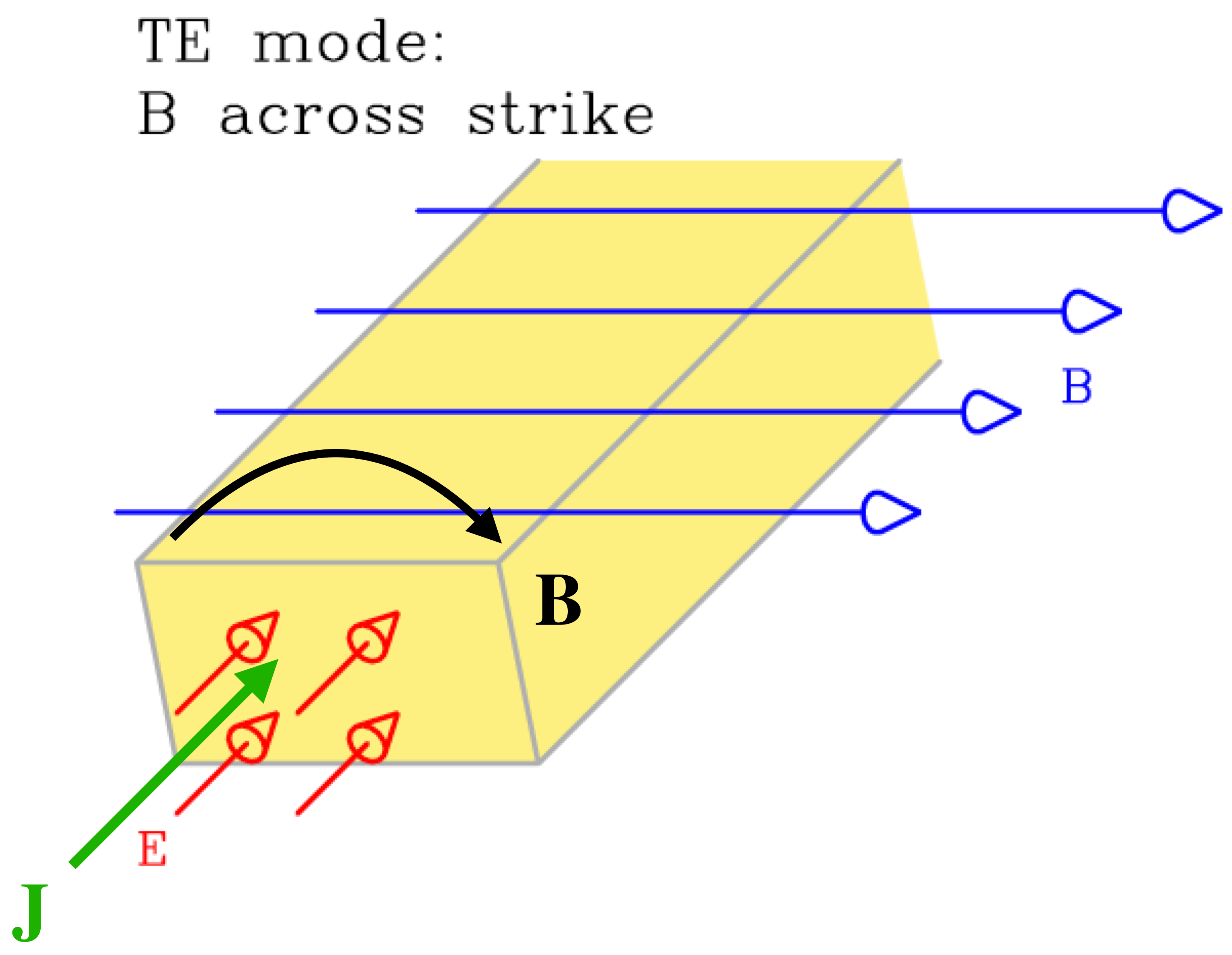
Transverse Electric (TE) mode



Boundary conditions: As before we put an infinite conductor at depth H. Because $\nabla \cdot \mathbf{E} = 0$, all components of $\mathbf{E}$ are continuous, so

$$E = 0 \quad \text{on} \quad z = H$$

Transverse Electric (TE) mode



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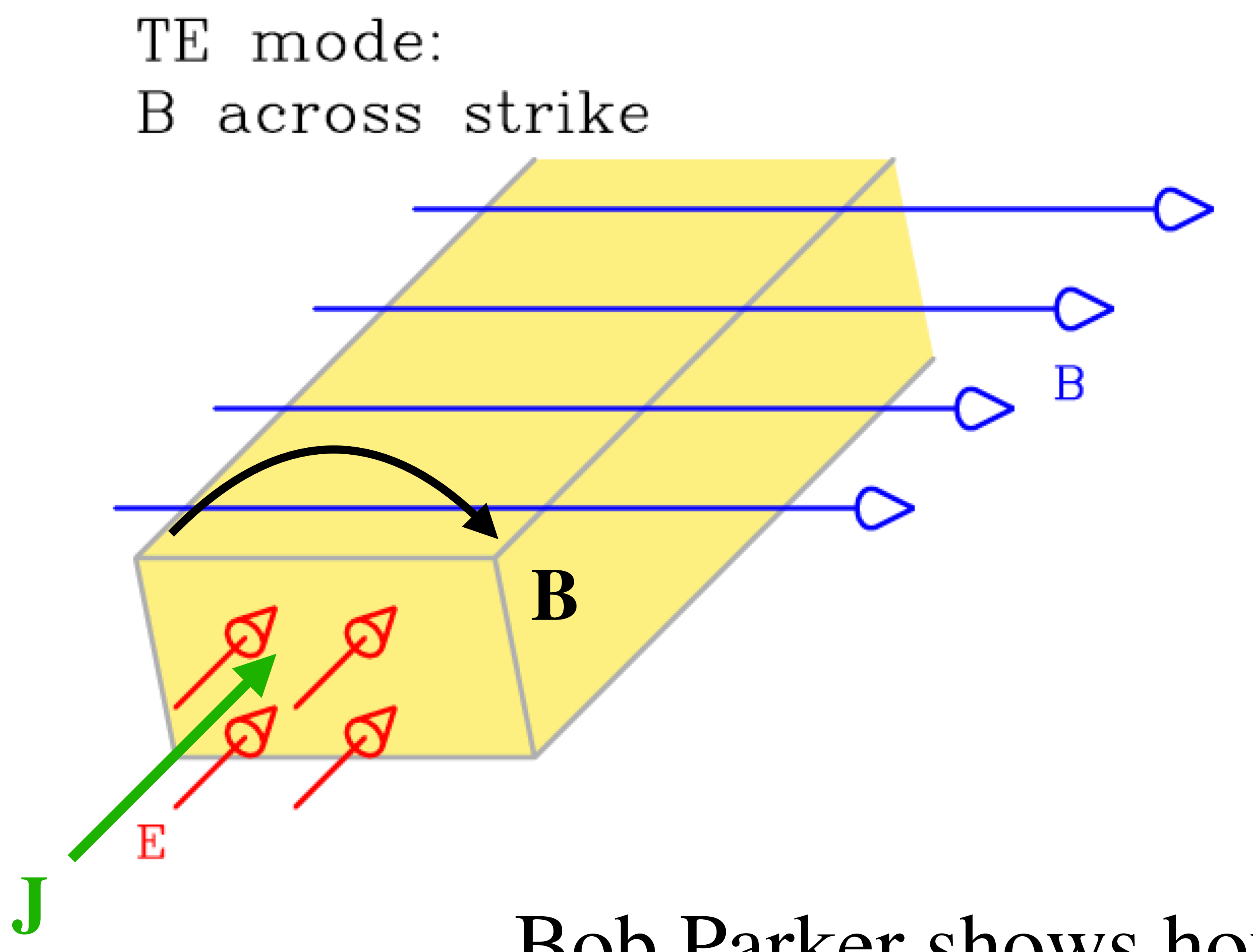
The fact that $\mathbf{J}_z = 0$ at $z = 0$ doesn't help us here, because $E_z = 0$ for all TE solutions. The top boundary condition is gnarly because the TE currents create secondary fields in the air layer. That is, the TE fields are *poloidal*.

Transverse Electric (TE) mode



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Bob Parker shows how you can get an exact boundary condition on $z = 0$, but it is rather complicated and in practice most people just include the air and consider $\mathbf{B} = B_o$ at some height above the ground.

As for the TM mode, you can use constant or 1D conductivities on the sides. Finally our impedance tensor element is given by

$$Z_{yx} = \frac{E_y}{B_x} = -\frac{E}{i\omega \partial E/\partial z} \quad \text{on} \quad z = 0$$

Fields in 3D:

In Lecture 3 we derived the diffusion equations by casting Maxwell's equations in only **E** or **B**: Substituting Ohm's Law into Ampere's Law we have

$$\mathbf{J} = \sigma \mathbf{E} \qquad \nabla \times \mathbf{B} = \mu_o \mathbf{J} \qquad \frac{1}{\sigma}\nabla \times \mathbf{B} = \mu_o \mathbf{E}$$

Take the curl $\nabla \times \frac{1}{\sigma}\nabla \times \mathbf{B} = \nabla \times \mu_o \mathbf{E}$ and use Faraday's Law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$

we get $\nabla \times \frac{1}{\sigma}\nabla \times \mathbf{B} = -\mu_o \partial \mathbf{B}/\partial t$ which for fixed frequency is

$$\nabla \times \frac{1}{\sigma}\nabla \times \mathbf{B} + i\omega\mu_o \mathbf{B} = 0$$

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we get $\nabla \times \frac{1}{\sigma} \nabla \times \mathbf{B} = -\mu_o \partial \mathbf{B} / \partial t$ which for fixed frequency is

$$\nabla \times \frac{1}{\sigma} \nabla \times \mathbf{B} + i\omega\mu_o \mathbf{B} = 0$$

Similarly, take the curl of Faraday's Law $\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}$ and use Ampere/Ohm

$$\nabla \times \nabla \times \mathbf{E} = -\sigma \mu_o \frac{\partial}{\partial t} \mathbf{E} \qquad \text{or} \qquad \nabla \times \nabla \times \mathbf{E} + i\omega\sigma\mu_o \mathbf{E} = 0$$

These are the curl-curl equations, which are completely general and can be solved in 2D or 3D.

$$\nabla \times \nabla \times \mathbf{E} + i\omega\sigma\mu_o\mathbf{E} = 0 \qquad \nabla \times \frac{1}{\sigma}\nabla \times \mathbf{B} + i\omega\mu_o\mathbf{B} = 0$$

They can be generalized to include current or magnetic source terms **J**s and **M**s

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu_0\sigma\mathbf{E} = -i\omega\mu_0\mathbf{J}_s - i\omega\nabla \times \mathbf{M}_s$$

$$\nabla \times \rho\nabla \times \mathbf{H} + i\omega\mu_0\sigma\mathbf{H} = \nabla \times \rho\mathbf{J}_s - i\omega\mathbf{M}_s$$

Both of these equations could be solved, but in practice it is cheaper to solve only one (usually **E**, to avoid having to deal with the gradients in σ), and compute the other using Faraday or Ampere.

