SIOG 231
GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 2

Classical Electrodynamics

1/11/2024

Today

- A bit of vector calculus, reminder of the gradient operator on a vector field, …
- Helmholtz's Theorem - writing vector fields as the sum of the gradient of a scalar potential, V, and the cross product of a vector potential, $\boldsymbol{A}$
- The origin of electric and magnetic and fields - electric field is produced by stationary charges, and the magnetic field by moving charges (currents)
- Force on a moving charge - the Lorentz Force
- Maxwell's Equations - differential and integral forms
- Static Case for Geomagnetic Modeling - i.e. no change in time
- Frequency dependence in electromagnetic physics

We are going to need a little calculus, and the use of the ∇ operator.

x,y,z Cartesian coordinate system

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

so $\nabla \mathbf{A}$ is just a **gradient**:

A is a vector,

$$\nabla \mathbf{A} = \left(\frac{\partial A_x}{\partial x}, \frac{\partial A_y}{\partial y}, \frac{\partial A_z}{\partial z}\right)$$

How would you express this in Einstein Summation Notation?
or in spherical coordinates?

$\nabla \cdot \mathbf{A}$ is the **divergence**:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$\nabla \times \mathbf{A}$ is the **curl**,

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Recall cross product of two vectors **A** and **B**

$$\mathbf{A} \times \mathbf{B} = [A_y B_z - A_z B_y \ , \ A_z B_x - A_x B_z \ , \ A_x B_y - A_y B_x]$$

and $\nabla \cdot \nabla \mathbf{A}$ is the **Laplacian**.

$$\nabla \cdot \nabla \mathbf{A} = \nabla^2 \mathbf{A} = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}$$

Useful Vector Identities: **A** is a vector, *s*,*t* are scalars

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$ Divergence of a curl of a vector is zero (I1)

$$\nabla \times (\nabla s) = 0$$ Curl of a scalar field is zero (I2)

$$\nabla (st) = s\nabla t + t\nabla s$$ Product rule for vector gradient of scalars (I3)

$$\nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s\nabla \cdot \mathbf{A}$$ Product rule for divergence (I4)

$$\nabla \times (s\mathbf{A}) = s\nabla \times \mathbf{A} + \nabla s \times \mathbf{A}$$ Product rule for curl (I5)

Dot product rule

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A} \quad \text{(I6)}$$

Cross product rules

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \quad \text{(I7)}$$

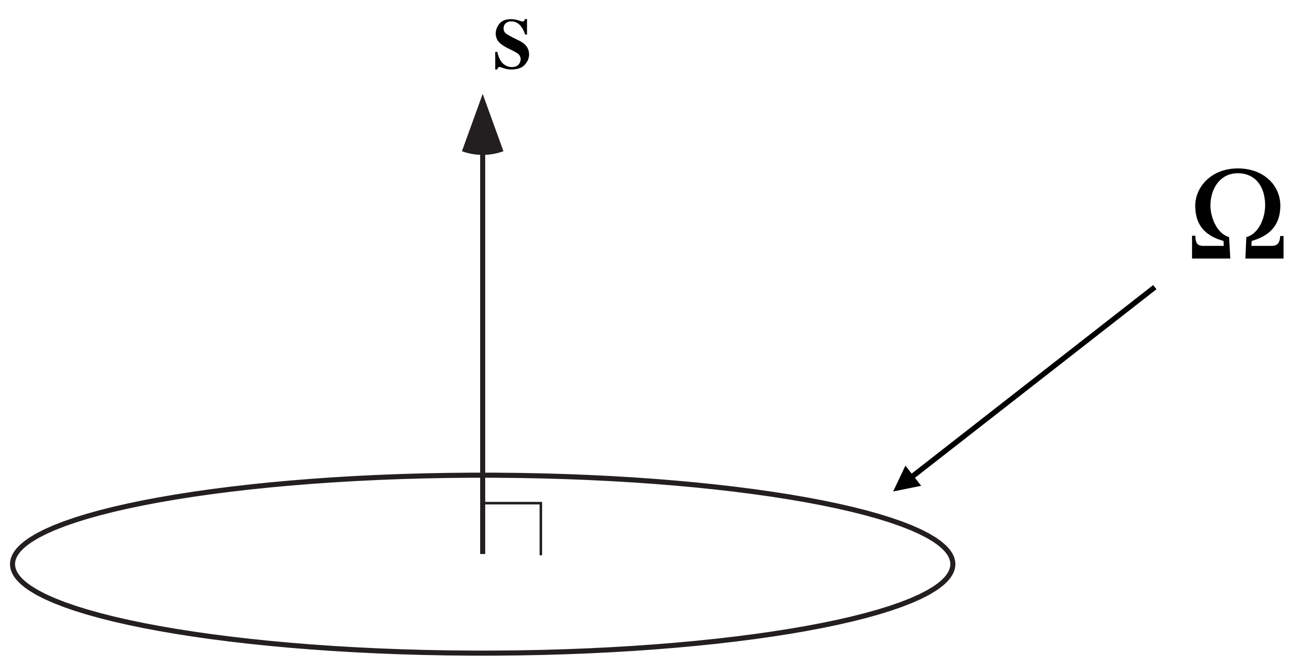
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \quad \text{(I8)}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$ Curl of curl (I9)

For more see wikipedia.org/wiki/Vector_calculus_identities

$$\mathbf{A} \times \mathbf{B} = [A_y B_z - A_z B_y \ , \ A_z B_x - A_x B_z \ , \ A_x B_y - A_y B_x]$$

Integrating over surfaces:



A surface vector, $\mathbf{s}$, has a direction that is the *outward normal* to the surface and a magnitude proportional to the area of the surface. An infinitesimal surface element, d$\mathbf{s}$, is useful for integrating things over surfaces. In particular, if we have a vector field $\mathbf{A}$, then the integral

$$\int_{\Omega} \mathbf{A} \cdot \mathrm{d}\mathbf{s} = \int_{\Omega} A.\cos\theta.\mathrm{ds}$$

over surface Ω is the *Flux* through the surface. θ is the angle between $\mathbf{A}$ and d$\mathbf{s}$.

The line integral of a vector field:

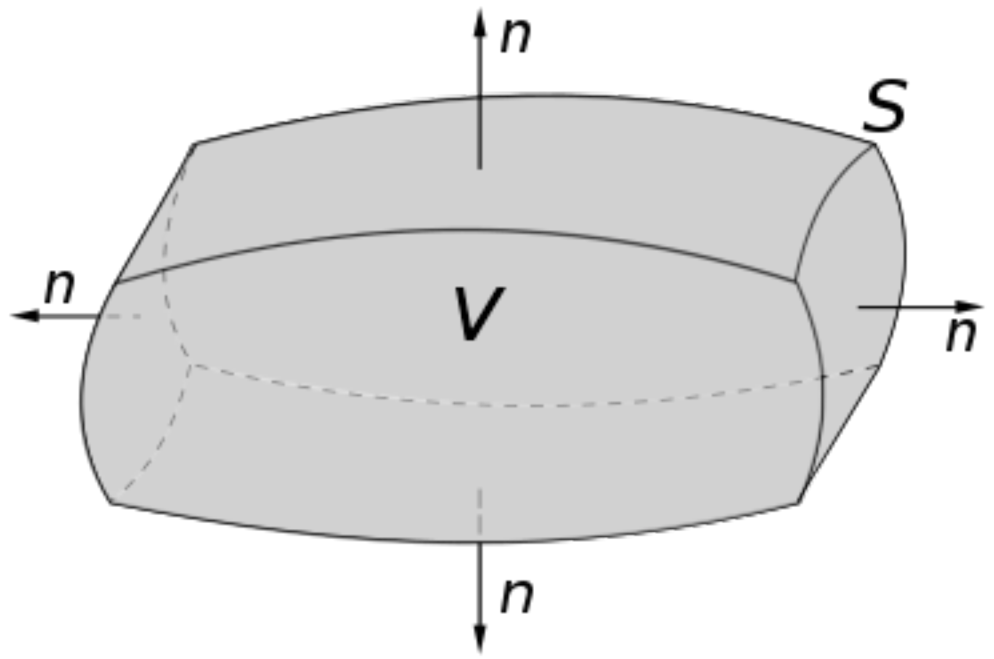
$$\int_C \mathbf{A}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{A}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ \mathrm{d}t$$

Gauss' Divergence Theorem

The divergence theorem links the flux of a (continuously differentiable) vector field **A** through a closed surface S to the divergence of the field in the volume V enclosed by the surface.

