SIOG 231
GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 3

Electromagnetic Induction and the MT Equation

1/16/2024

Review of terminology:

σ is electrical conductivity, units S/m (S = 1/Ω). Relates $\mathbf{J} = \sigma \mathbf{E}$ through Ohm's Law.

ρ is electrical resistivity, units Ωm. Just the reciprocal of conductivity.

$\mathbf{B}$ is magnetic field, units of Tesla, although nT is more useful in geophysics. Also called flux density.

$\mathbf{H}$ is magnetizing field, units of A/m. A mathematical construct. Also called magnetic field.

$\mathbf{E}$ is the electric field, units V/m. The field created by a charge.

$\mathbf{J}$ is electric current density, units A/m^2 . Flow of charge through a material.

μ is magnetic permeability. A measure of how well a material magnetizes. Relates $\mathbf{B} = \mu \mathbf{H}$.

μ_o is permeability of free space. Almost exactly $4\pi \times 10^{-7}$ Tm/A (H/m).

ϵ is electric permittivity. A measure of how charges polarize in a material. Relates $\mathbf{D} = \epsilon \mathbf{E}$.

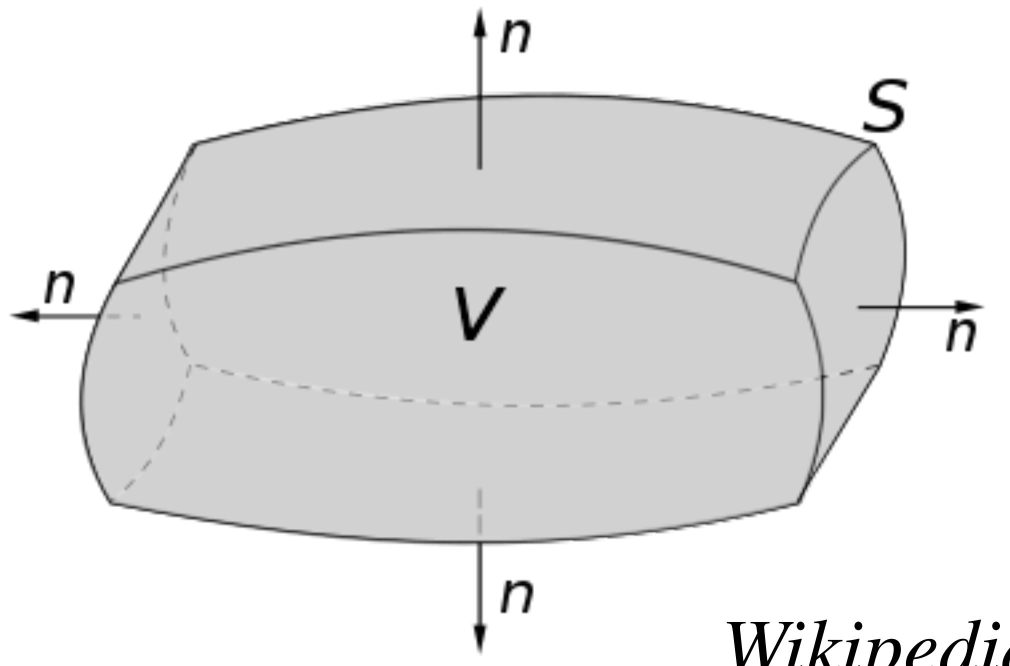
ϵ_o is permittivity of free space. It is $1/c^2\mu_o \approx$ 8.85 $\times 10^{-12}$ C/(Vm) (F/m).

Gauss' Divergence Theorem

The divergence theorem links the flux of a (continuously differentiable) vector field **A** through a closed surface *S* to the divergence of the field in the volume *V* enclosed by the surface.

At any point on the surface $S = \partial V$ we can define the outward pointing unit normal vector $\hat{\mathbf{n}}$. Then the divergence theorem states

$$\int_V (\nabla.\mathbf{A})dV = \int_{\partial V} (\mathbf{A} \cdot \hat{\mathbf{n}})dS$$



Wikipedia

In words, we are relating the sum (integral) of all the sources in the volume V to the total flow across the boundary S.

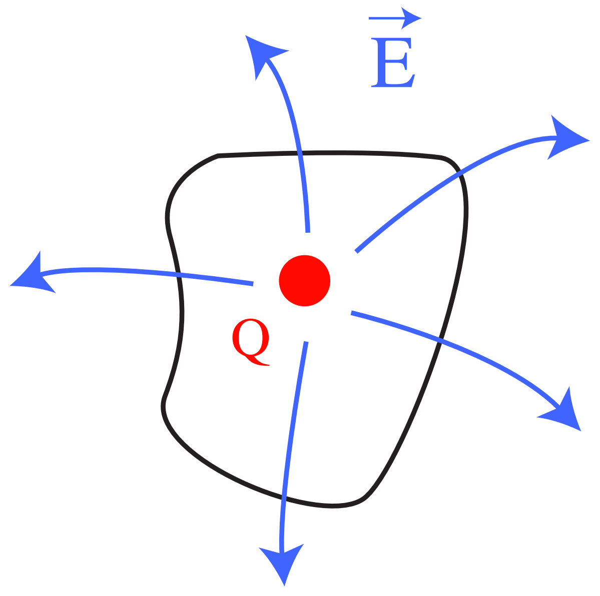
The divergence theorem allows us to write some physical laws in two ways: (1) a differential form - one quantity is the divergence of another) (2) an integral form - flux one quantity through a closed surface is equal to another quantity

e.g. Gauss's laws in electrostatics, magnetism, and gravity.

Maxwell in a vacuum:

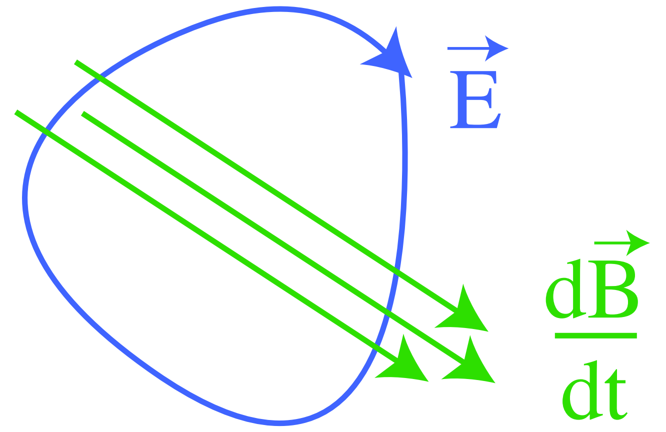
Gauss' Law:

$$\int_\Omega \mathbf{E} \cdot \mathrm{d}\mathbf{s} = \frac{Q}{\epsilon_o} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$$