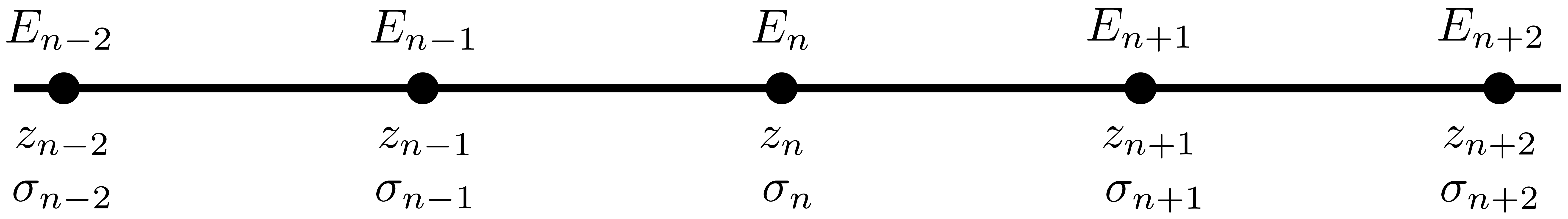
In lecture 12 we showed that the governing equation for the 1D problem was the same as for a half-space of constant conductivity:

$$\nabla^2 \mathbf{E} = i\omega\mu_o\sigma\mathbf{E}$$

which reduces to solving

$$\text{Given} \quad \sigma(z) \quad \text{solve} \quad \frac{d^2E}{dz^2} = i\omega\mu_o\sigma(z)E(z) \quad \text{for} \quad E(\infty) = 0 \quad \text{and} \quad E'(0) = -1$$

where *E(0)* at the surface was our admittance, c. We can solve this equation numerically using a **finite difference** approach where conductivity is defined on nodes



and we cast the solution as a linear system of equations

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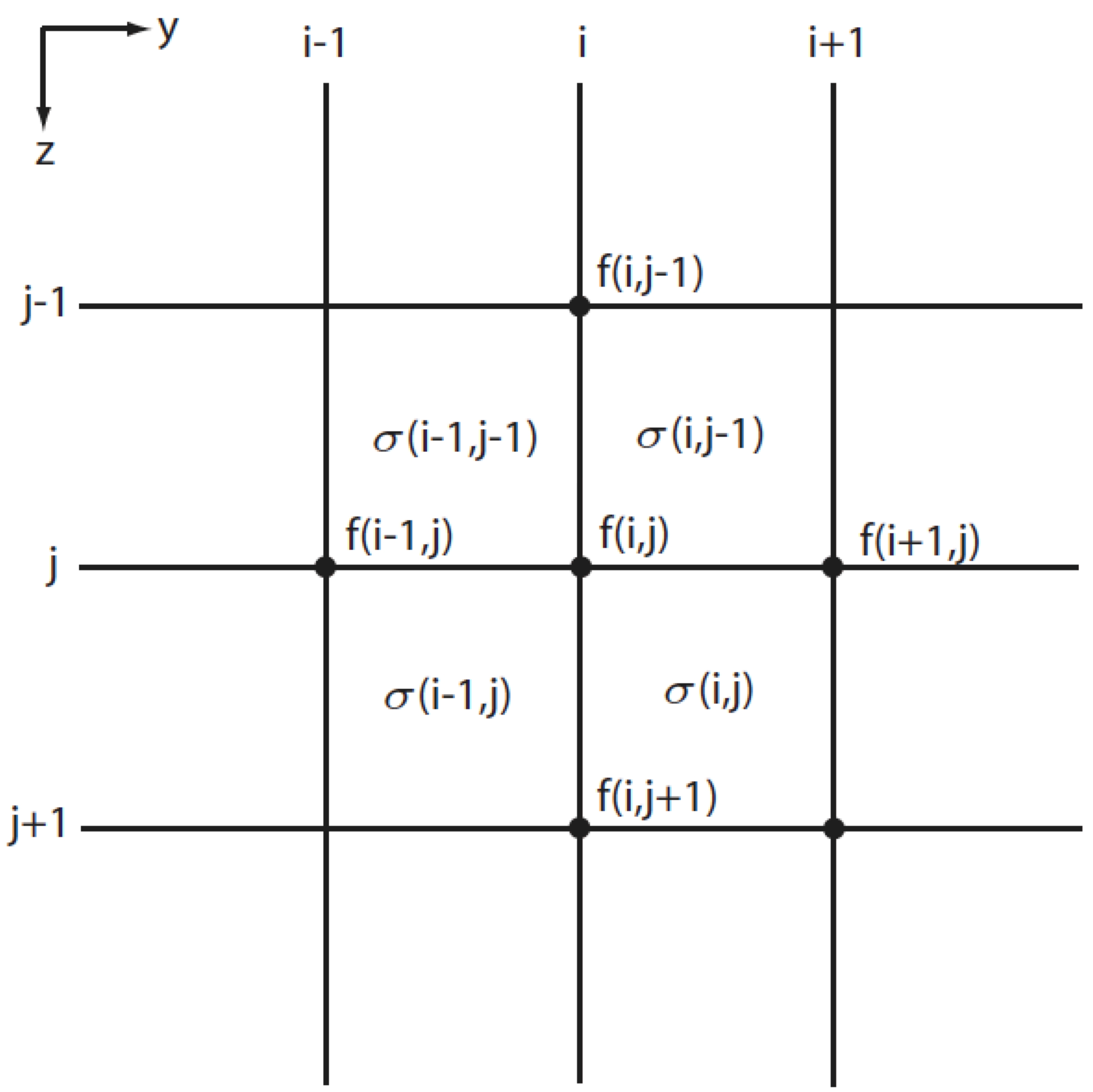
Finite differences in 2D



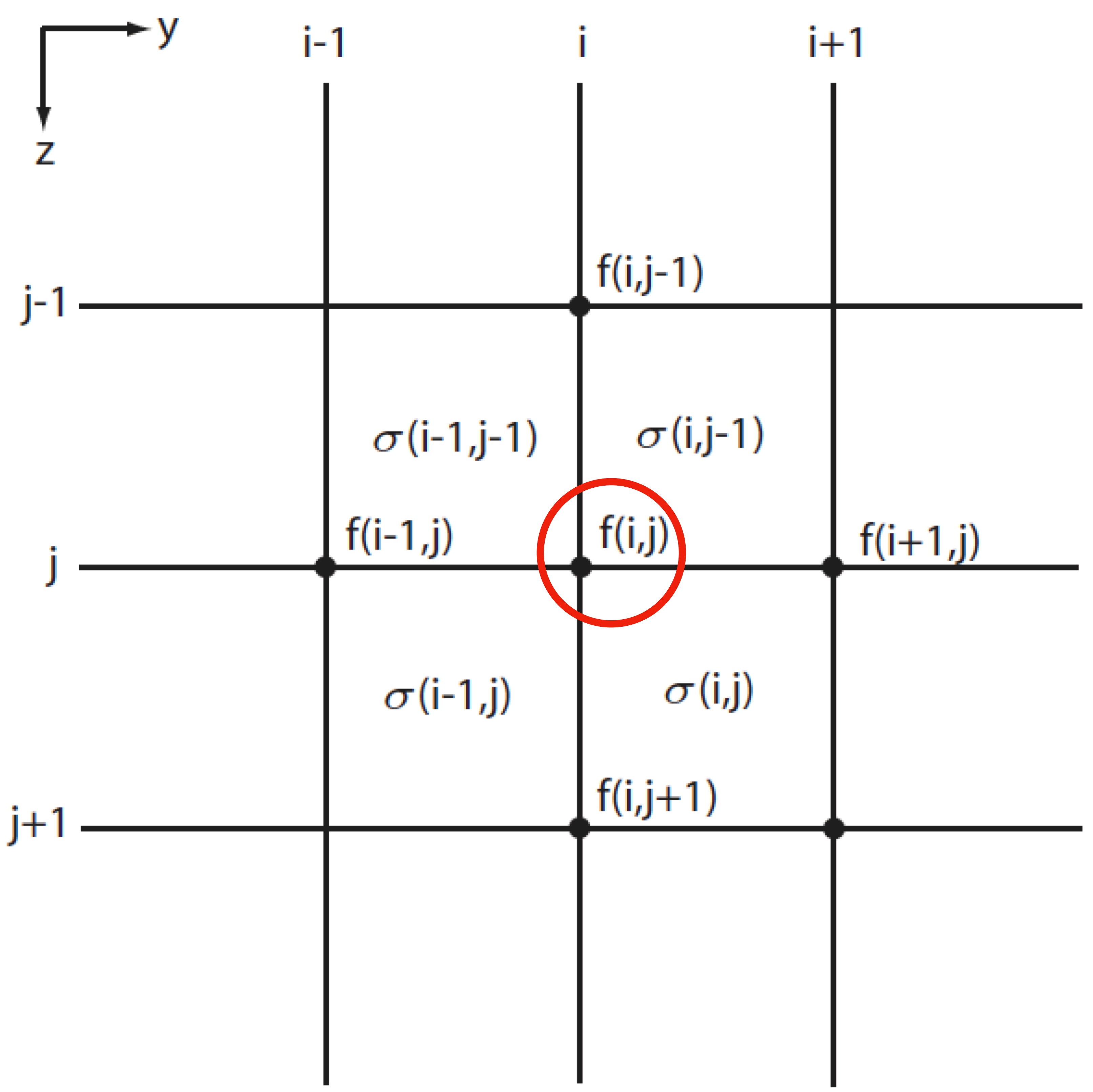
Here conductivities are constant in the cells and the fields are calculated on the nodes. We saw that the governing equation for the **TE mode** is

$$\nabla^2 E(y,z) - i\omega\mu_o\sigma E(y,z) = 0$$

which is

$$\frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - i\omega\mu_o\sigma E = 0$$

Finite differences in 2D



Here conductivities are constant in the cells and the fields are calculated on the nodes. We saw that the governing equation for the **TE mode** is

$$\nabla^2 E(y,z) - i\omega\mu_o\sigma E(y,z) = 0$$

which is

$$\frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - i\omega\mu_o\sigma E = 0$$

At the (i, j) node this can be approximated by

$$\frac{E_{i+1,j} - 2E_{i,j} + E_{i-1,j}}{\Delta y^2} + \frac{E_{i,j+1} - 2E_{i,j} + E_{i,j-1}}{\Delta z^2} - i\omega\mu_o\sigma E = 0$$

or if the mesh is equal in *y* and *z*

$$\frac{1}{h^2}(E_{i+1,j} + E_{i-1,j} + E_{i,j+1} + E_{i,j-1} - 4E_{i,j}) = -i\omega\mu_o\sigma E = 0$$

If the 4 conductivities surrounding (i,j) are different, we have to use the average.

Our TE finite difference equation

$$\frac{1}{h^2}(E_{i+1,j} + E_{i-1,j} + E_{i,j+1} + E_{i,j-1} - 4E_{i,j}) = -i\omega\mu_o\sigma E = 0$$

becomes one row of our linear system $\mathbf{A}E = b$

which looks like

$$\left[\frac{1}{h^2}, \frac{1}{h^2}, \frac{1}{h^2}, \frac{1}{h^2}, -\frac{4}{h^2} - i\omega\mu_o\sigma_{ave}\right] \begin{bmatrix} E_{i+1,j} \\ E_{i-1,j} \\ E_{i,j+1} \\ E_{i,j-1} \\ E_{i,j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Our TE finite difference equation

$$\frac{1}{h^2}(E_{i+1,j} + E_{i-1,j} + E_{i,j+1} + E_{i,j-1} - 4E_{i,j}) = -i\omega\mu_o\sigma E = 0$$