At any point on the surface $S = \partial V$ we can define the outward pointing unit normal vector $\hat{\mathbf{n}}$. Then the divergence theorem states

$$\int_V (\nabla.\mathbf{A})dV = \int_{\partial V} (\mathbf{A} \cdot \hat{\mathbf{n}})dS$$



In words, we are relating the sum (integral) of all the sources in the volume V to the total flow across the boundary S.

The divergence theorem allows us to write some physical laws in two ways: (1) a differential form - one quantity is the divergence of another) (2) an integral form - flux one quantity through a closed surface is equal to another quantity

e.g. Gauss's laws in electrostatics, magnetism, and gravity.

Suppose *F* is any vector field -
this could be ***E*** or ***B*** or in gravity studies **g**…

Helmholtz's theorem

Hermann von Helmholtz 1821-1894

$$\boldsymbol{F} = -\nabla V + \nabla \times \boldsymbol{A}$$

V is called a scalar potential and **A** a vector potential.
These two potentials can be explicitly computed from the following two integrals:

$$V(\boldsymbol{r}) = \frac{1}{4\pi} \int \frac{\nabla \cdot \boldsymbol{F}(\boldsymbol{s})}{|\boldsymbol{r} - \boldsymbol{s}|} d^3 \boldsymbol{s}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi} \int \frac{\nabla \times \boldsymbol{F}(\boldsymbol{s})}{|\boldsymbol{r} - \boldsymbol{s}|} d^3 \boldsymbol{s}$$

This also applies to gravity

Gravitational Potential $V(\mathbf{r})$ at location $\mathbf{r}$ is generated by density distribution $\rho(\mathbf{s})$ as a function of position $\mathbf{s}$

$$V(\mathbf{r}) = -G \int d^3\mathbf{s} \frac{\rho(\mathbf{s})}{|\mathbf{r} - \mathbf{s}|}$$

$$\nabla^2 V = 4\pi G \rho$$ Poisson's Equation

and

$$\mathbf{F} = \mathbf{g} = -\nabla V$$

Moving on to *E* and *B*

Electrostatics and Magnetic Field Review

- https://en.wikipedia.org/wiki/Electrostatics
    - Coulomb's law, Electric field, Gauss' Law, Poisson and Laplace's Equations, Electrostatic Approx'n
- https://en.wikipedia.org/wiki/Magnetic_field
    - The B-field, the H-field, Units, Biot-Savart and Ampère's laws
- See also Introduction to Electrodynamics by Griffiths doi:10.1017/9781108333511

Basic Assumptions

Our mathematical description of electromagnetism will be interns of macroscopic continuum physical quantities measured in SI units:

$\mathbf{B}(r, \theta, \phi, t)$: Magnetic field in Tesla (T)

$\mathbf{E}(r, \theta, \phi, t)$: Electric field in Volts/meter (V/m)

$\mathbf{J}(r, \theta, \phi, t)$: Electric Current Density in Amperes /square meter (A/m^2)

$\rho(r, \theta, \phi, t)$: Electric Charge Density in Coulombs /cubic meter (C/m^3)

Constants:

ϵ_0: Electrical permittivity of free space $= 8.85 \times 10^{-12} F/m$

μ_0: Magnetic permeability of free space $= 4\pi \times 10^{-7} H/m$

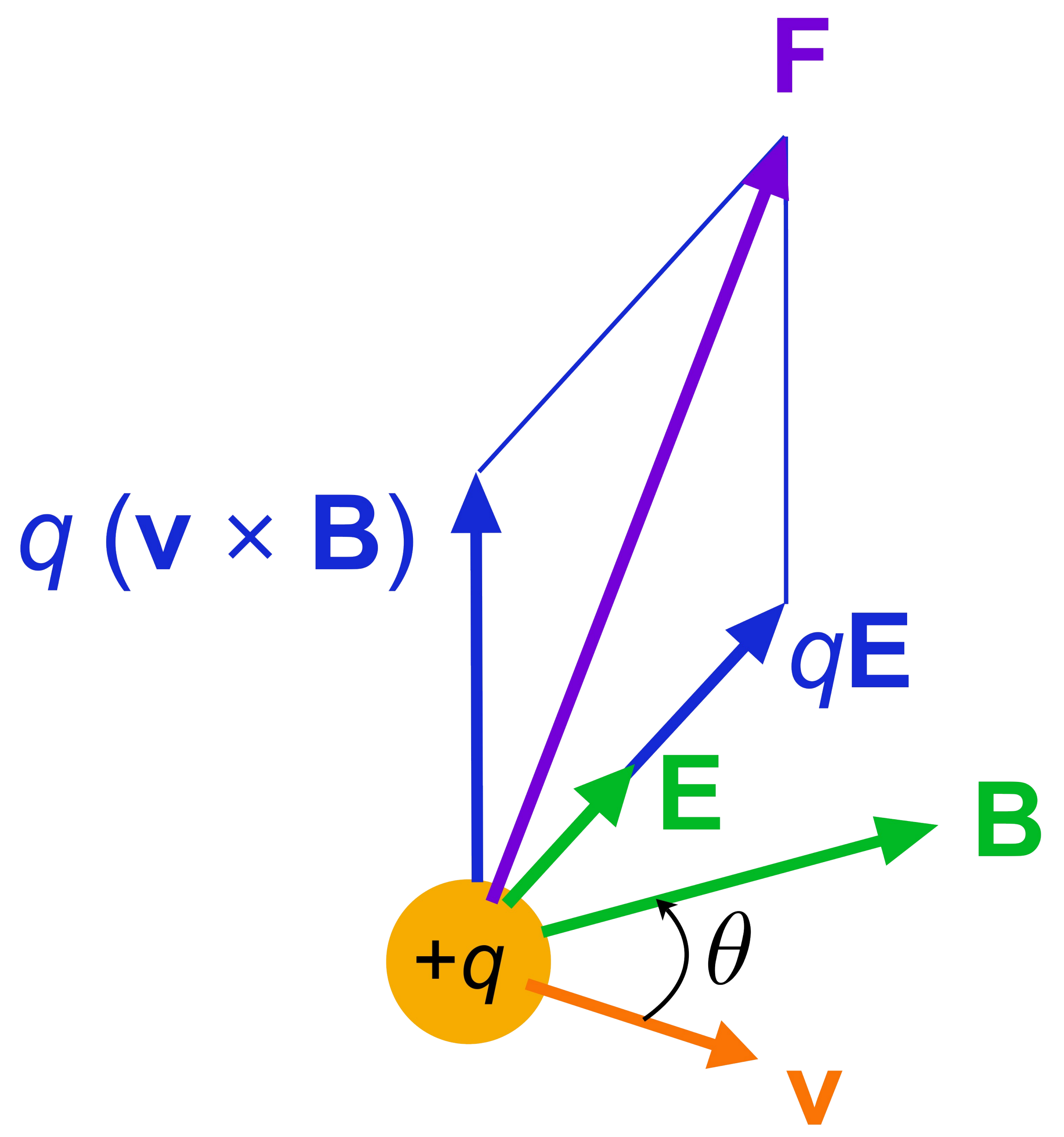
$(\epsilon_0 \mu_0)^{-\frac{1}{2}} = c = 3 \times 10^8 m/s$

Actually in the new SI system since 2019 μ_0 is determined experimentally; $4\pi \times 1.00000000055(15) \times 10^{-7} H/m$ is a recently measured value in the new system.