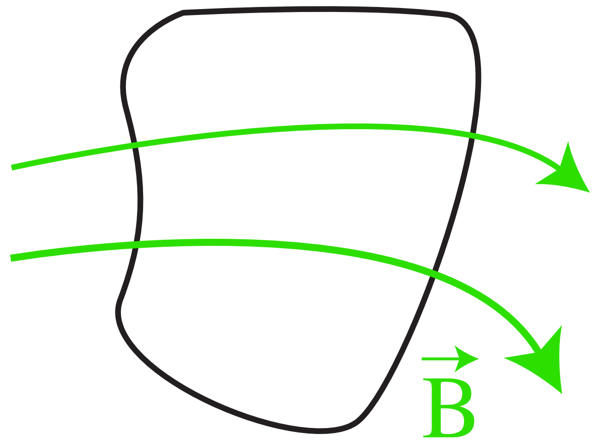
Faraday's Law:

$$\oint_c \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$



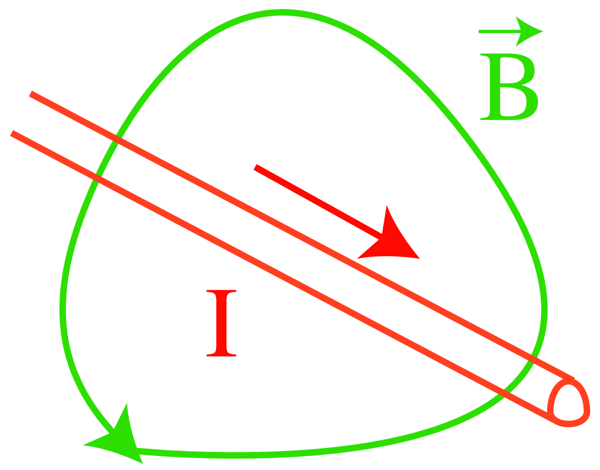
Gauss' Law (magnetism):

$$\int_\Omega \mathbf{B} \cdot \mathrm{d}\mathbf{s} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$



Ampère's Law + displacement current:

$$\oint_c \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_o(I + \epsilon_o \frac{\mathrm{d}\Phi_E}{\mathrm{d}t}) \qquad \nabla \times \mathbf{B} = \mu_o\left(\mathbf{J} + \epsilon_o \frac{\partial \mathbf{E}}{\partial t}\right)$$



All we really need is Faraday, Ampère, Gauss, and Ohm's Law:

Faraday $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Coulomb (Gauss) $$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$$

Ampère $$\nabla \times \mathbf{B} = \mu_o \mathbf{J}$$

Gauss $$\nabla \cdot \mathbf{B} = 0$$

(These are called the "pre-Maxwell equations")

Ohm $$\mathbf{J} = \sigma \mathbf{E}$$

and a few of the nine standard vector identities

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{I1}$$

$$\nabla \times (\nabla s) = 0 \tag{I2}$$

$$\nabla(st) = s\nabla t + t\nabla s \tag{I3}$$

$$\nabla \cdot (s\mathbf{A}) = \mathbf{A} \cdot \nabla s + s\nabla \cdot \mathbf{A} \tag{I4}$$

$$\nabla \times (s\mathbf{A}) = s\nabla \times \mathbf{A} + \nabla s \times \mathbf{A} \tag{I5}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A} \tag{I6}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{I7}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{I8}$$

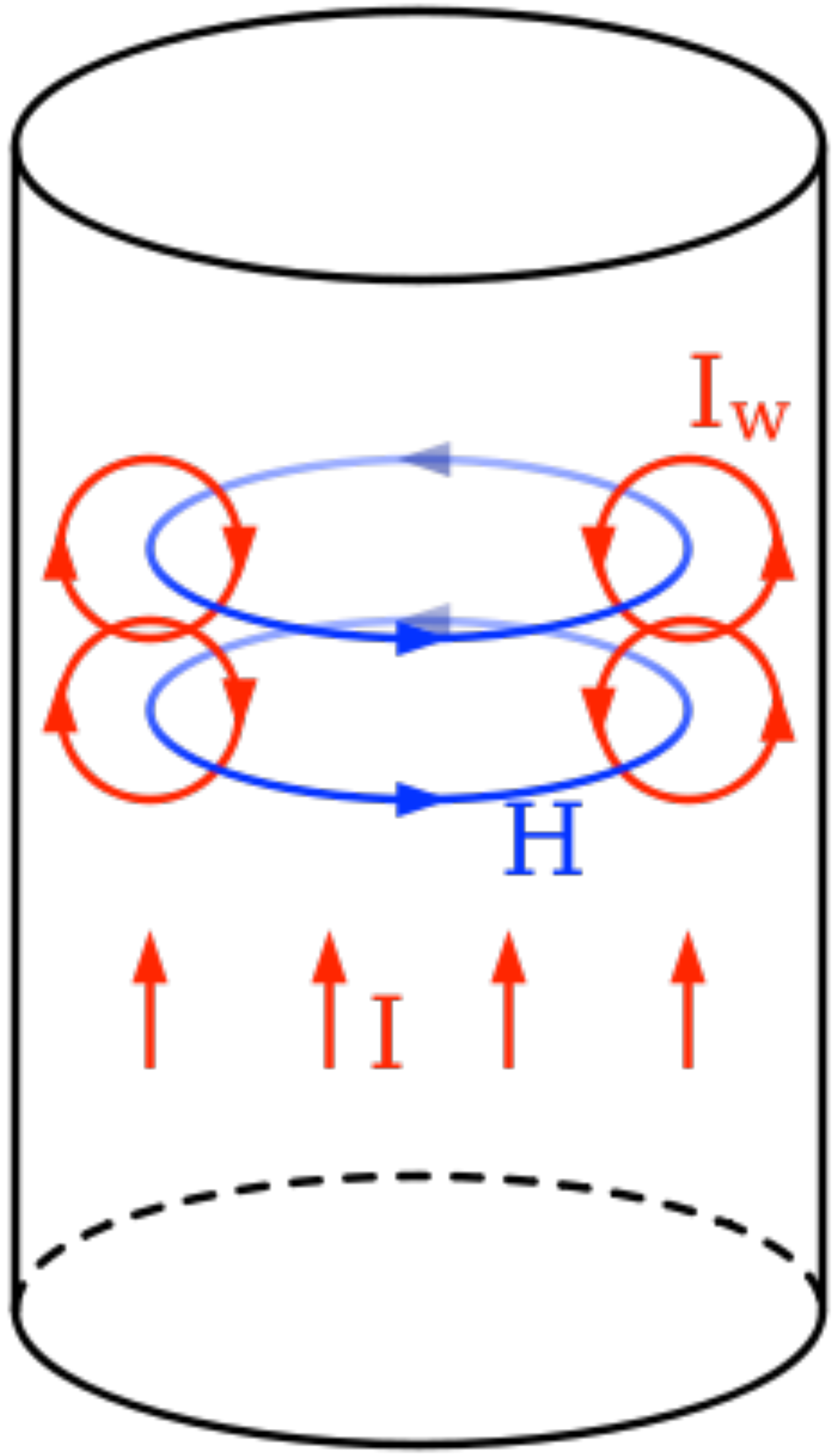
$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{I9}$$

as well as the definition of the cross product:

$$\mathbf{A} \times \mathbf{B} = [A_y B_z - A_z B_y \ , \ A_z B_x - A_x B_z \ , \ A_x B_y - A_y B_x]$$

“Skin effect” describes the tendency for current to flow in the skin of a conductor. I (and many others) used to think that it was the reason that the high voltage from an RF Tesla coil didn’t kill people, but that’s not the case. The skin depth in people at RF frequencies is 25 cm or more.