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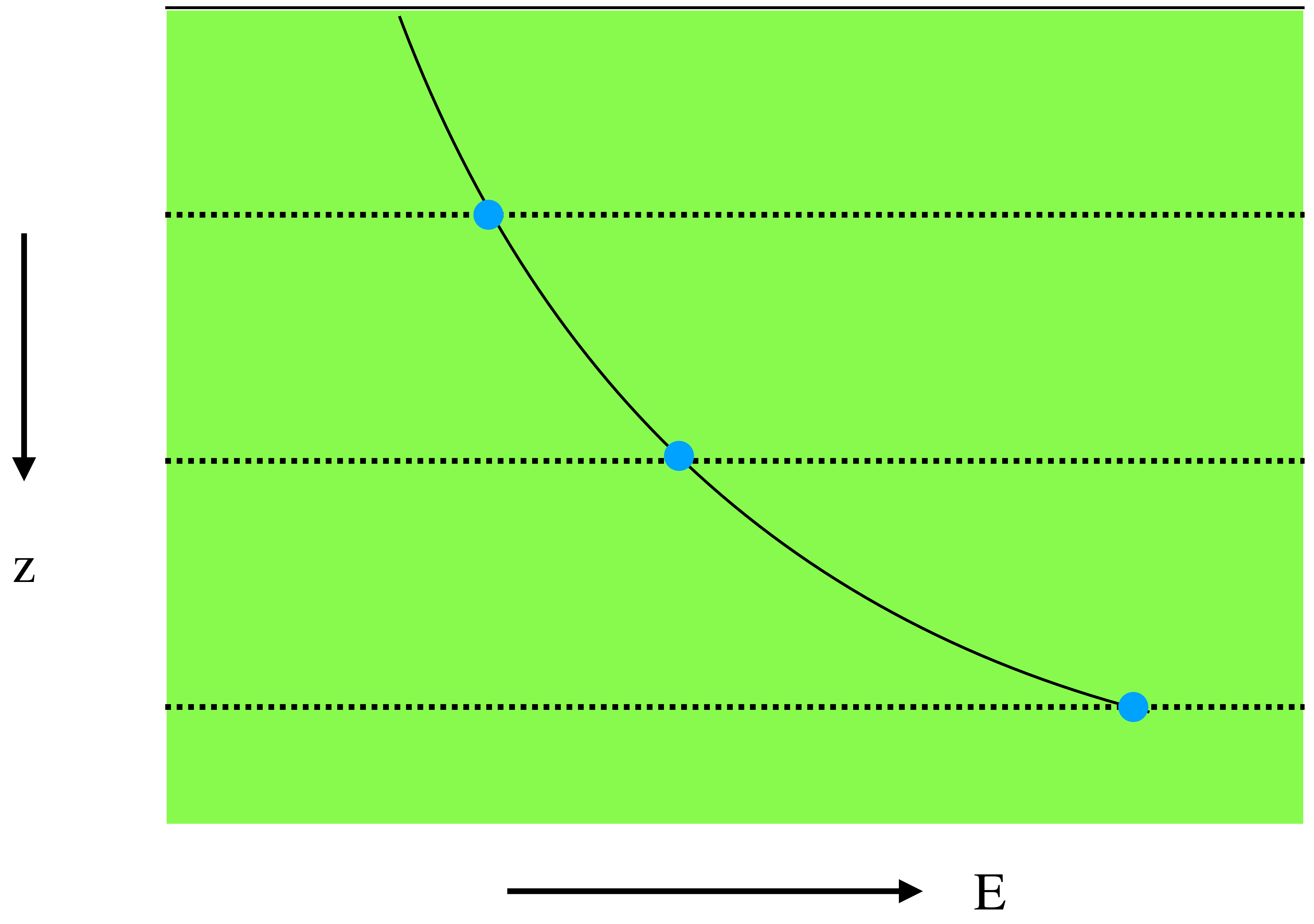
$$\left[\frac{1}{h^2}, \frac{1}{h^2}, \frac{1}{h^2}, \frac{1}{h^2}, \; -\frac{4}{h^2} - i\omega\mu_o\sigma_{ave}\right] \begin{bmatrix} E_{i+1,j} \\ E_{i-1,j} \\ E_{i,j+1} \\ E_{i,j-1} \\ E_{i,j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

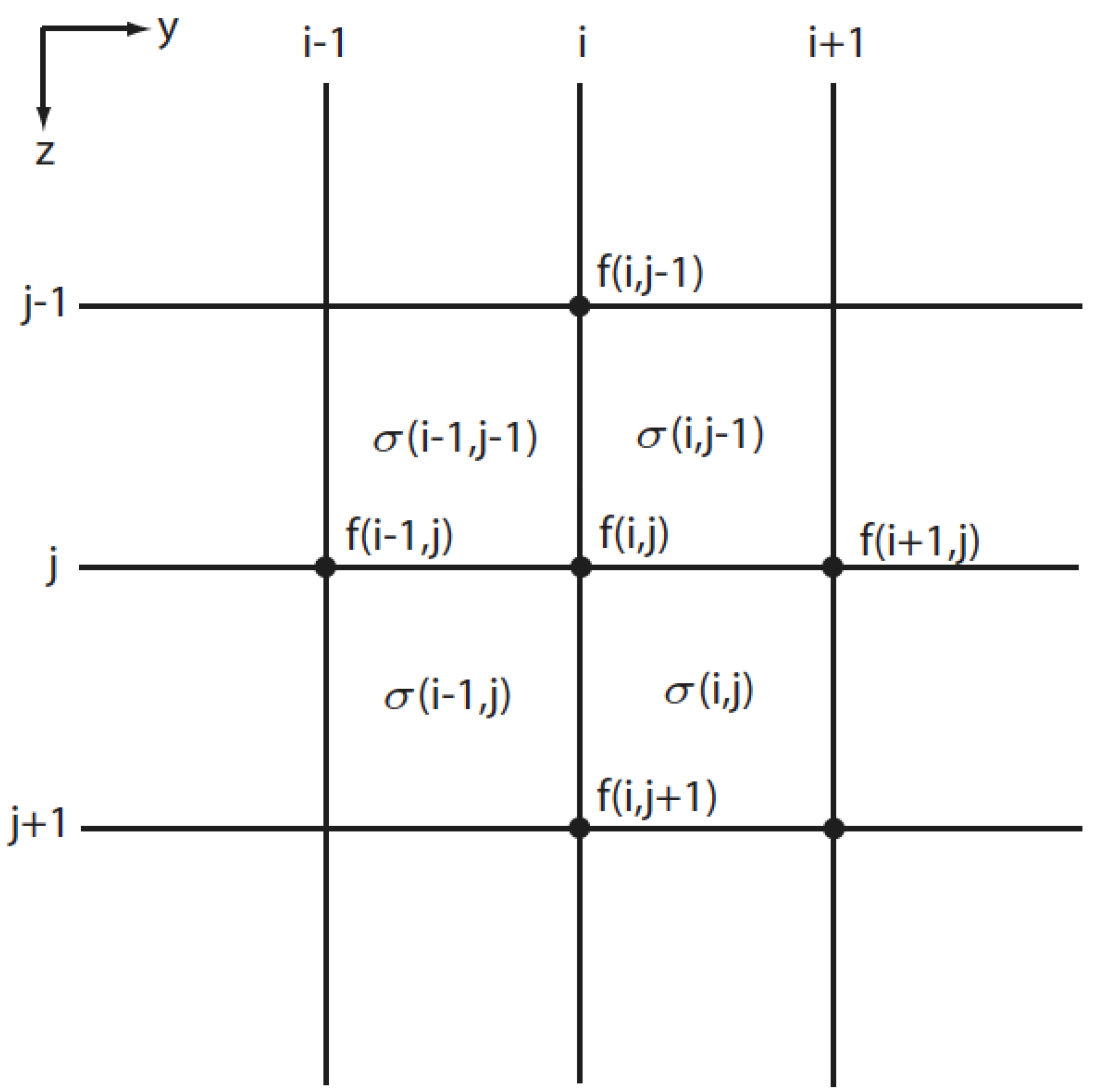
We need an entry like this for every node (i,j) along with the boundary conditions. Finally we need the vertical derivative at the surface to compute the impedance:

$$Z_{yx} = \frac{E_y}{B_x} = -\frac{E}{i\omega\partial E/\partial z} \quad \text{on} \quad z = 0$$

Since the vertical derivative changes in the atmosphere, the practice is to fit a parabola to the electric fields at the three nodes closest to the surface and use the analytical derivative at $z = 0$.


z
E

Finite differences in 2D



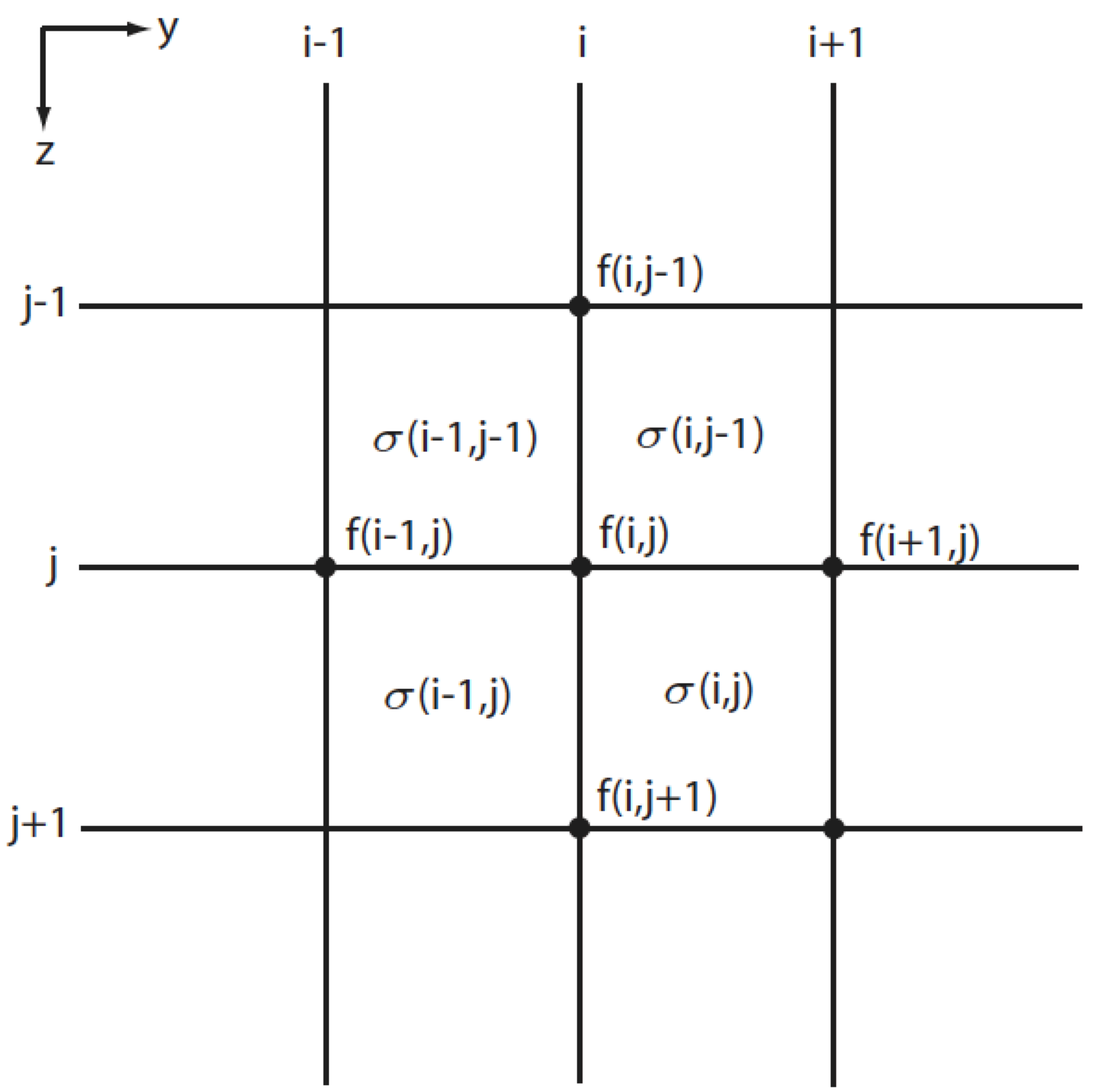
The governing equation for the **TM mode** is

$$i\omega\mu_o B = \nabla \cdot (\rho \nabla B)$$

which is

$$\rho \frac{\partial^2 B}{\partial y^2} + \rho \frac{\partial^2 B}{\partial z^2} + \frac{\partial \rho}{\partial y}\frac{\partial B}{\partial y} + \frac{\partial \rho}{\partial z}\frac{\partial B}{\partial z} - i\omega\mu_o B = 0$$