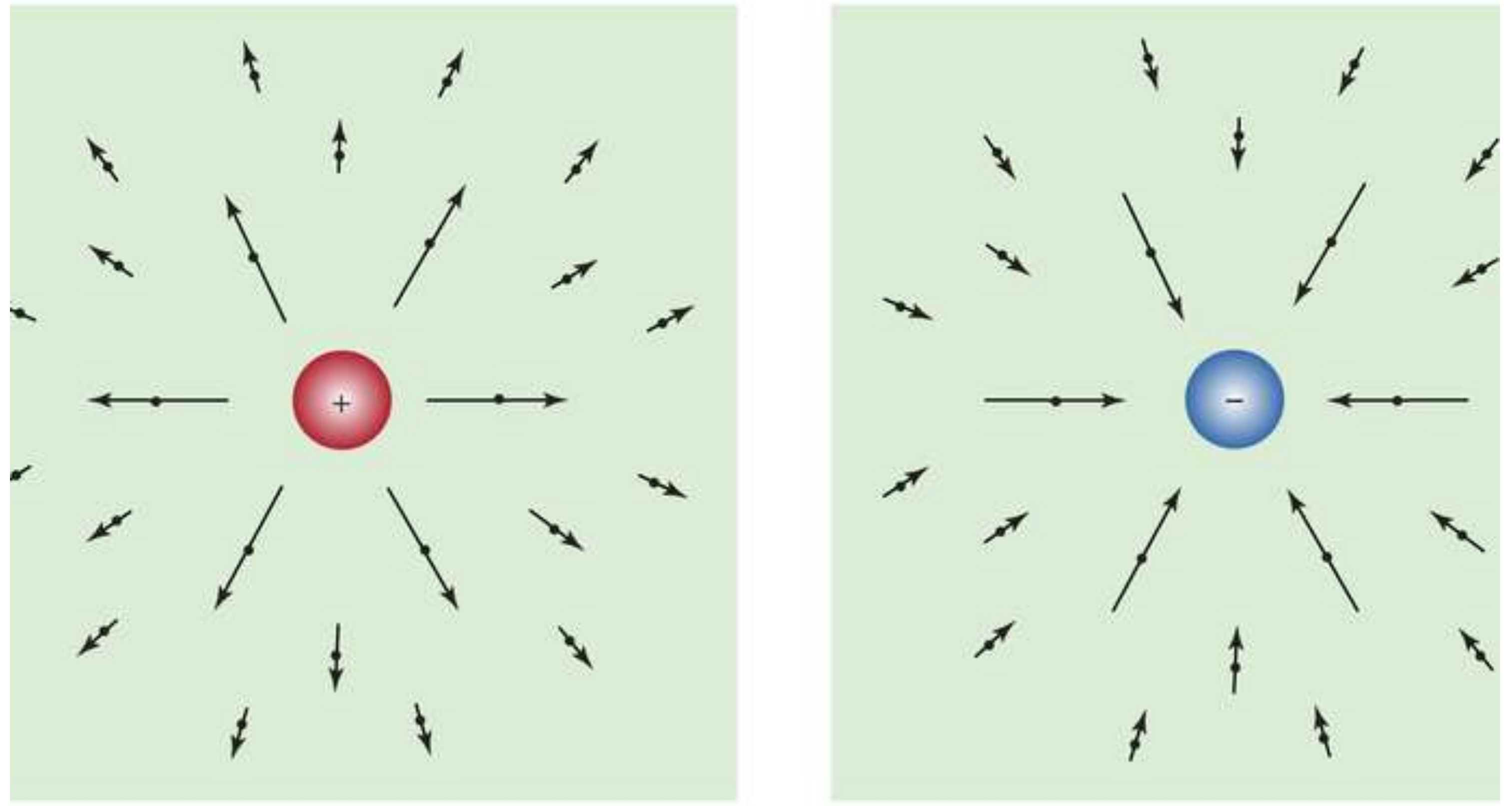
The Lorentz Force

Hendrik Antoon Lorentz
1853 -1928

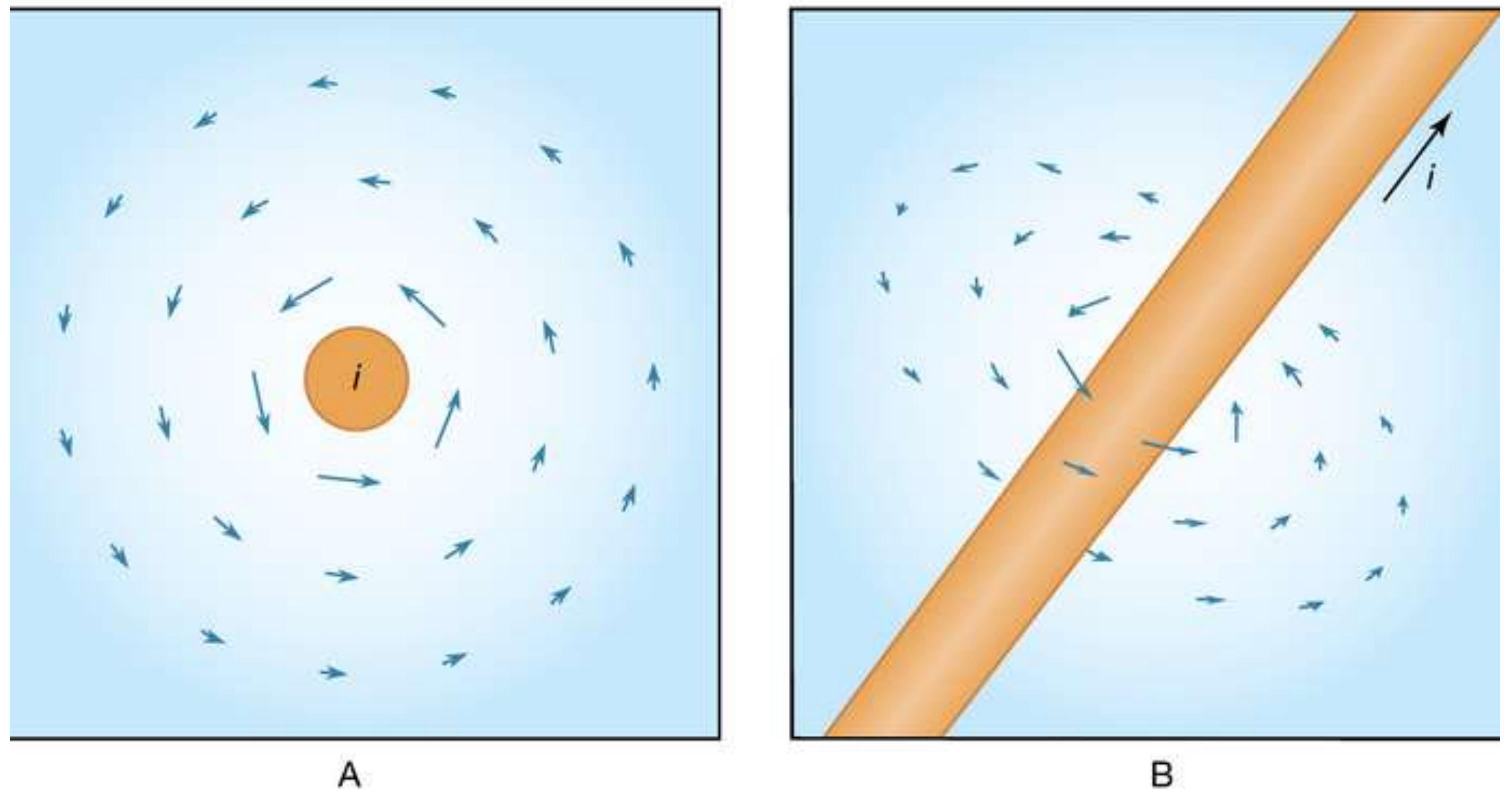
Electric and magnetic force on a point charge in motion