Turns out that RF the nervous system is insensitive to RF currents.



Wikipedia commons

Let's derive the skin depth equation:

Substitute Ohm's Law into Ampère's Law:

$$\mathbf{J} = \sigma \mathbf{E} \qquad \nabla \times \mathbf{B} = \mu_o \mathbf{J}$$

to get

$$\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E}$$

Take the curl of this and use Faraday's Law ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$):

$$\nabla \times \nabla \times \mathbf{B} = \mu_o \sigma \nabla \times \mathbf{E} \quad \rightarrow \quad \nabla \times \nabla \times \mathbf{B} = -\mu_o \sigma \frac{\partial \mathbf{B}}{\partial t}$$

Now we need the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ to get

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\mu_o \sigma \frac{\partial \mathbf{B}}{\partial t}$$

But, the no-monopoles law says that $\nabla \cdot \mathbf{B} = 0$, so ...

$$\boxed{\nabla^2 \mathbf{B} = \mu_o \sigma \frac{\partial \mathbf{B}}{\partial t}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Similarly, we can take the curl of Faraday's Law and substitute Ampère's and Ohm's Laws to get

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}$$

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \quad \rightarrow \quad \nabla \times \nabla \times \mathbf{E} = -\mu_o \sigma \frac{\partial \mathbf{E}}{\partial t}$$

To pull the same vector identity trick we need to use $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ on Ampère's Law ($\nabla \times \mathbf{B} = \mu_o \mathbf{J}$) to get

$$\nabla \cdot \mathbf{J} = 0 \quad \text{which for constant } \sigma_o \text{ gives} \quad \nabla \cdot \mathbf{E} = 0$$

so

$$\nabla^2 \mathbf{E} = \mu_o \sigma_o \frac{\partial \mathbf{E}}{\partial t}$$

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$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \quad \longrightarrow \quad \nabla \times \nabla \times \mathbf{E} = -\mu_o \sigma \frac{\partial \mathbf{E}}{\partial t}$$

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$$\boxed{\nabla^2 \mathbf{E} = \mu_o \sigma_o \frac{\partial \mathbf{E}}{\partial t}}$$

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so

$$\nabla^2 \mathbf{E} = \mu_o \sigma_o \frac{\partial \mathbf{E}}{\partial t}$$

now we need the idea of a half-space

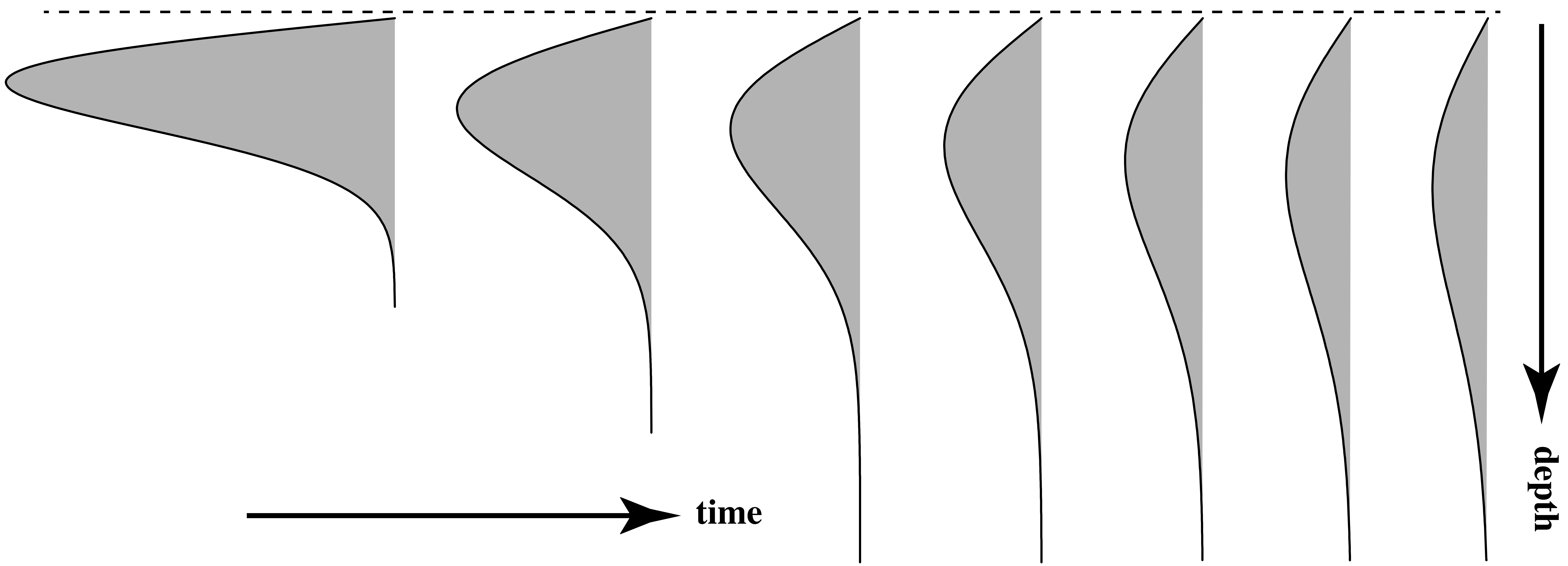
air, $\sigma = 0$

earth, $\sigma = \sigma_o$

These equations in E and B are Diffusion Equations :

$$\nabla^2 \mathbf{B} = \mu_o \sigma \frac{\partial \mathbf{B}}{\partial t} \qquad \nabla^2 \mathbf{E} = \mu_o \sigma \frac{\partial \mathbf{E}}{\partial t}$$

We can define a diffusivity $\eta = 1/(\mu_o \sigma_o)$ so that $\dfrac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$



Amplitude decays as t^{-1} and depth as $t^{1/2}$. Not as bad as heat flow: We can choose time dependence through frequency to alter the decay rate through skin depth.

Now it is time to consider a single frequency ω, so

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$\mathbf{B}(t) = \mathbf{B}e^{i\omega t} \qquad \text{and} \qquad \frac{\partial \mathbf{B}}{\partial t} = i\omega\mathbf{B}$$

$$\nabla^2\mathbf{B} = \mu_o\sigma_o\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla^2\mathbf{E} = \mu_o\sigma\frac{\partial \mathbf{E}}{\partial t}$$

and the same for **E**, so our diffusion equations become

$$\nabla^2\mathbf{E} = i\omega\mu_o\sigma_o\mathbf{E} \qquad \text{and} \qquad \nabla^2\mathbf{B} = i\omega\mu_o\sigma_o\mathbf{B}$$

For external sources of **B**, at Earth's surface **B** is purely horizontal and uniform. Then

$$\nabla^2\mathbf{B} = \cancel{\frac{\partial^2 B_x}{\partial x^2}} + \cancel{\frac{\partial^2 B_y}{\partial y^2}} + \frac{\partial^2 B_z}{\partial z^2} \longrightarrow \frac{d^2 B}{dz^2} = i\omega\mu_o\sigma_o B(z)$$

air, $\sigma = 0$ $\longrightarrow$ **B**

earth, $\sigma = \sigma_o$

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We define a *complex wavenumber* $k^2 = i\omega\mu_o\sigma_o$ so $\dfrac{d^2 B}{dz^2} = k^2 B(z)$