Finite differences in 2D



The governing equation for the **TM mode** is

$$i\omega\mu_o B = \nabla \cdot (\rho \nabla B)$$

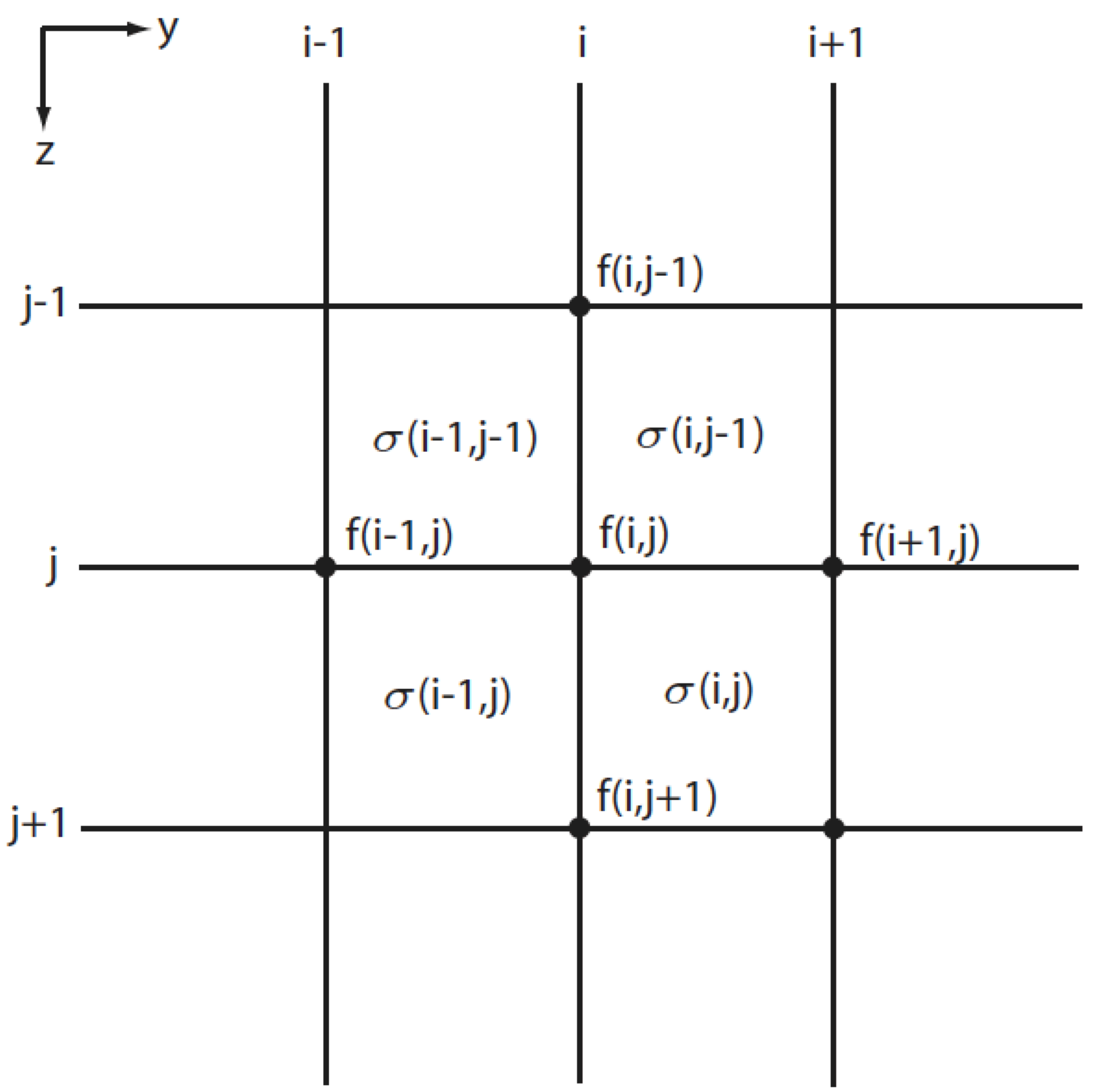
which is

$$\boxed{\rho\frac{\partial^2 B}{\partial y^2} + \rho\frac{\partial^2 B}{\partial z^2}} + \frac{\partial \rho}{\partial y}\frac{\partial B}{\partial y} + \frac{\partial \rho}{\partial z}\frac{\partial B}{\partial z} - i\omega\mu_o B = 0$$

The first two terms can be approximated as before

$$\rho_{ave}\frac{B_{i+1,j} - 2B_{i,j} + B_{i-1,j}}{\Delta y^2} + \rho_{ave}\frac{B_{i,j+1} - 2B_{i,j} + B_{i,j-1}}{\Delta z^2}$$

Finite differences in 2D



The governing equation for the **TM mode** is

$$i\omega\mu_o B = \nabla \cdot (\rho \nabla B)$$

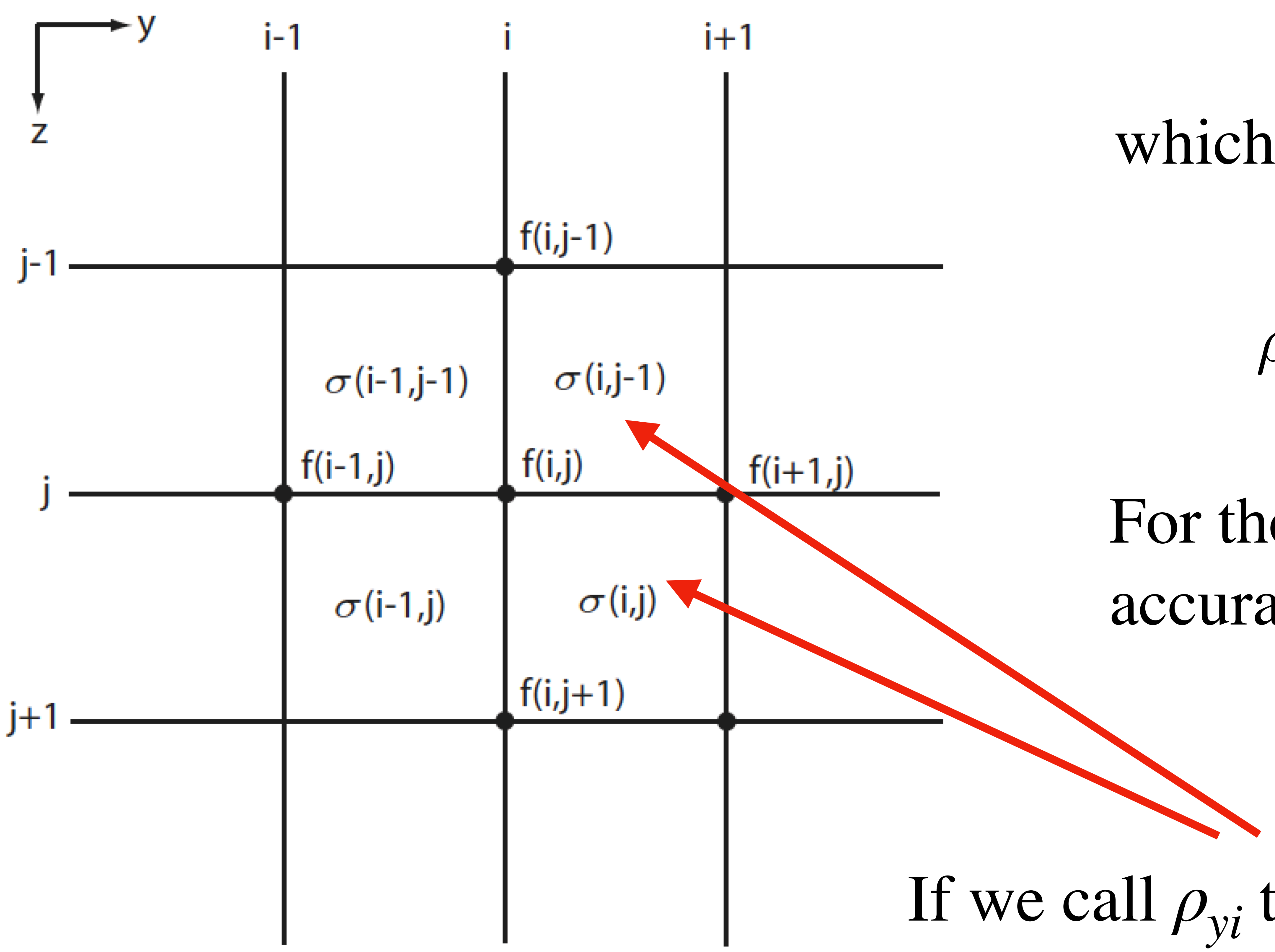
which is

$$\rho\frac{\partial^2 B}{\partial y^2} + \rho\frac{\partial^2 B}{\partial z^2} + \boxed{\frac{\partial \rho}{\partial y}\frac{\partial B}{\partial y}} + \frac{\partial \rho}{\partial z}\frac{\partial B}{\partial z} - i\omega\mu_o B = 0$$

For the next two terms we note that central derivatives are more accurate

$$\frac{\partial B}{\partial y} \approx \frac{B_{i+1,j} - B_{i-1,j}}{2\Delta y}$$

Finite differences in 2D



The governing equation for the **TM mode** is

$$i\omega\mu_o B = \nabla \cdot (\rho \nabla B)$$

which is

$$\rho \frac{\partial^2 B}{\partial y^2} + \rho \frac{\partial^2 B}{\partial z^2} + \frac{\partial \rho}{\partial y}\frac{\partial B}{\partial y} + \frac{\partial \rho}{\partial z}\frac{\partial B}{\partial z} - i\omega\mu_o B = 0$$

For the next two terms we note that central derivatives are more accurate

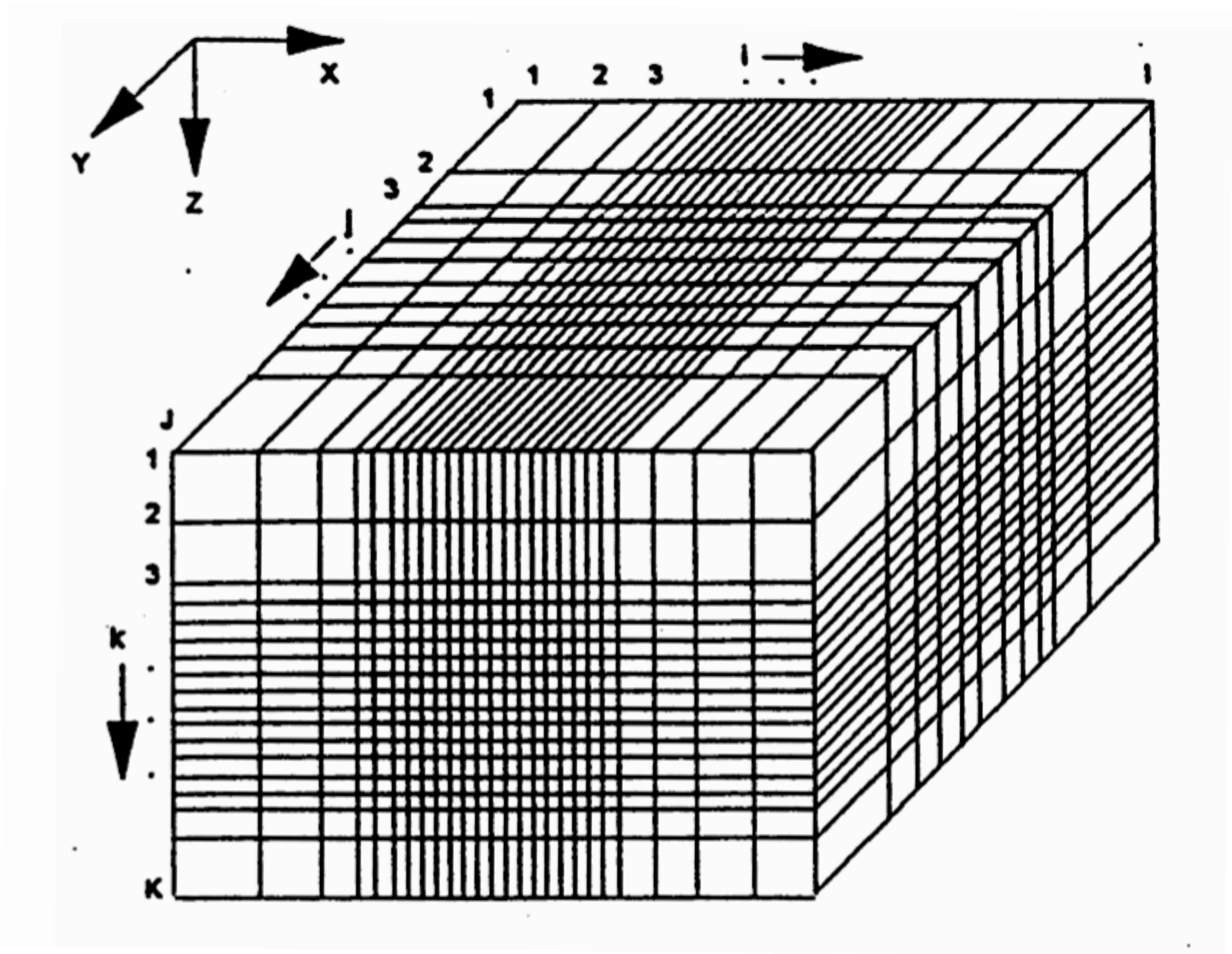
$$\frac{\partial B}{\partial y} \approx \frac{B_{i+1,j} - B_{i-1,j}}{2\Delta y}$$