$$\boldsymbol{F} = q[\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r},t)]$$



Electric field due to static positive (left) or negative (right) electric charge



Magnetic field of a long wire carrying current i - i.e., charge in motion

Maxwell’s equations: An Overview

- The relationship between magnetic fields, electric fields and electrical currents are encapsulated in just 4 simple equations:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

Gauss's Law: E-fields produced by charge density

$$\nabla \cdot \boldsymbol{B} = 0$$

Lack of monopole sources of B-field so B-field is solenoidal

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Faraday's Law of Induction:

E-field induced by changing B-field

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Ampere-Maxwell Law: B-field produced

by currents or by changing E-field

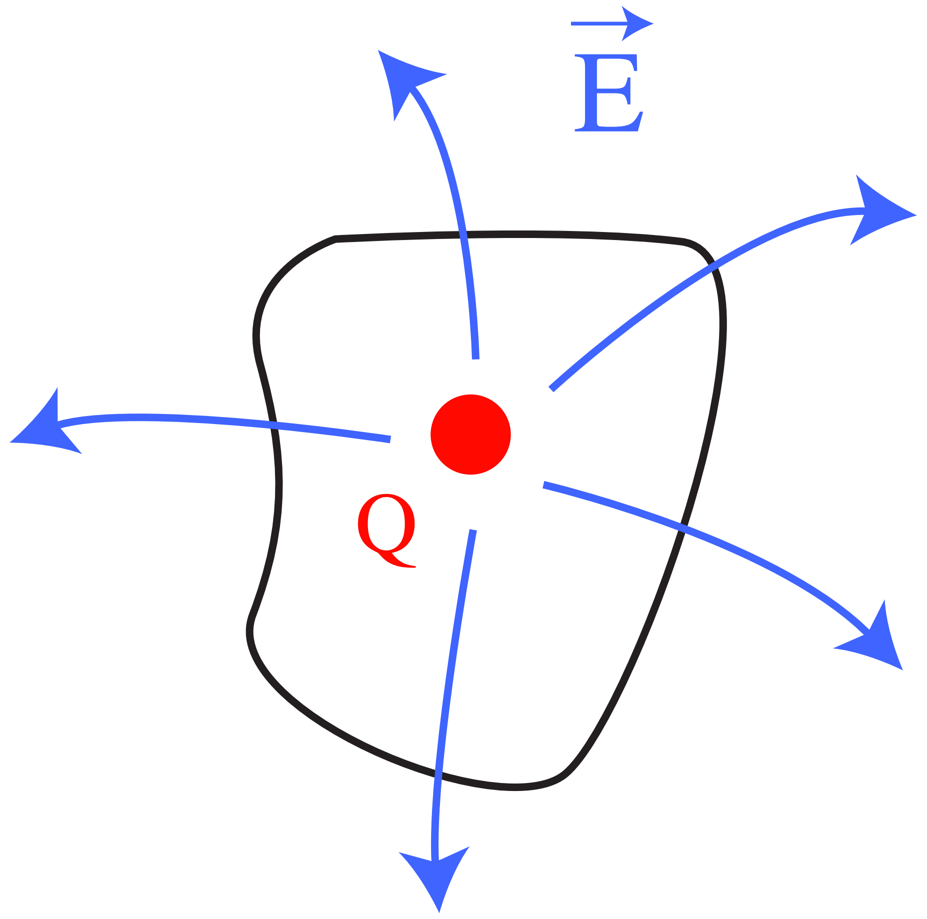
- Note, this is the form of Maxwell’s equations for materials with no permanent magnetization or electric polarization

Gauss' Law:

$$\int_\Omega \mathbf{E} \cdot \mathbf{ds} = \frac{Q}{\epsilon_o} \qquad\qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$$

Q is charge, C
ρ is charge density, C/m^3
ϵ_o is permittivity of free space, = 8.895×10^{-12} F/m.

Gauss' Law says that the electric field leaving a volume is proportional to the enclosed charge.

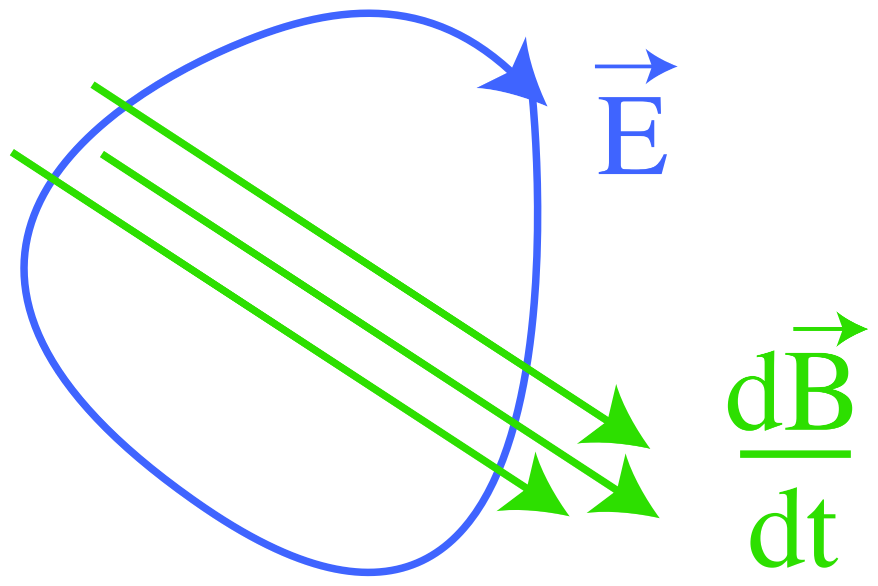


Faraday's Law:

$$\oint_c \mathbf{E} \cdot \mathbf{dl} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Φ_B is magnetic flux