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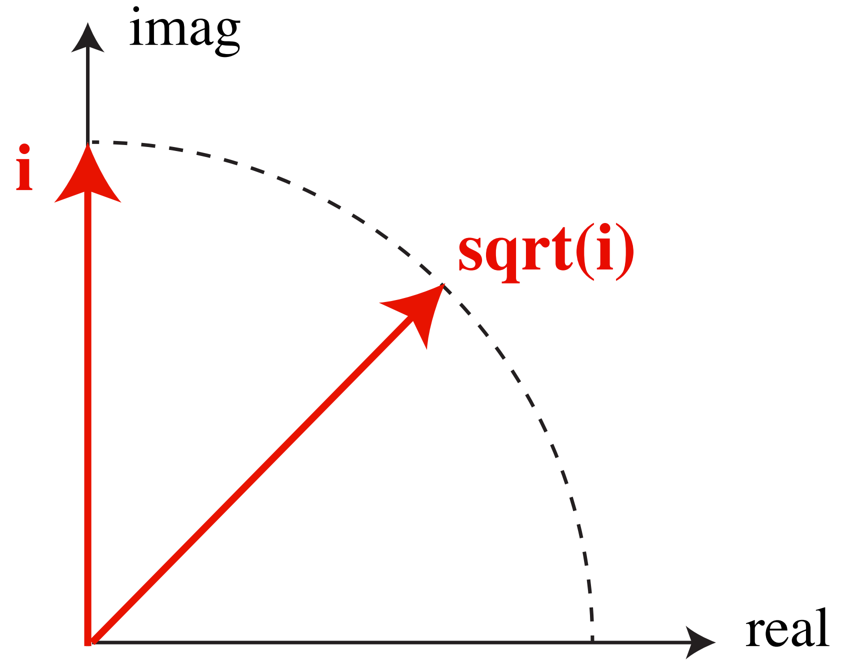
We define a *complex wavenumber* $k^2 = i\omega\mu_o\sigma_o$ so $\dfrac{d^2B}{dz^2} = k^2 B(z)$

This is a second order linear ODE with solutions of the form $B(z) = c_1 e^{kz} + c_2 e^{-kz}$

The first term grows with depth so $c_1 = 0$ and setting $z = 0$ we can infer that $c_2 = B_o e^{i\omega t}$.

This all gives $B(z) = B_o e^{i\omega t} e^{-kz}$



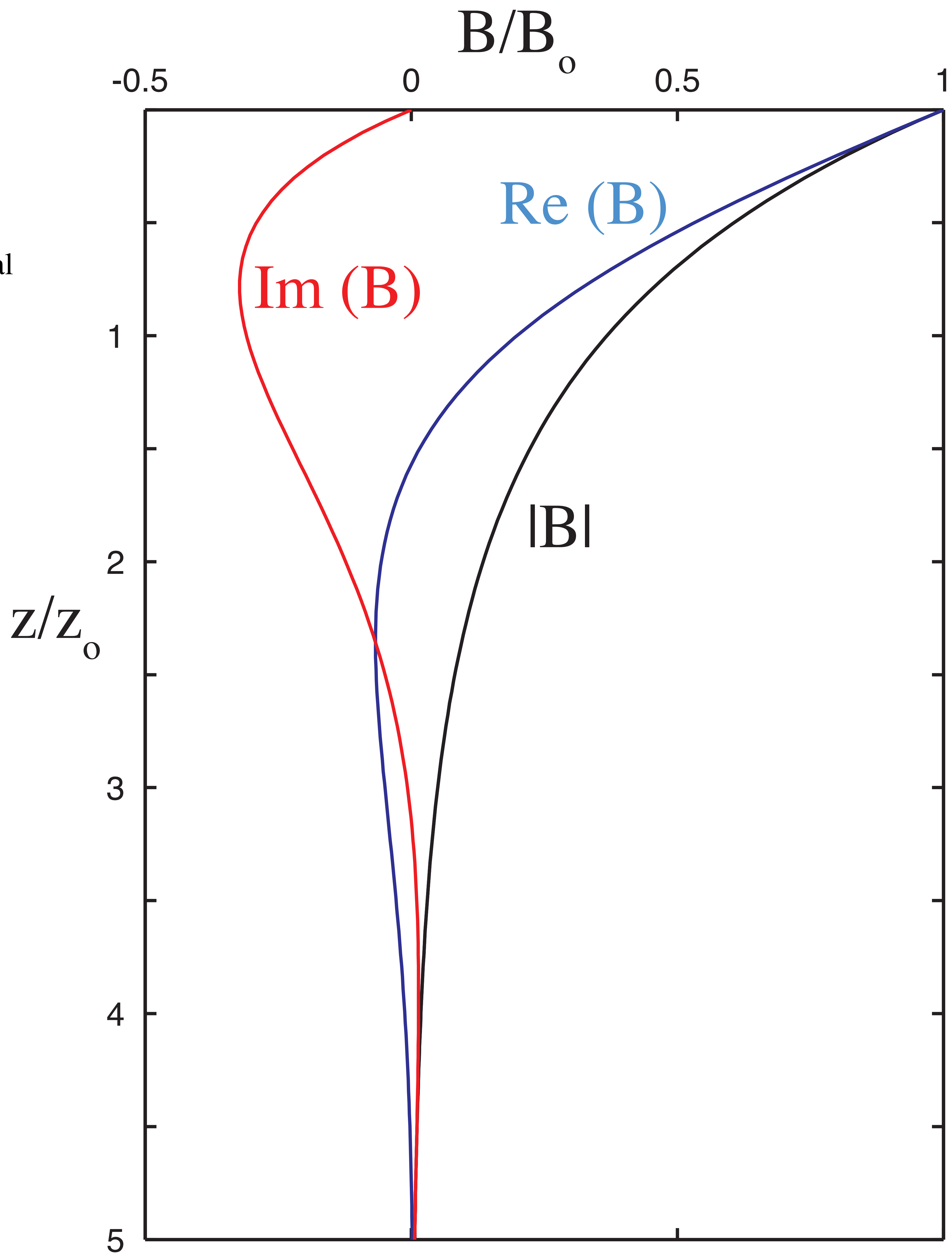
Recalling that $k^2 = i\omega\mu_o\sigma_o$ we can write k as

$$k = \sqrt{i\omega\mu_o\sigma_o} = (1+i)\sqrt{\frac{\omega\mu_o\sigma_o}{2}} = \frac{1+i}{z_o} \quad \text{where} \quad z_o = \sqrt{\frac{2}{\omega\mu_o\sigma_o}}$$

to get

$$B(z) = B_o e^{i\omega t} e^{-z(1+i)/z_o}$$

So $B(z)$ falls off exponentially with z_o, which is the **skin depth**.