If we call ρ_{yi} the average of ρ(i,j-1) and ρ(i,j) then

$$\frac{\partial \rho}{\partial y}\frac{\partial B}{\partial y} \approx \left(\frac{\rho_{yi-1} - \rho_{yi}}{\Delta y}\right)\left(\frac{B_{i+1,j} - B_{i-1,j}}{2\Delta y}\right) = \frac{\rho_{yi-1}B_{i+1,j} - \rho_{yi-1}B_{i-1,j} - \rho_{yi}B_{i+1,j} + \rho_{yi}B_{i-1,j}}{2\Delta y^2}$$

$$= \frac{(\rho_{yi-1} - \rho_{yi})B_{i+1,j} - (\rho_{yi-1} - \rho_{yi})B_{i-1,j}}{2\Delta y^2}$$

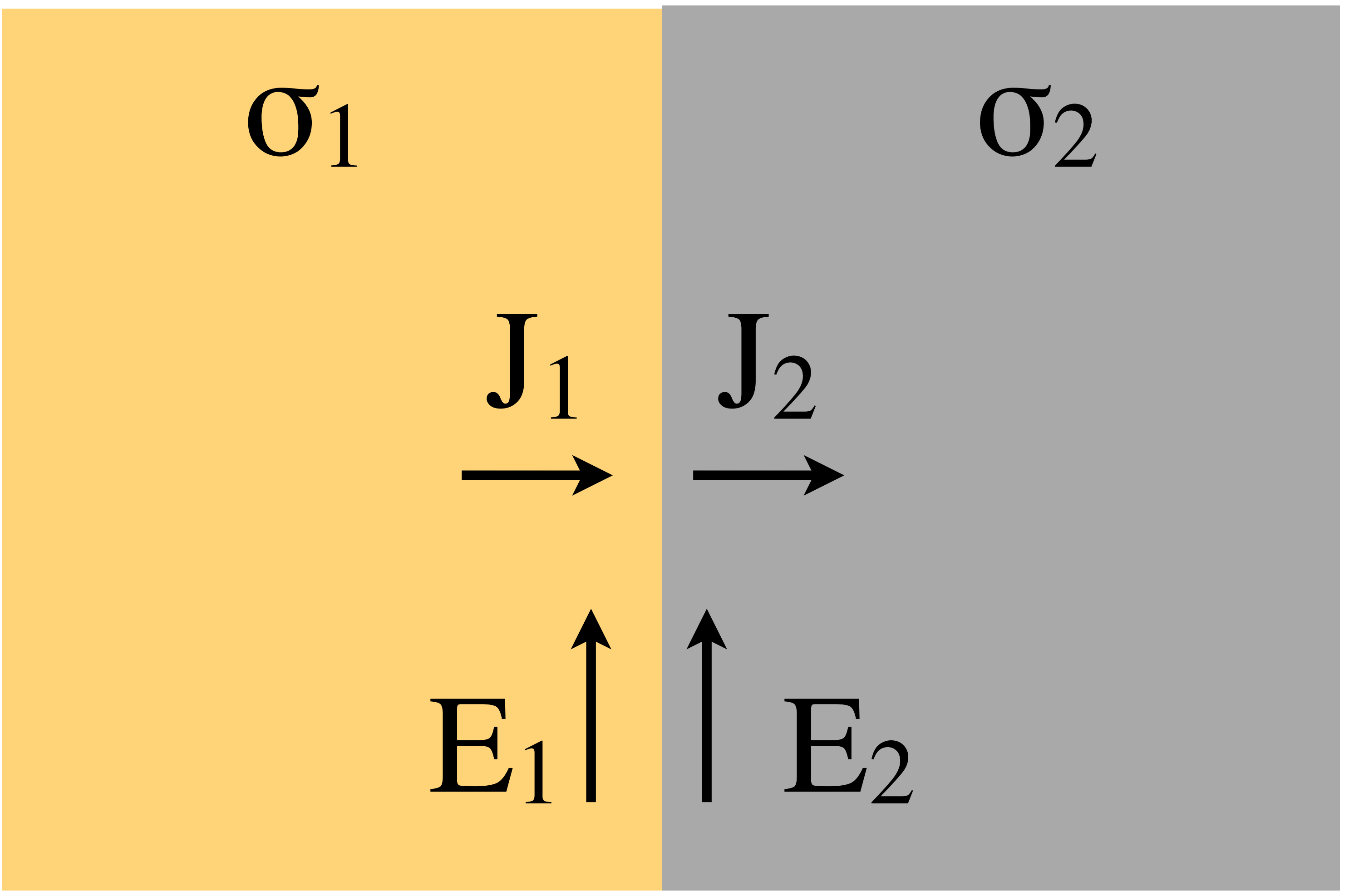
A finite difference grid in 3D. Note that the node spacing in the center of the mesh propagates to the edges.



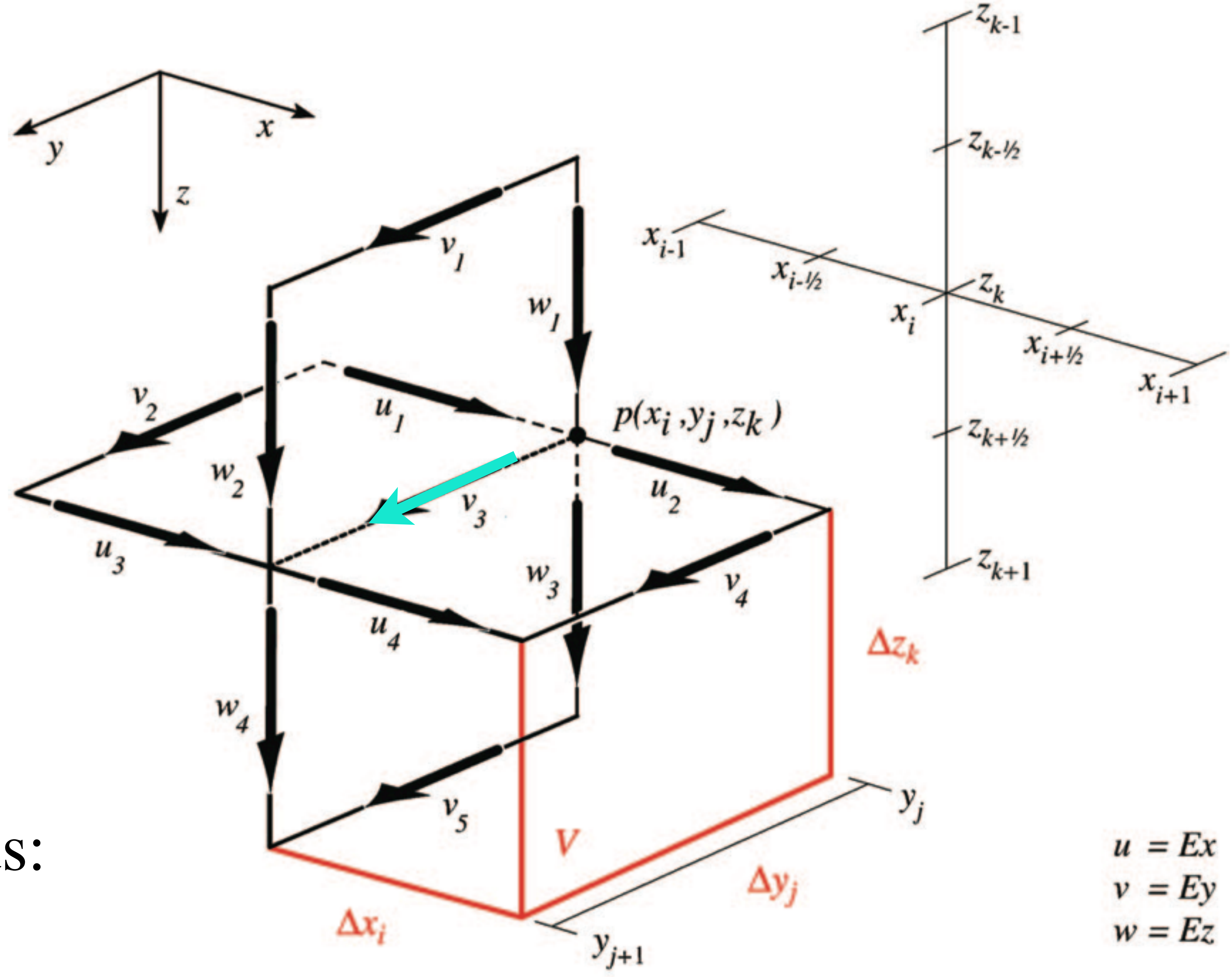
Some Boundary Conditions

Continuity of tangential electric field: $E_1 = E_2$

Continuity of normal electric current: $J_1 = J_2$

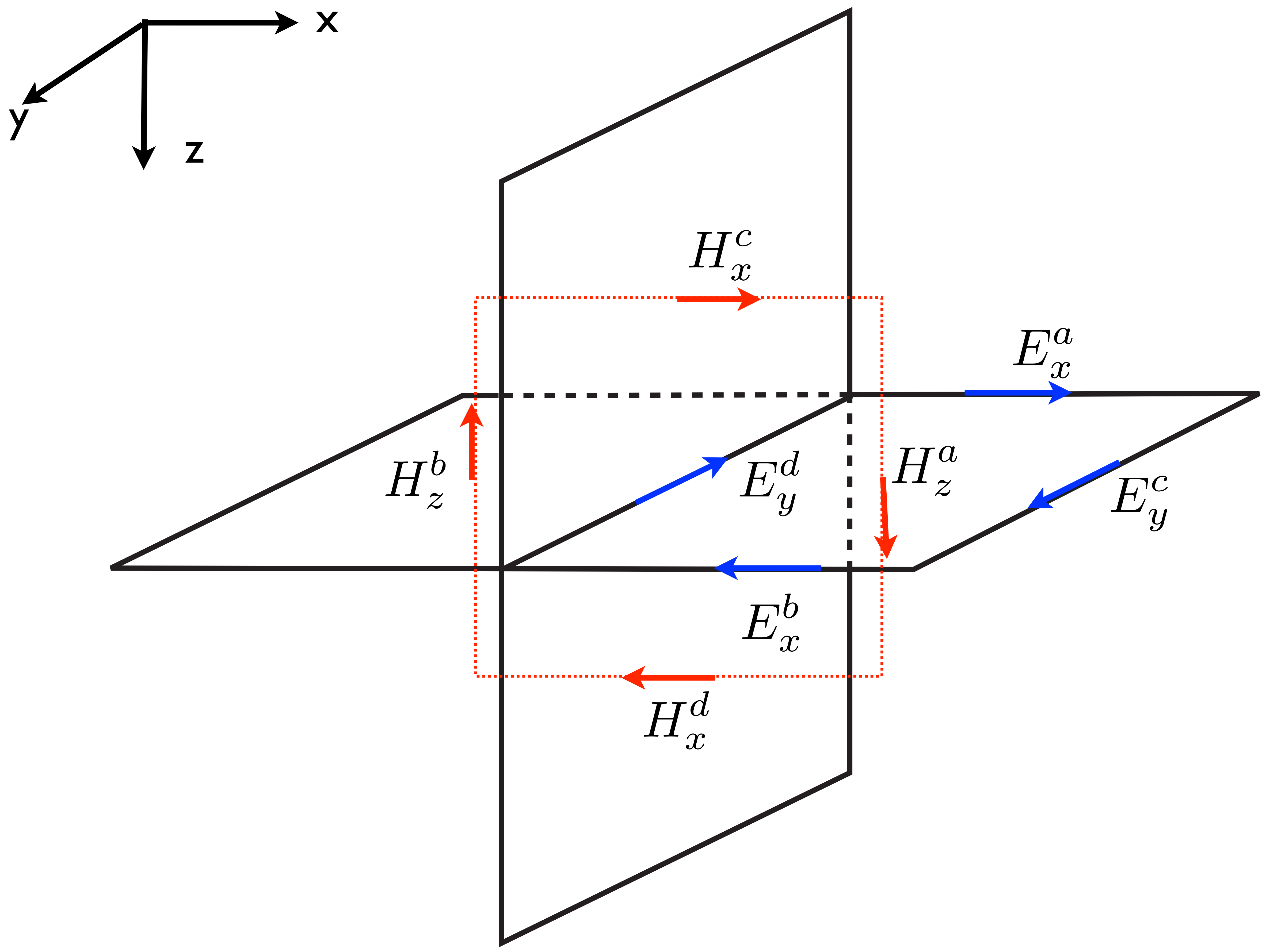


Using a staggered-grid of **E** along cell edges ensures continuity of tangential electric fields:



Curl-curl operator approximated using finite differences defined on a paddle wheel around each edge.

$$\nabla \times \nabla \times \mathbf{E} + i\omega\sigma\mu_o\mathbf{E} = 0$$



$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z$$

$$\frac{\partial E_x}{\partial y} \approx \frac{E_x^a - E_x^b}{\Delta y}$$

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma\mathbf{E}$$

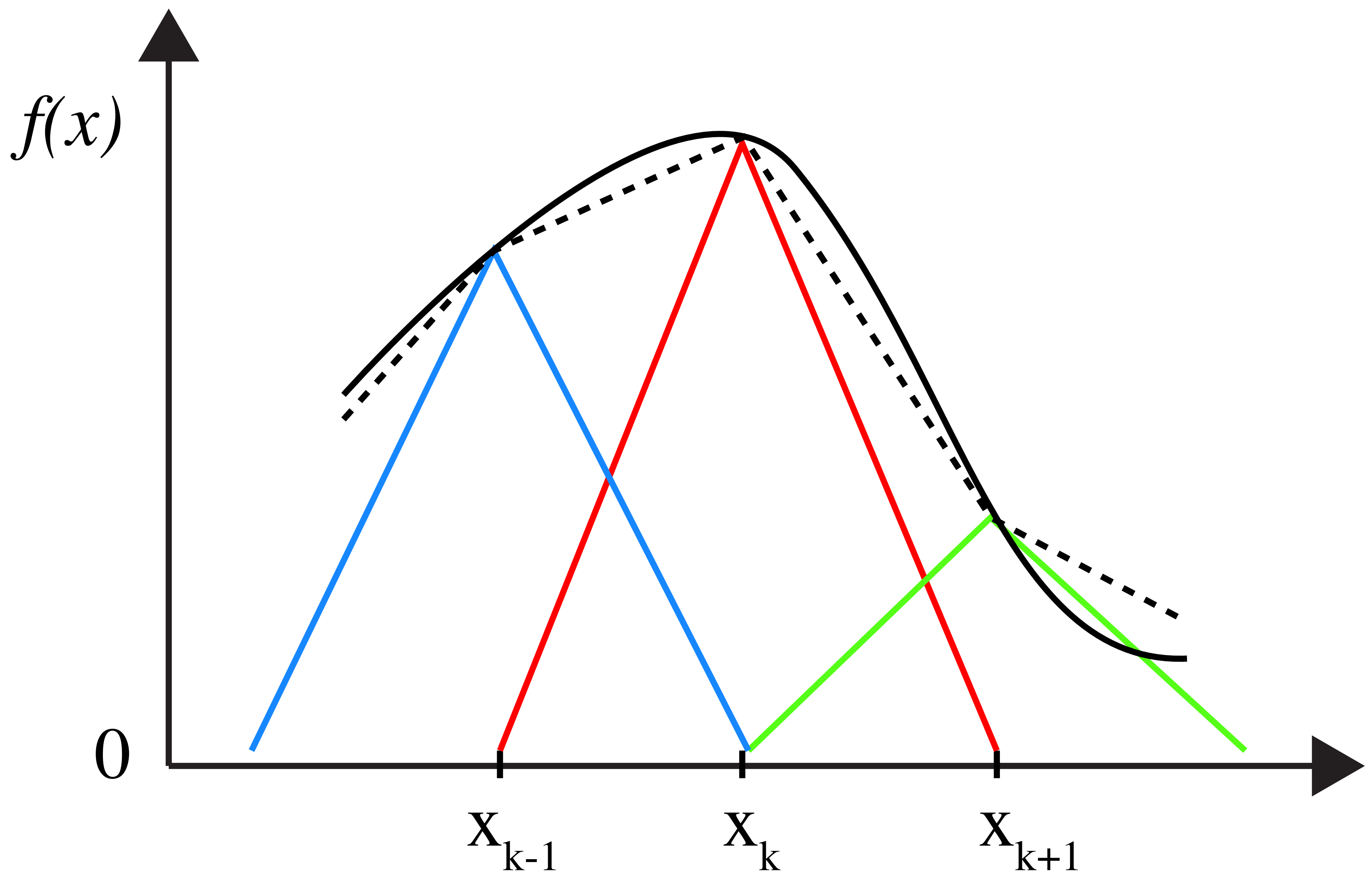
$$\mathbf{Le} = 0$$

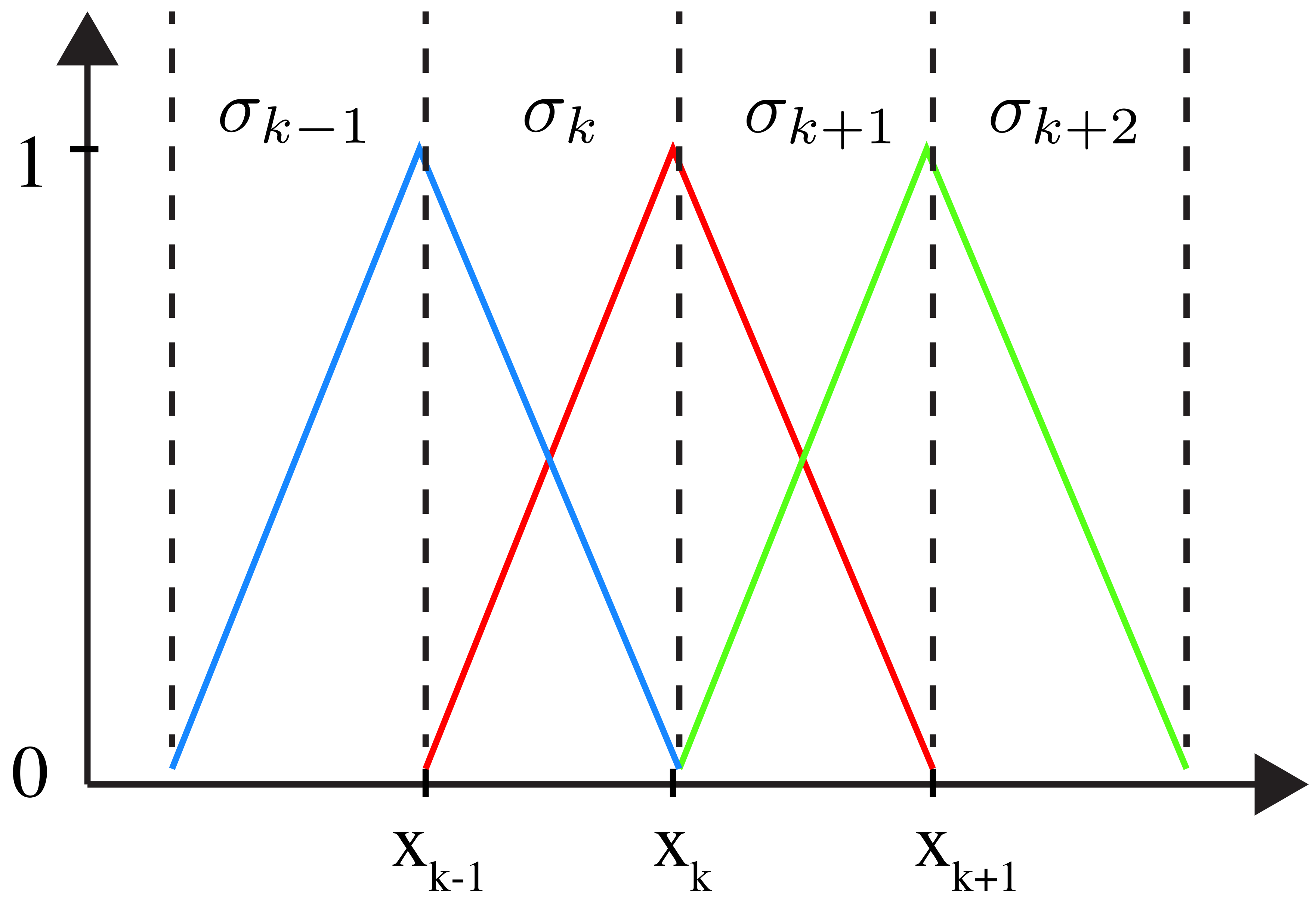
difference operators are placed into the matrix L

e is the unknown electric field on cell edges

rhs

Alternatively, in the **finite element** approach, we defined conductivity on elements, and used linear basis functions to describe how *E* behaves