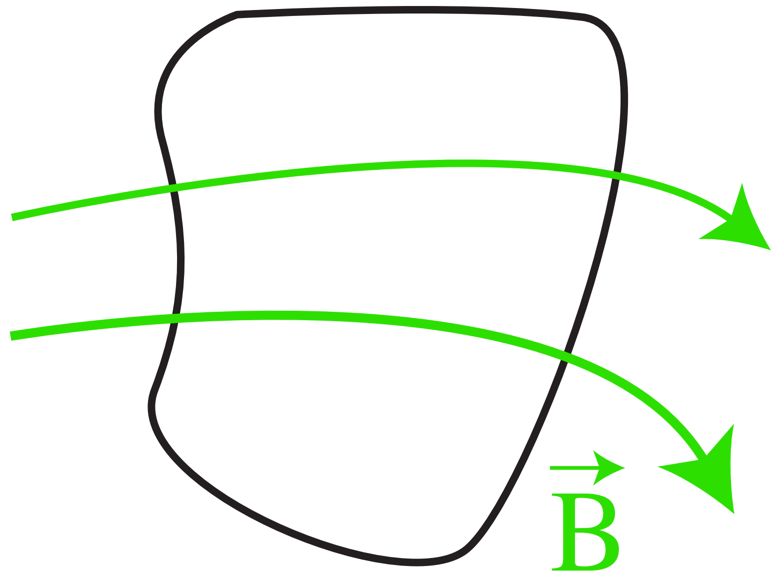
Faraday's Law says that the electric field integrated around a loop (i.e. the voltage) is given by the time rate of change of the enclosed magnetic flux.



Gauss' Law (magnetism):

$$\int_{\Omega} \mathbf{B} \cdot d\mathbf{s} = 0 \qquad\qquad \nabla \cdot \mathbf{B} = 0$$

Gauss' Law for magnetism says that there are no magnetic monopoles. Any flux entering a volume has to leave it.

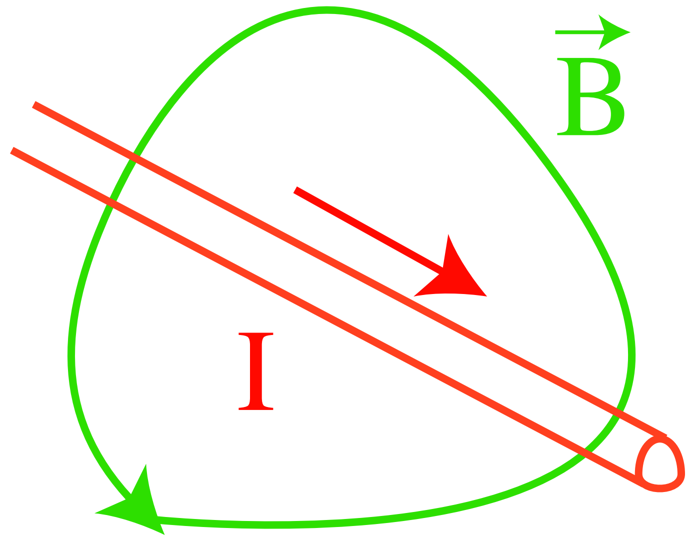


Ampère's Law:

$$\oint_c \mathbf{B} \cdot \mathbf{dl} = \mu_o I \qquad\qquad \nabla \times \mathbf{B} = \mu_o \mathbf{J}$$

I is electric current, A
$\mathbf{J}$ is current density, A/m^2
μ_o is permeability of free space, = $4\pi \times 10^{-7}$ H/m.

Ampère's Law says that an electric current will generate a circulating magnetic field.

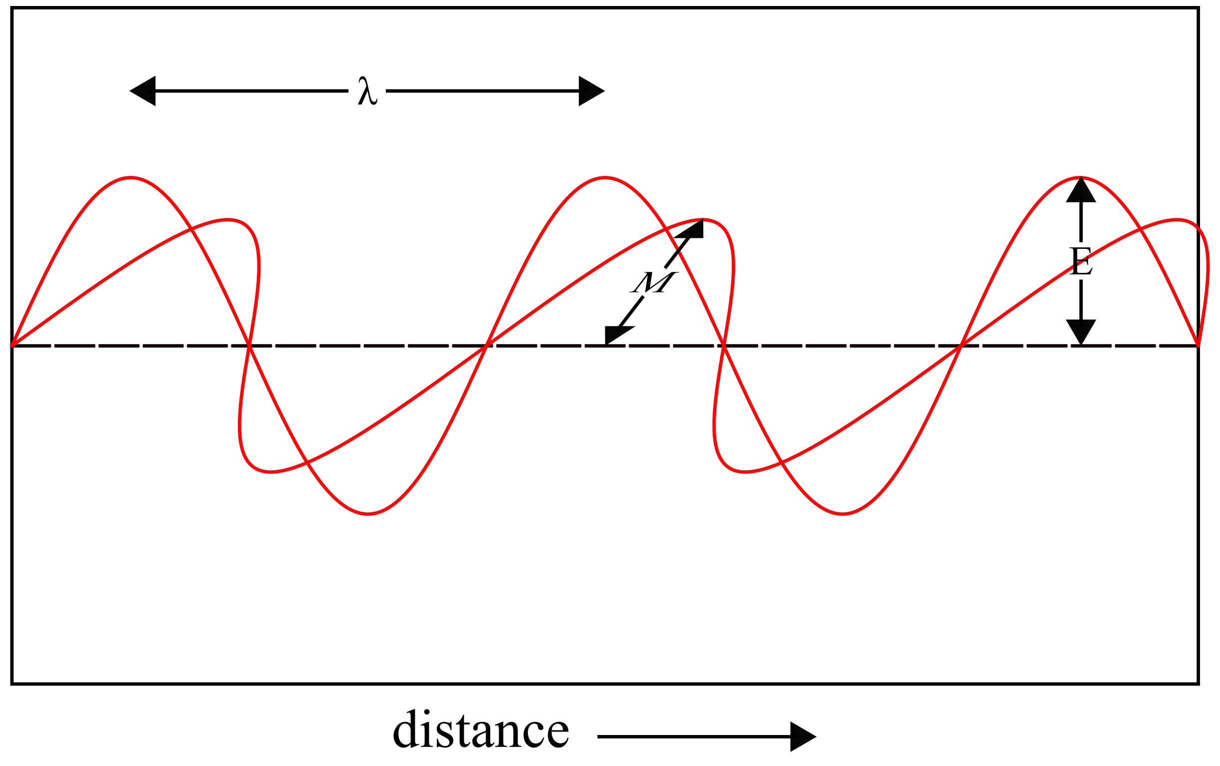


Maxwell's equations (in a vacuum):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$$

$$\nabla \times \mathbf{B} = \mu_o \left(\mathbf{J} + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \right) \qquad \nabla \cdot \mathbf{B} = 0$$

The extra term in Ampère's Law was added by Maxwell. It allows fields to exist without charges or currents, and allows electromagnetic radiation to propagate in a vacuum at speed c, where $c^2 = 1/(\mu_o \epsilon_o)$.