To a good approximation skin depth is given by

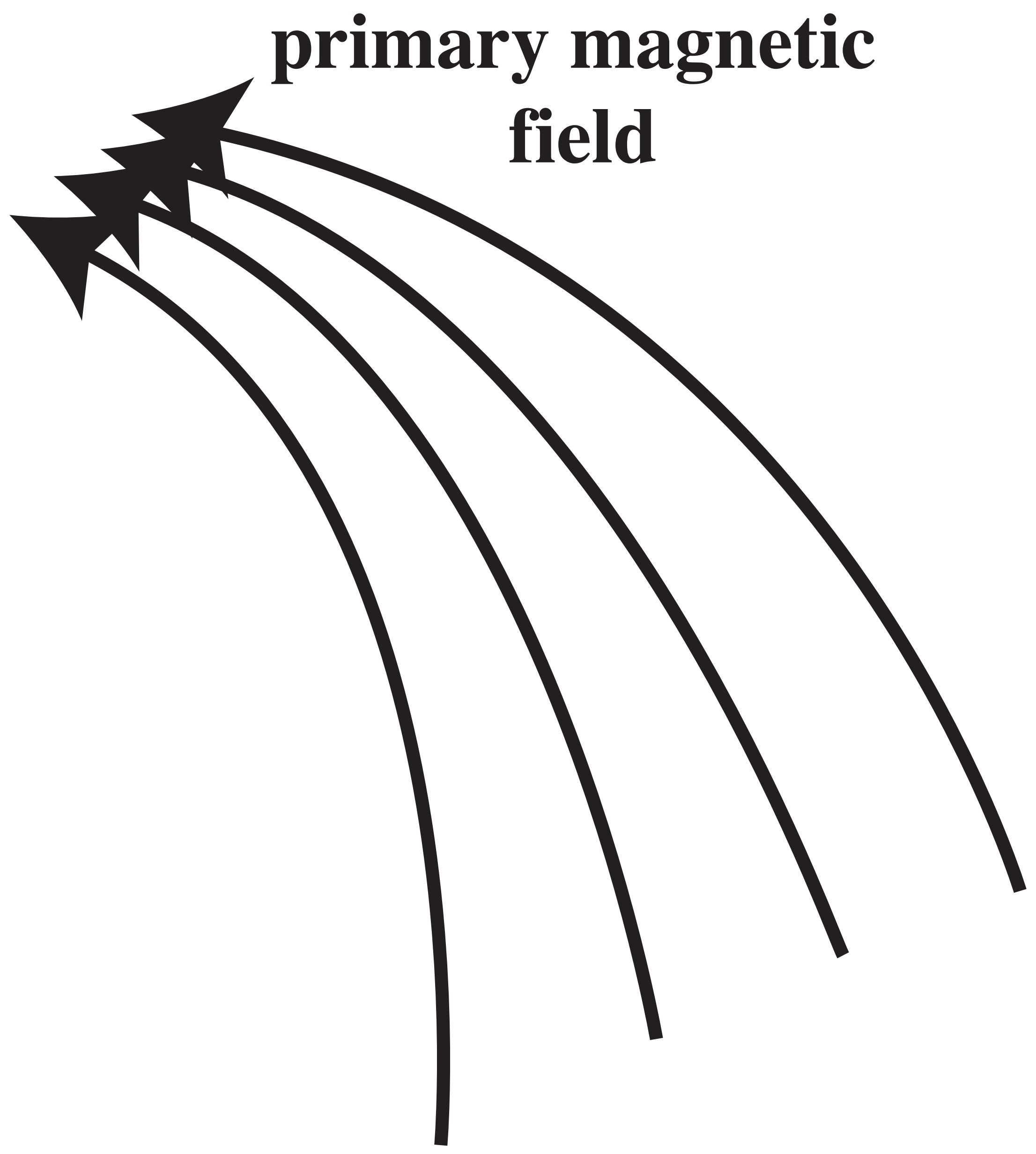
$$\text{skin depth} = \sqrt{\frac{2\rho}{\omega\mu_o}} = \sqrt{\frac{2\rho}{2\pi f\mu_o}} \approx 500\sqrt{\rho T} \quad \text{metres.}$$

where T is period in seconds. Some typical skin depths:

		Period						
material	σ, S/m	1 year	1 month	1 day	1 hour	1 min	1 s	1 ms
outer core	10^5	8.9 km	2.6 km	470 m	95 m	12 m	1.6 m	50 mm
lower mantle	10	890 km	260 km	47 km	9.5 km	1.2 km	160 m	5 m
seawater, basaltic lava	3	1.6 Mm	470 km	85 km	17 km	2.3 km	290 m	9 m
marine sediments	1	2.8 Mm	820 km	150 km	30 km	3.9 km	500 m	16 m
cont. sediments	0.1	8.9 Mm	2.6 Mm	470 km	95 km	12 km	1.6 km	50 m
warm upper mantle	10^{-3}	89 Mm	26 Mm	4.7 Mm	950 km	120 km	16 km	500 m
cool mantle, granite	10^{-5}	890 Mm	260 Mm	47 Mm	9.5 Mm	1.2 Mm	160 km	5 km

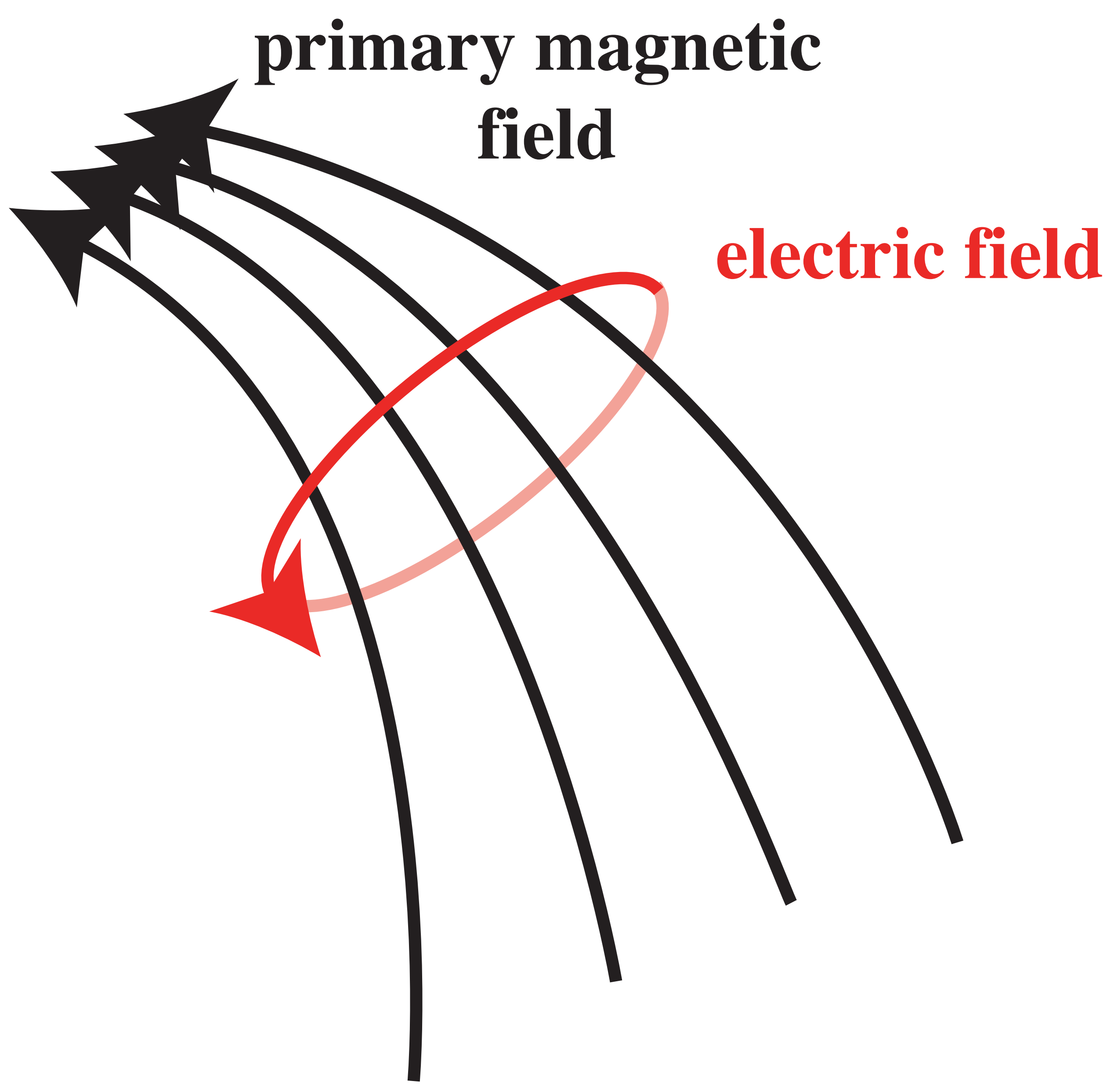
Skin depth is not a measure of resolution, but is a guide to the maximum distance that EM energy can propagate.