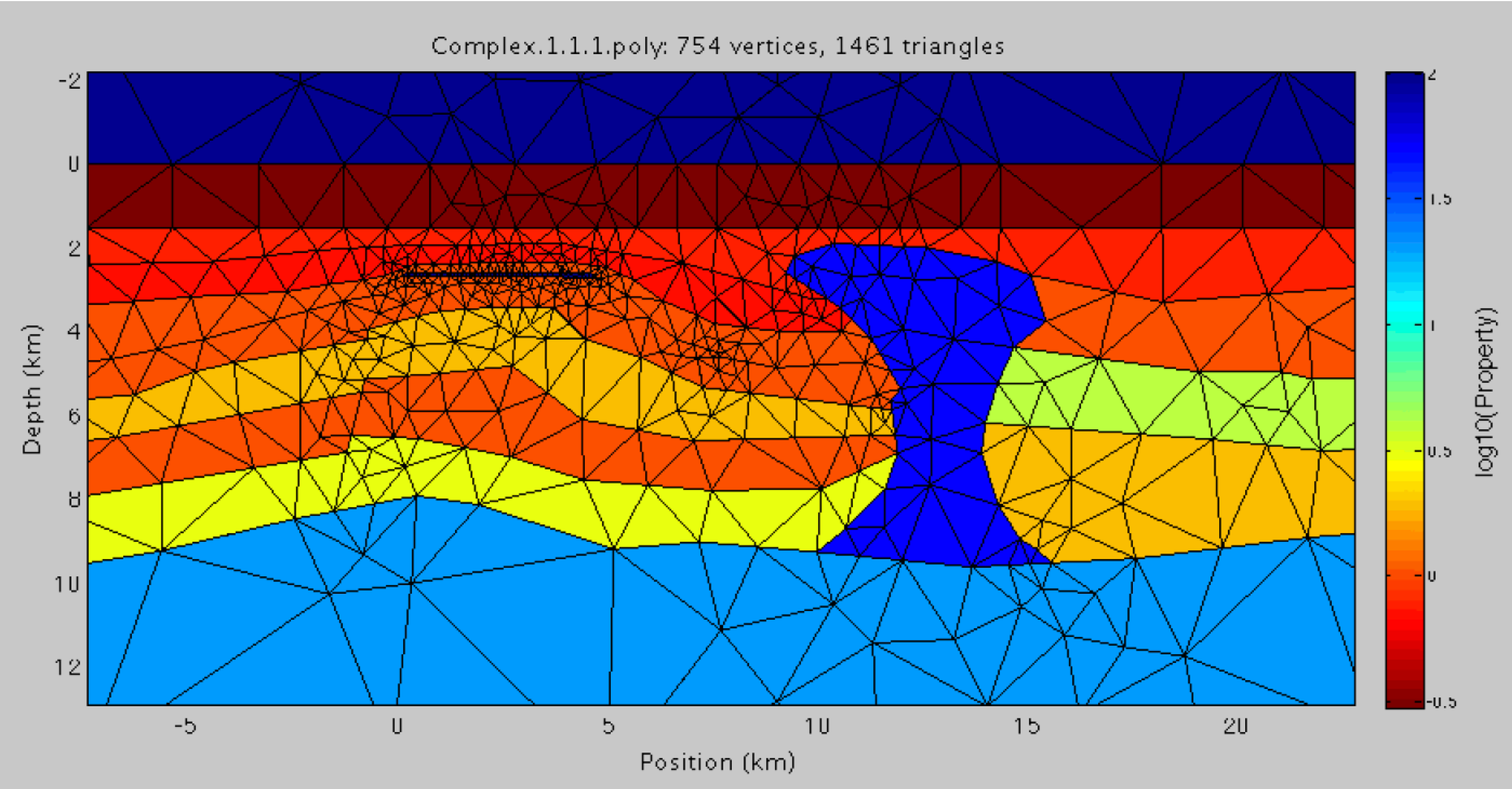


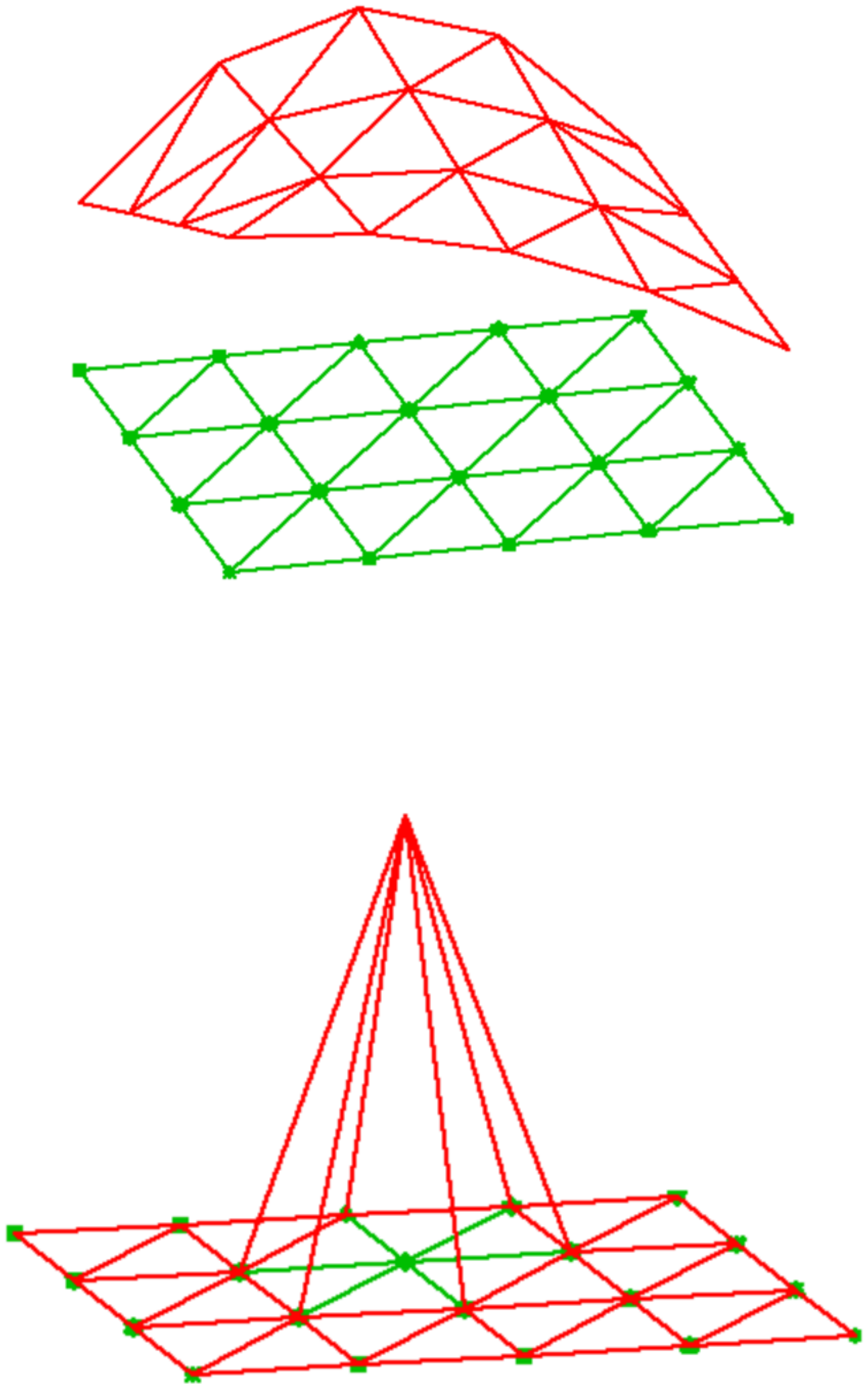
This produces a more complicated but still linear system

$$\sum_{k=1}^{n} \int_{\Omega_j} \left[v_k'(z) v_j'(z) + i\omega\mu_o \sigma(z) v_k(z) v_j(z) \right] dz E_k = 0 \quad \text{for } j = 1, ..., n$$

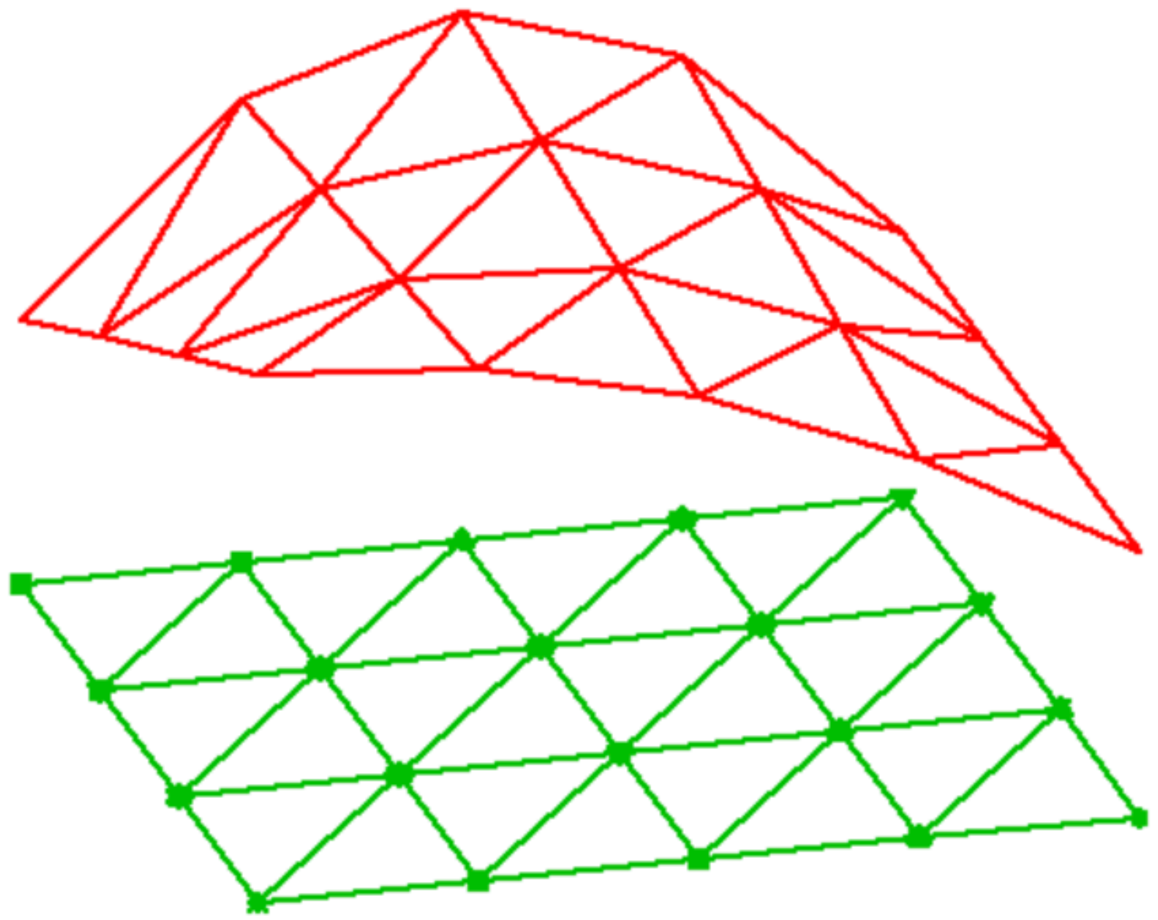
$$\mathbf{A}E = b$$

Finite elements in 2D. Electric fields defined on the nodes, conductivity is constant within each element.





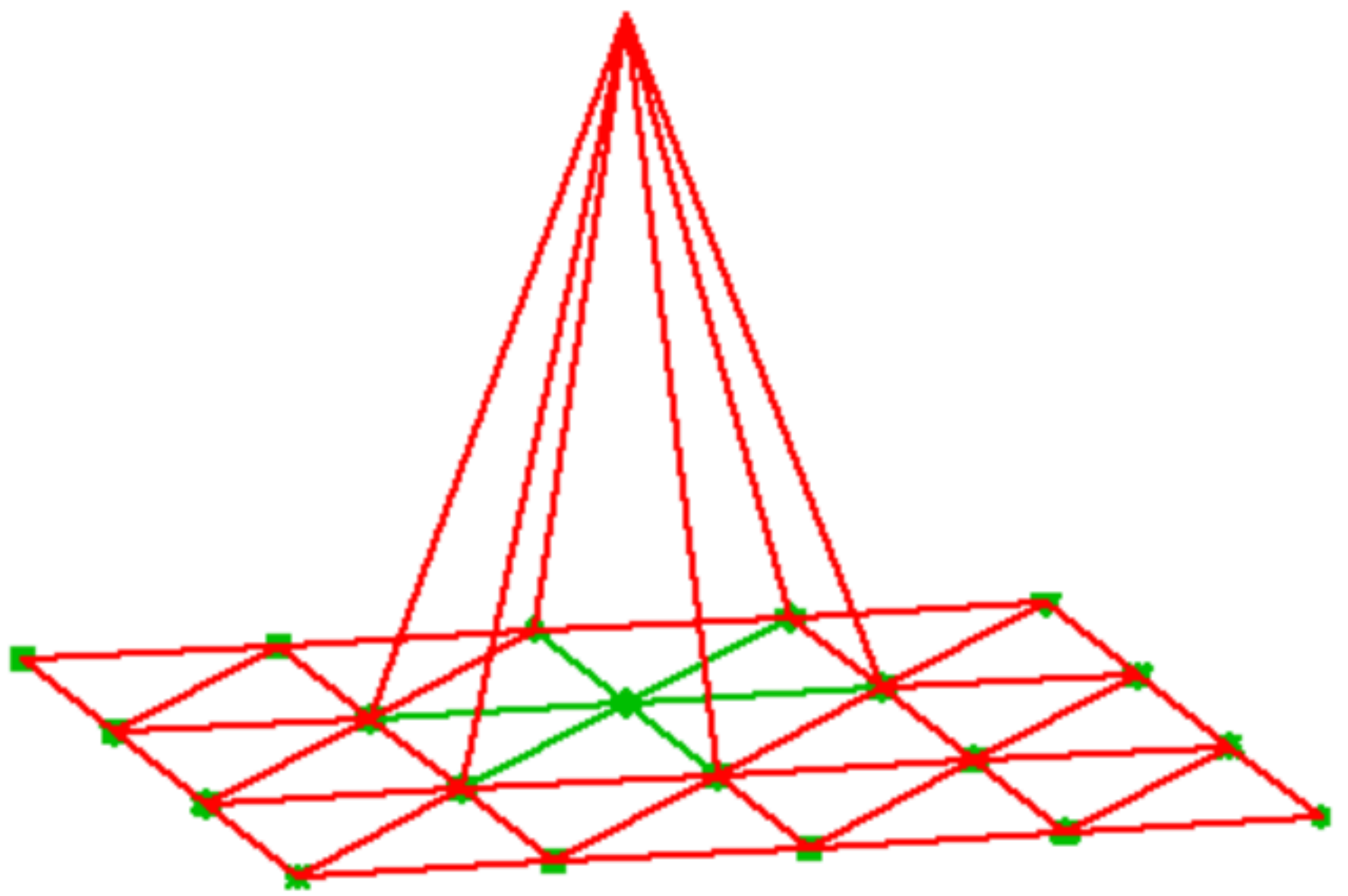
Finite elements in 2D. Electric fields defined on the nodes, conductivity is constant within each element.