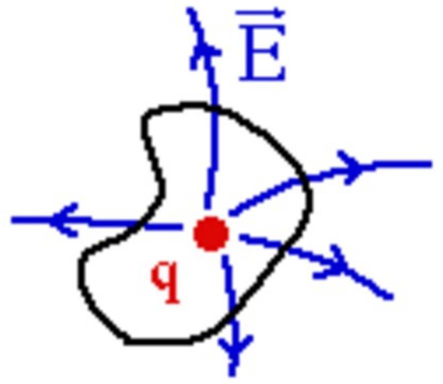


Integral Forms

Maxwell's Equations

$$\oiint \vec{E} \bullet \hat{\mathbf{n}}\, dS = \frac{q}{\varepsilon_o}$$

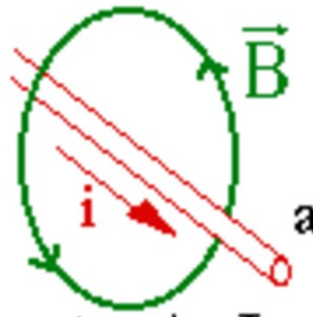
Gauss's Law



$$\oiint \vec{B} \bullet \hat{\mathbf{n}}\, dS = 0$$

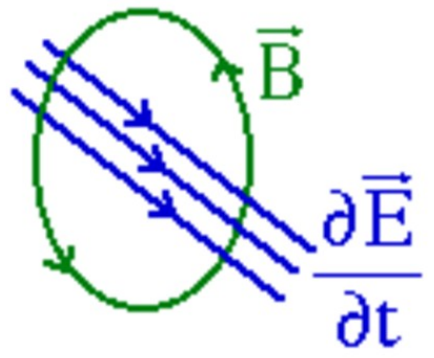
(no monopoles)

$$\oint \vec{B} \bullet d\vec{l} = \mu_o \left(i + \varepsilon_o \frac{d}{dt} \Phi_E \right)$$



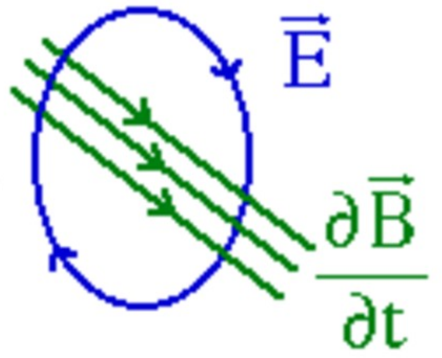
and

Ampère's Law



$$\oint \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \Phi_B$$

Faraday's Law



$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o} \qquad \vec{\nabla} \times \vec{B} = \mu_o \left(\vec{j} + \varepsilon_o \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} \bullet \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(Differential Forms)

Back to F as a vector field and Helmholtz

Force on a moving charge:

$$\mathbf{f} = q[\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, \mathbf{t})]$$

That is, **E** and **B** are defined by their effects on a charged particle.

Helmholtz Theorem:

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$

Scalar potential:

$$V(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{F}(\mathbf{s})}{|\mathbf{r} - \mathbf{s}|} \, \mathrm{d}^3\mathbf{s}$$

Vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\nabla \times \mathbf{F}(\mathbf{s})}{|\mathbf{r} - \mathbf{s}|} \, \mathrm{d}^3\mathbf{s}$$

Maxwell's equations provide expressions for the divergence and curl of the electric and magnetic fields.

E and B sources are charge density and current density

- The relationship between magnetic fields, electric fields and electrical currents are encapsulated in just 4 simple equations:

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- Note, this is the form of Maxwell's equations for materials with no permanent magnetization or electric polarization

Empirical relations are needed for describing *E* and *B* in physical media

Some basic electromagnetic theory. Let's start with

E is the electric field, measured in volts/meter.

B is the magnetic field, measured in tesla.

H is sometimes also called the magnetic field, and has units of A/m, and then **B** is called magnetic induction vector, or the magnetic flux density. But here we call **B** the magnetic field, and **H** the magnetizing field, in recognition of its relationship to magnetic polarization.

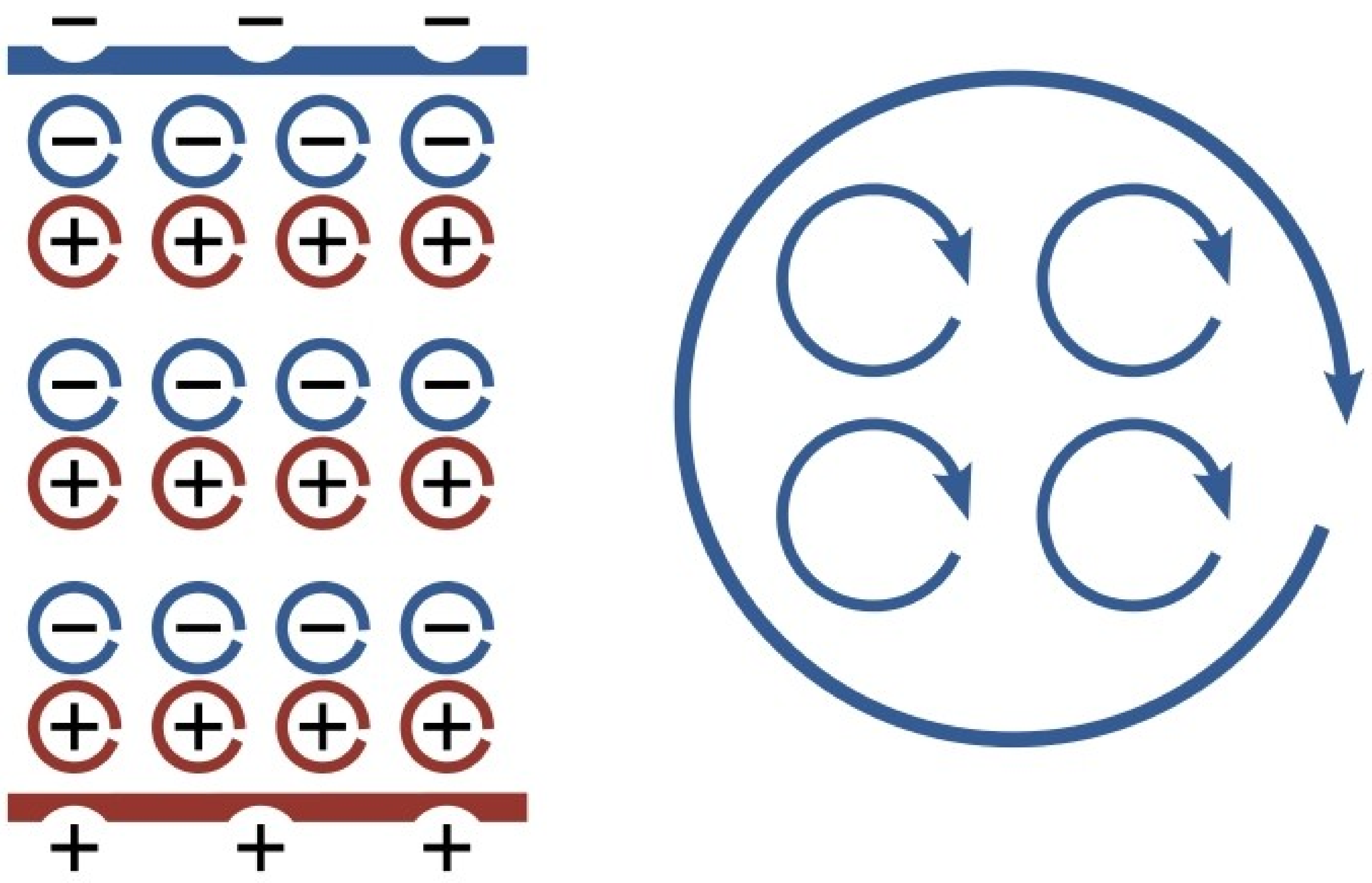
B and **H** are related through the **permeability** μ of a material.

$$\mathbf{B} = \mu\mathbf{H}$$

In a vacuum, $\mu = \mu_o = 4\pi \times 10^{-7}$ H/m

Adding Constitutive Relations

Polarized Medium



Bound charge and current

Left: A schematic view of how an assembly of microscopic dipoles produces opposite surface charges as shown at top and bottom.