What is physically going on?



A primary magnetic field varies in time (or moves in space)

A time varying magnetic field will generate an electric field.

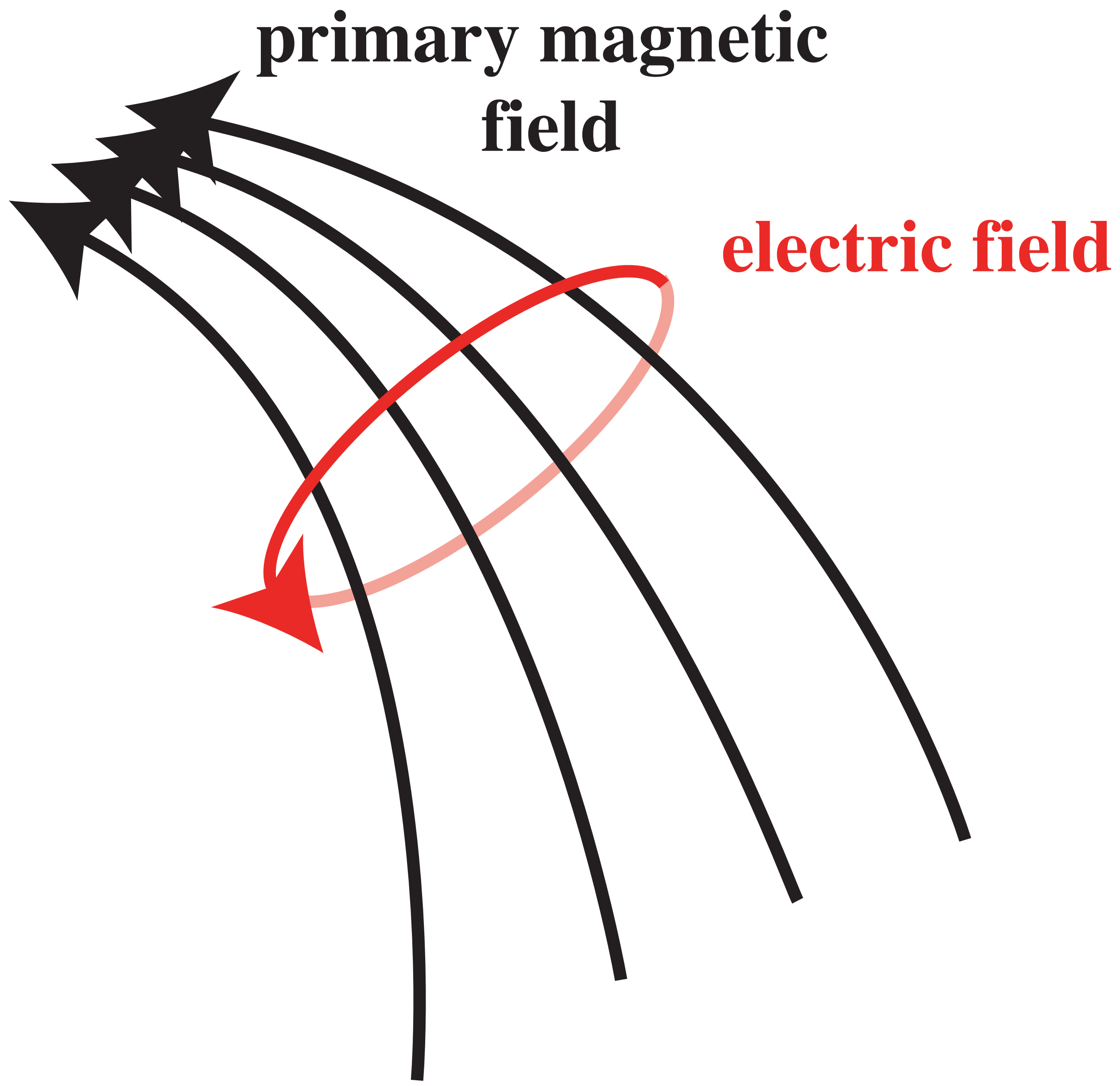


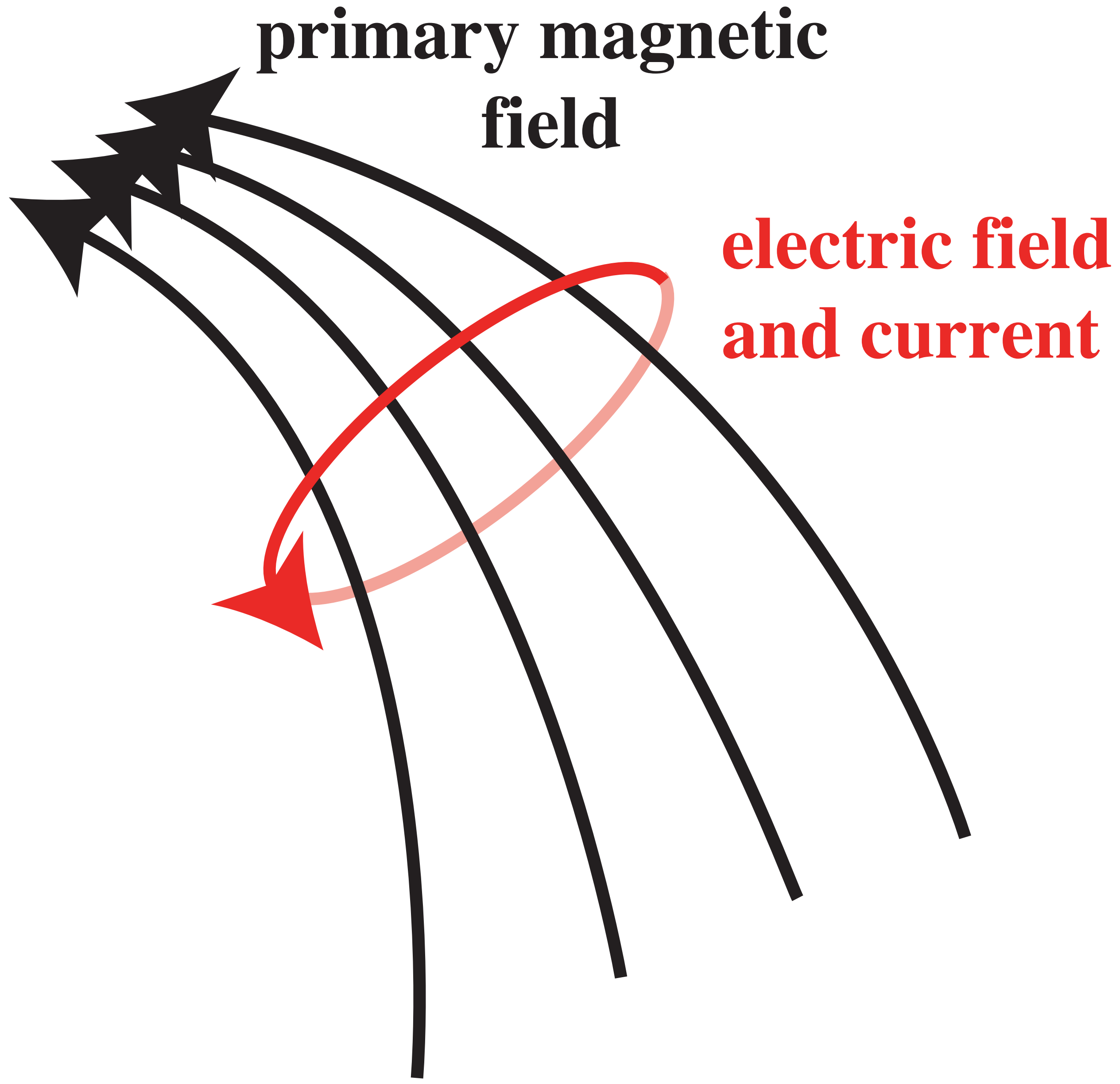
Faraday's Law

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