$$\sum_{e=1}^{m}\sum_{j=1}^{n}\int_{\Omega_e}(\nabla v_i \cdot \nabla v_j + i\omega\mu_o\sigma_e v_i v_j)E_j \; d\Omega_e = 0, \quad i = 1, ..., n$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A}_{ij} = \sum_{e=1}^{m}\int_{\Omega_e}(\nabla v_i \cdot \nabla v_j + i\omega\mu_o\sigma_e v_i v_j)$$

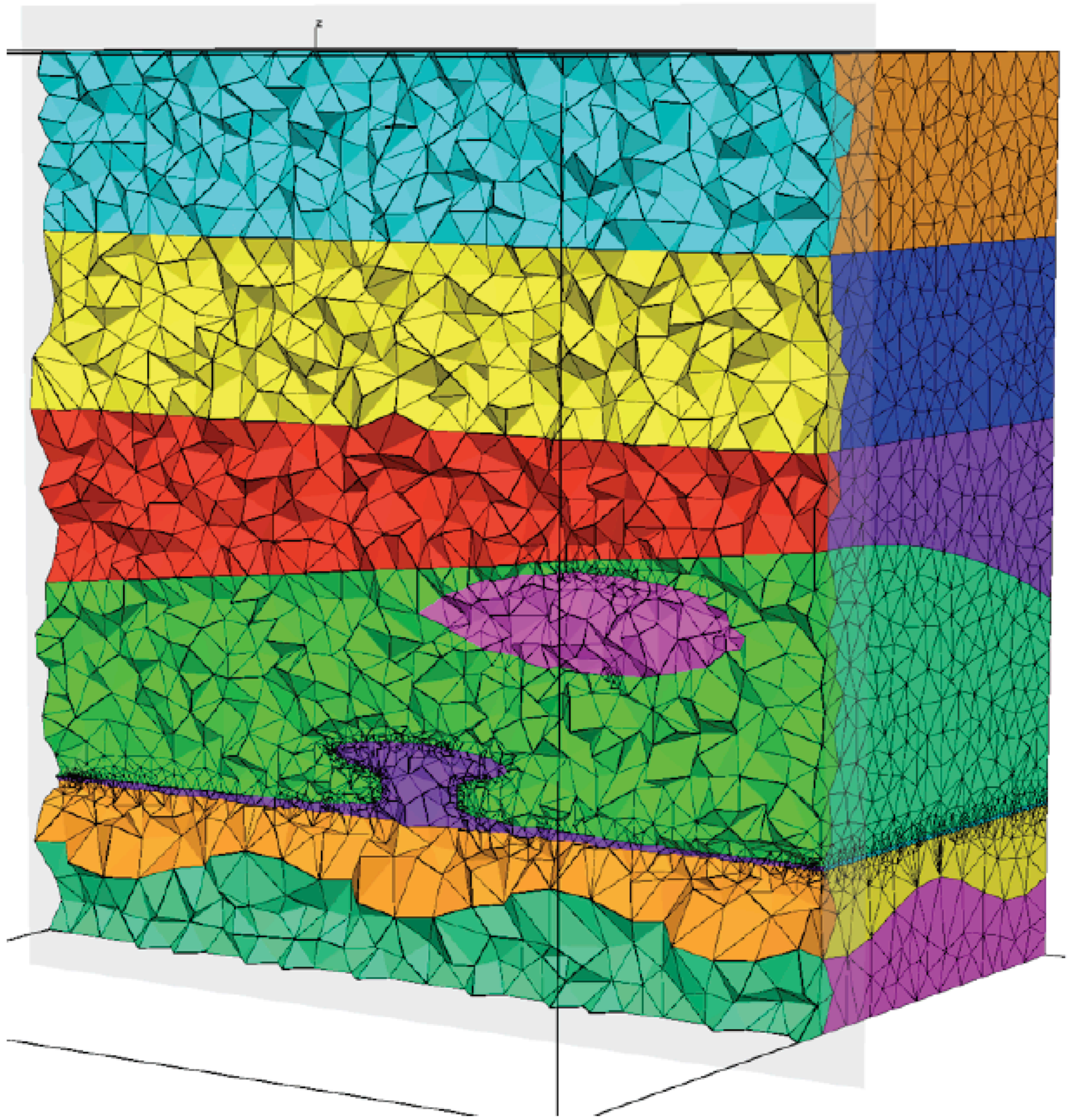


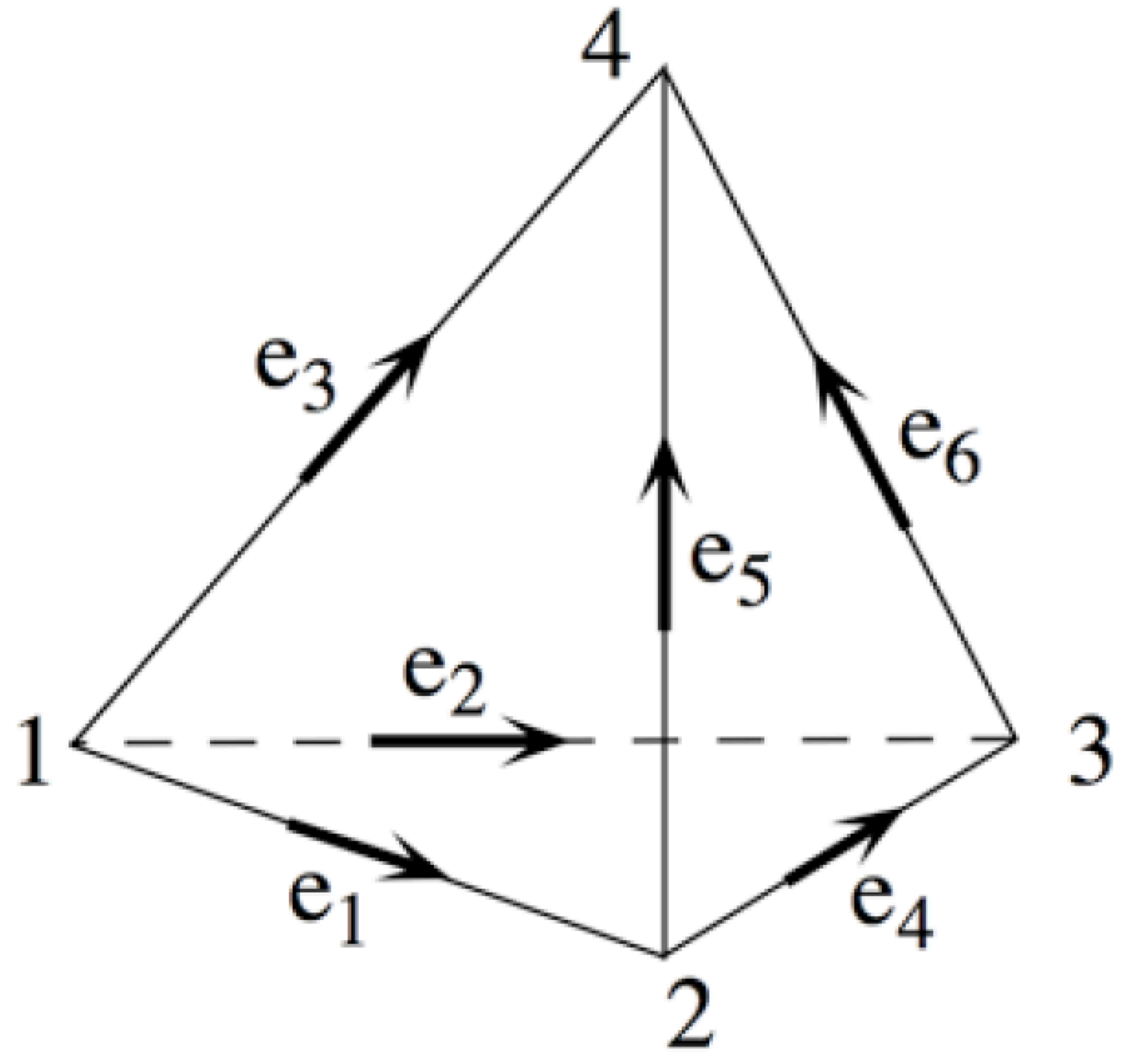
$$x_j = E_j$$

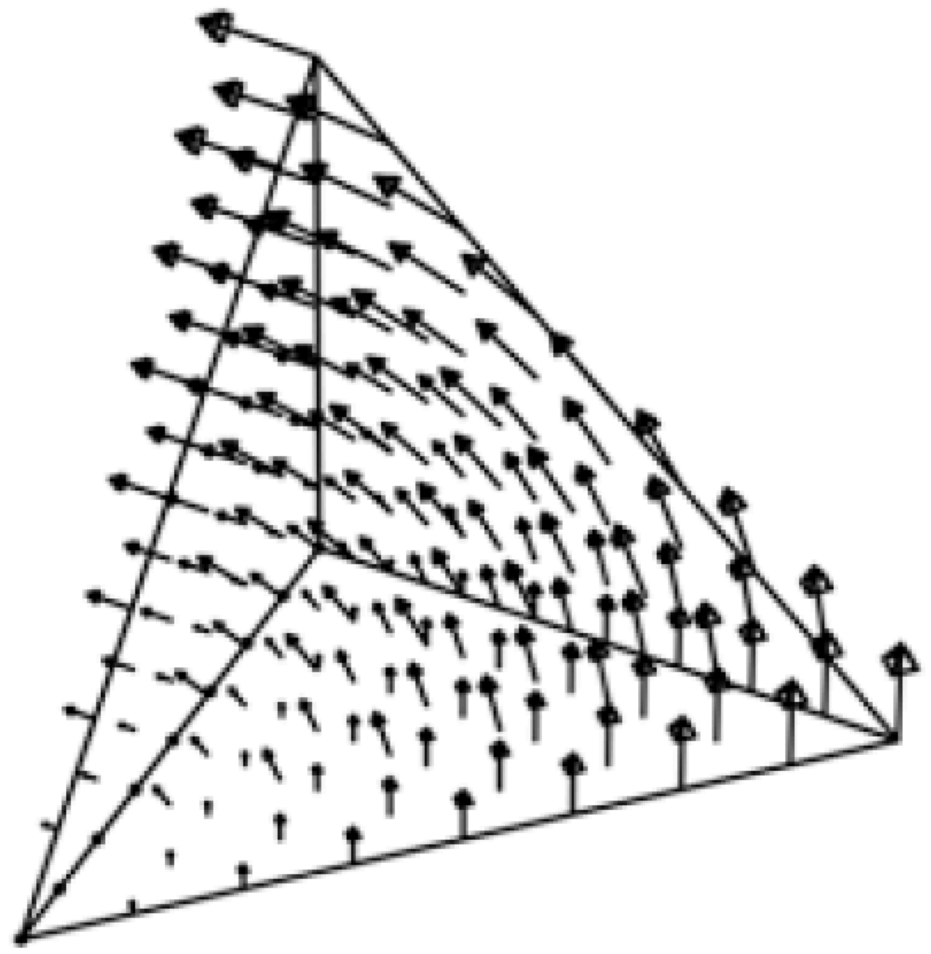
$$b_j = 0$$

(plus boundary conditions)

Finite elements in 3D. The electric fields are now defined on the element edges.







Linear basis for e6

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