Right: How an assembly of microscopic current loops add together to produce a macroscopically circulating current loop. Inside the boundaries, the individual contributions tend to cancel, but at the boundaries no cancellation occurs.

Electric and magnetic polarization per unit volume:

$$\boldsymbol{P}(\boldsymbol{r}) = \frac{1}{|B(\boldsymbol{r},a)|} \sum_{B(\boldsymbol{r},a)} \boldsymbol{p}$$

$$\boldsymbol{M}(\boldsymbol{r}) = \frac{1}{|B(\boldsymbol{r},a)|} \sum_{B(\boldsymbol{r},a)} \boldsymbol{m}$$

In a material:

$$\langle \rho \rangle = \rho^{(F)} - \nabla \boldsymbol{P}$$

$$\langle \boldsymbol{J} \rangle = \boldsymbol{J}^{(F)} + \nabla \times \boldsymbol{M} + \partial_t \boldsymbol{P}$$

Maxwell's equations in matter:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0$$

where

$\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$ is the electric displacement field, and
$\mathbf{H} = \mathbf{B}/\mu_o - \mathbf{M}$ is the magnetizing field. **H** has units of A/m.

Constitutive Relations

- Maxwell's Equations in a vacuum and a polarized medium:

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \varepsilon_0 \partial_t \boldsymbol{E} \right)$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = \rho - \nabla \cdot \boldsymbol{P}$$

$$\nabla \times \boldsymbol{B} / \mu_0 = \left(\boldsymbol{J} + \partial_t \boldsymbol{P} + \nabla \times \boldsymbol{M} + \varepsilon_0 \partial_t \boldsymbol{E} \right)$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\langle \rho \rangle = \rho^{(F)} - \nabla \boldsymbol{P}$$

$$\langle \boldsymbol{J} \rangle = \boldsymbol{J}^{(F)} + \nabla \times \boldsymbol{M} + \partial_t \boldsymbol{P}$$

Constitutive Relations

- The electric and magnetic flux densities **D**, **B** are related to the field intensities ***E***, ***H*** via the so-called constitutive relations, whose precise form depends on the material in which the fields exist.

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}$$

In a vacuum:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}$$

where ε_0, μ_0 are the electric permittivity and magnetic permeability of vacuum:

$$\varepsilon_0 = 8.854 \times 10^{-12}\ F/m$$

$$\mu_0 = 4\pi \times 10^{-7}\ H/m$$

They are related to the electric and magnetic susceptibilities of the material as follows:

$$\varepsilon = \varepsilon_0 (1 + \chi_E)$$

$$\mu = \mu_0 (1 + \chi_m)$$

The susceptibilities χ_E, χ_m are measures of the electric and magnetic polarization properties of the material.

In matter:

$$\mathbf{P} = \epsilon_o \chi_E \mathbf{E}$$

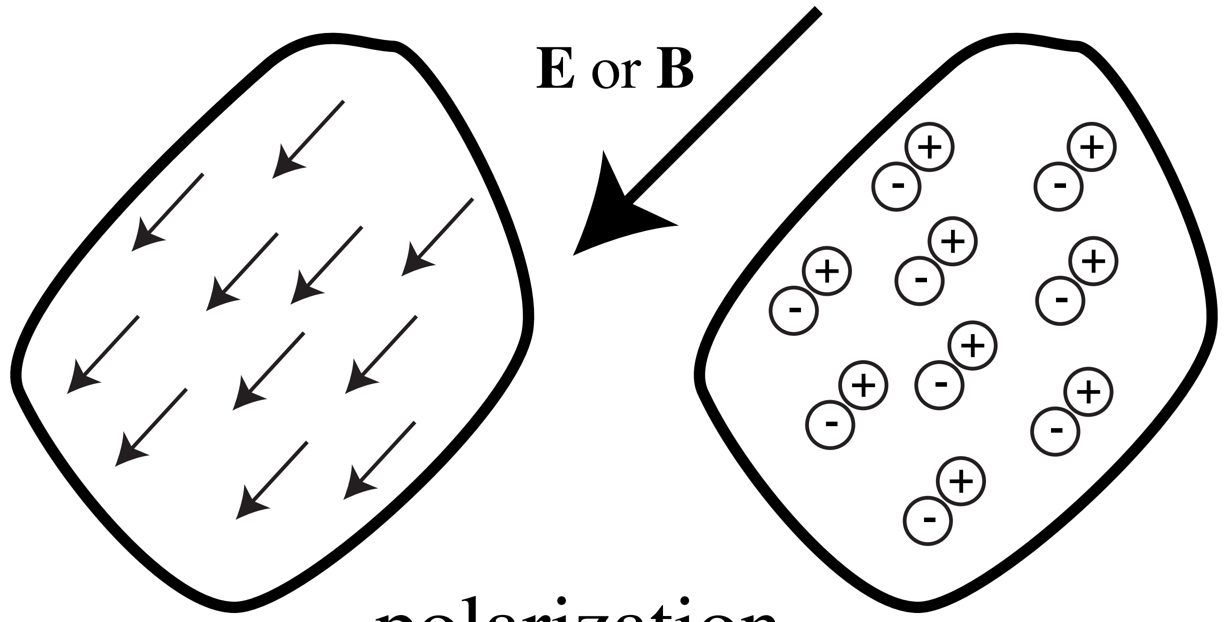
$$\mathbf{M} = \frac{\chi_M}{\mu_o} \mathbf{B}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

Georg Ohm 1789-1854

χ_E is electric susceptibility
χ_M is magnetic susceptibility
σ electrical conductivity, S/m

The last equation is, of course, Ohm's Law, but these are approximations! Matter doesn't have to be linear and isotropic. Clearly, there will be saturation phenomena.



Constitutive Relations

$$\boldsymbol{D} = \varepsilon\,\boldsymbol{E} = \varepsilon_0(1+\chi_E)\,\boldsymbol{E} = \varepsilon_0\,\boldsymbol{E} + \varepsilon_0\chi_E\,\boldsymbol{E} = \varepsilon_0\,\boldsymbol{E} + \boldsymbol{P}$$

$$\boldsymbol{B} = \mu\,\boldsymbol{H} = \mu_0(1+\chi_m)\,\boldsymbol{H} = \mu_0\,\boldsymbol{H} + \mu_0\chi_m\,\boldsymbol{H} = \mu_0\,\boldsymbol{H} + \mu_0\,\boldsymbol{M}$$

$$\boldsymbol{D} = \varepsilon_0\,\boldsymbol{E} + \boldsymbol{P}$$

$\boldsymbol{D}$ is the electric displacement vector

$$\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}$$

$\boldsymbol{H}$ is the magnetic displacement vector

$$\boldsymbol{P} = \varepsilon_0\chi_E\,\boldsymbol{E}$$

$$\boldsymbol{M} = \chi_m\,\boldsymbol{H}$$

where $\boldsymbol{P}$ and $\boldsymbol{M}$ are the electric polarization per unit volume and the magnetization, or magnetic polarization per unit volume of the material.

Constitutive Relations

- Maxwell's Equations in a vacuum and a polarized medium:

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \varepsilon_0 \partial_t \boldsymbol{E} \right)$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \partial_t \boldsymbol{D}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

Maxwell's equations in matter:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0$$

where

$\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$ is the electric displacement field, and
$\mathbf{H} = \mathbf{B}/\mu_o - \mathbf{M}$ is the magnetizing field.