A time varying magnetic field will generate an electric field.

In a conductor this drives an electric current.





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Ohm's Law

$$\mathbf{J} = \sigma \mathbf{E}$$

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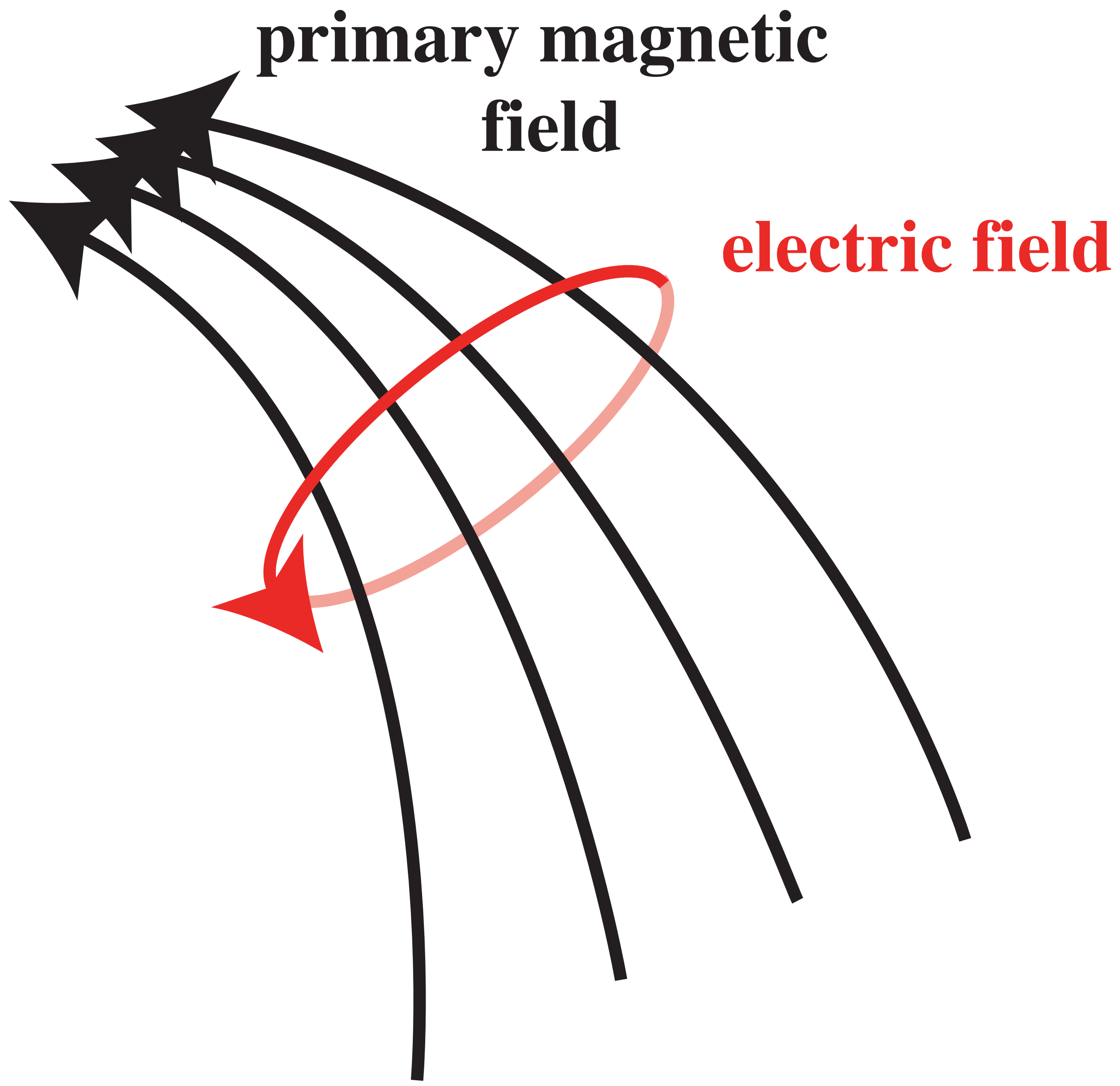