$\partial \mathbf{D}/\partial t$ is called the displacement current, and can be ignored at the frequencies, length scales, and conductivities that are relevant to geomagnetic induction. Similarly, we won't concern ourselves with polarizable media, so $\mathbf{E} = \mathbf{D}/\epsilon_o$ and $\mathbf{B} = \mu_o \mathbf{H}$. **H** has units of A/m.

Application to the geomagnetic field

- Neglect of the displacement current

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \partial_t \boldsymbol{D} \qquad / \ \partial_t$$

$$\nabla \times \partial_t \boldsymbol{B} / \mu_0 = \sigma \partial_t \boldsymbol{E} + \varepsilon_0 \partial_t^2 \boldsymbol{E}$$

L, T are the characteristic length and time scales associated with EM field changes we wish to study. We can roughly replace space derivatives by 1/L and time derivatives by 1/T:

$$\nabla \times \nabla \times \boldsymbol{E} + \mu_0 \sigma \partial_t \boldsymbol{E} + \mu_0 \varepsilon_0 \partial_t^2 \boldsymbol{E} = 0$$

$$|\boldsymbol{E}| [1 + \mu_0 \sigma (L^2 / T) + (L / cT)^2] = 0$$

If $L=10^3$km; $T=10$s and $\sigma \approx 10^{-3}$S/m

$$\mu_0 \sigma (L^2 / T) \approx 120$$

$$(L / cT)^2 \approx 10^{-5}$$

Substituting Ohm's Law into Faraday's Law:

$$\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E} + \mu_o \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \quad .$$

The time derivative is proportional to frequency:

$$\nabla \times \mathbf{B} = \mu_o \mathbf{E}(\sigma + \epsilon_o \omega) \quad .$$

$$\sigma \approx 10^{-4}\ \mathrm{S/m} \qquad \epsilon_o \approx 10^{-11}\ \mathrm{F/m}$$

so unless frequency is of order megahertz (GPR) or sigma is zero (air), the second term is negligible.

All we really need is Faraday, Ampere, Gauss, and Ohm.

Pre-Maxwell Equations are what we will use

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \cancel{\varepsilon_0 \partial_t \boldsymbol{E}} \right)$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = \rho - \nabla \cdot \boldsymbol{P}$$

$$\nabla \times \boldsymbol{B} / \mu_0 = \left(\boldsymbol{J} + \partial_t \boldsymbol{P} + \nabla \times \boldsymbol{M} + \cancel{\varepsilon_0 \partial_t \boldsymbol{E}} \right)$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\langle \rho \rangle = \rho^{(F)} - \nabla \boldsymbol{P}$$

$$\langle \boldsymbol{J} \rangle = \boldsymbol{J}^{(F)} + \nabla \times \boldsymbol{M} + \partial_t \boldsymbol{P}$$

Back once again to *F* as a vector field and Helmholtz

Helmholtz's theorem

$$\boldsymbol{F} = -\nabla V + \nabla \times \boldsymbol{A}$$

V is called a scalar potential and **A** a vector potential.
These two potentials can be explicitly computed from the following two integrals:

$$V(\boldsymbol{r}) = \frac{1}{4\pi} \int \frac{\nabla \cdot \boldsymbol{F}(\boldsymbol{s})}{|\boldsymbol{r} - \boldsymbol{s}|} d^3 \boldsymbol{s}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi} \int \frac{\nabla \times \boldsymbol{F}(\boldsymbol{s})}{|\boldsymbol{r} - \boldsymbol{s}|} d^3 \boldsymbol{s}$$

Static case for geomagnetic field modeling

- Assumption: Neglect time variation in geomagnetic processes and imagine a system of stationary charges and steady current flows $\nabla \times \mathbf{E} = 0$ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

$$\boldsymbol{E} = -\nabla \phi \qquad \phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{s})}{|\boldsymbol{r}-\boldsymbol{s}|} d^3\boldsymbol{s}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{s})}{|\boldsymbol{r}-\boldsymbol{s}|} d^3\boldsymbol{s}$$

Application to the geomagnetic field

- Magnetic field representation in terms of scalar potential

Approximation: Earth's atmosphere is an insulator with no electrical currents

$$\nabla \times \boldsymbol{B} = 0$$ **since J=0**

$$\boldsymbol{B} = -\nabla \phi$$ **See vector identity number I2**

Since $\nabla \cdot \boldsymbol{B} = 0$ **no monopoles**

$$\boxed{\nabla^2 \phi = 0}$$

Useful Vector Identities: $\mathbf{A}$ is a vector, s,t are scalars

$\nabla \cdot (\nabla \times \mathbf{A}) = 0$ Divergence of a curl of a vector is zero (I1)

$\nabla \times (\nabla s) = 0$ Curl of a scalar field is zero (I2)

$\nabla (st) = s\nabla t + t\nabla s$ Product rule for vector gradient of scalars (I3)

$\nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s\nabla \cdot \mathbf{A}$ Product rule for divergence (I4)

$\nabla \times (s\mathbf{A}) = s\nabla \times \mathbf{A} + \nabla s \times \mathbf{A}$ Product rule for curl (I5)

Dot product rule $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A}$ (I6)

Cross product rules

$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$ (I7)

$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$ (I8)

$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ Curl of curl (I9)

For more see wikipedia.org/wiki/Vector_calculus_identities

$$\mathbf{A} \times \mathbf{B} = [A_y B_z - A_z B_y \ , \ A_z B_x - A_x B_z \ , \ A_x B_y - A_y B_x]$$

Application to the geomagnetic field

- Neglect of the displacement current

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}+\partial_t\boldsymbol{D} \qquad / \ \partial_t$$

$$\nabla\times\partial_t\boldsymbol{B}/\mu_0=\sigma\partial_t\boldsymbol{E}+\varepsilon_0\partial_t^2\boldsymbol{E}$$

L, T are the characteristic length and time scales associated with EM field changes we wish to study. We can roughly replace space derivatives by 1/L and time derivatives by 1/T:

$$\nabla\times\nabla\times\boldsymbol{E}+\mu_0\sigma\partial_t\boldsymbol{E}+\mu_0\varepsilon_0\partial_t^2\boldsymbol{E}=0$$

$$|\boldsymbol{E}|[1+\mu_0\sigma(L^2/T)+(L/cT)^2]=0$$

If L=10^3km; T=10s and $\sigma\approx10^{-3}$S/m

$$\mu_0\sigma(L^2/T)\approx120$$

$$(L/cT)^2\approx10^{-5}$$

Does this work at all frequencies?

Application in Electromagnetism

High frequency (megahertz)

Wave equation: Resolution ~ wavelength

Radar $$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Seismics $$\nabla^2 u = \epsilon \frac{\partial u}{\partial t} + \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Mid frequency (0.001 - 1000 Hz)

Diffusion equation: Resolution ~ size/depth

Inductive EM $$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$

Zero frequency

Laplace equation: Resolution ~ bounds only

DC Resistivity $$\nabla^2 \mathbf{E} = 0$$

Gravity/ Magnetism $$\nabla^2 U = 0$$

σ = electrical conductivity ~ $3 - 10^{-6}$ S/m

μ = magnetic permeability ~ $10^{-4} - 10^{-6}$ H/m

ϵ = electric permittivity ~ $10^{-9} - 10^{-11}$ F/m