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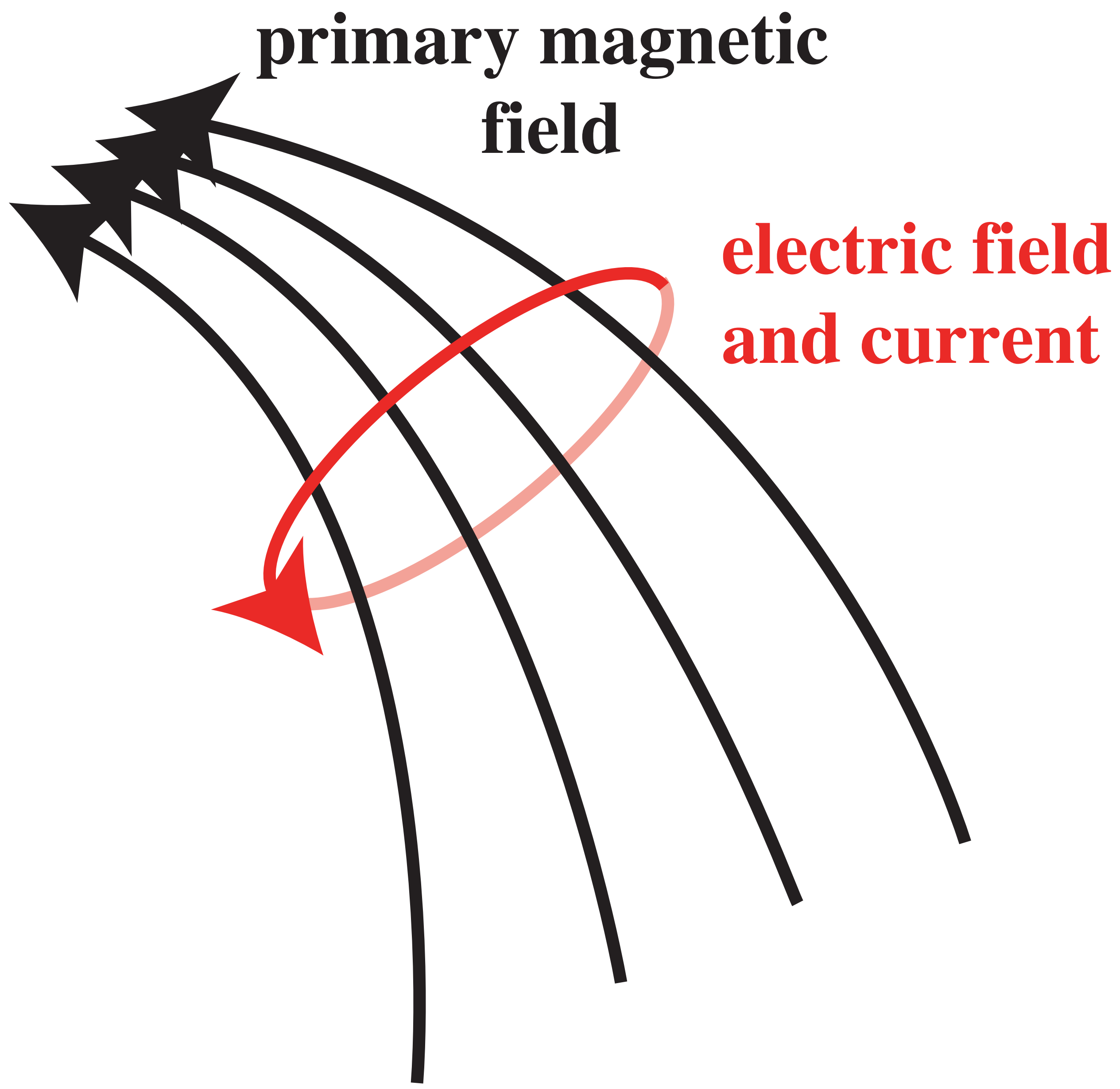
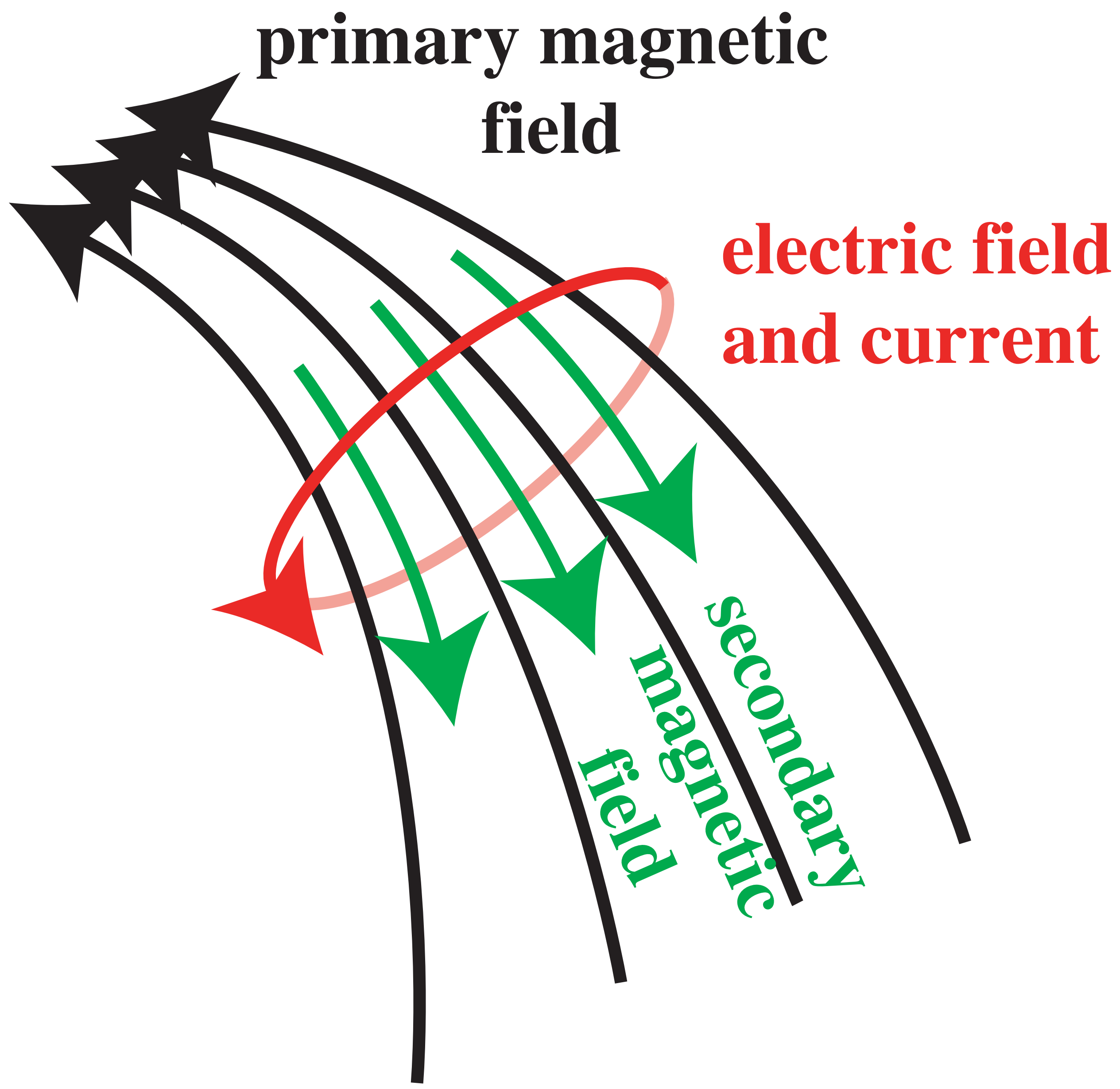


Ohm’s Law

$$\mathbf{J} = \sigma \mathbf{E}$$

Which generates another magnetic field.



Ampere’s Law

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu I$$

Because of the negative sign in Faraday’s Law, these secondary magnetic fields act to oppose the primary field, weakening it

The Magnetotelluric Method:

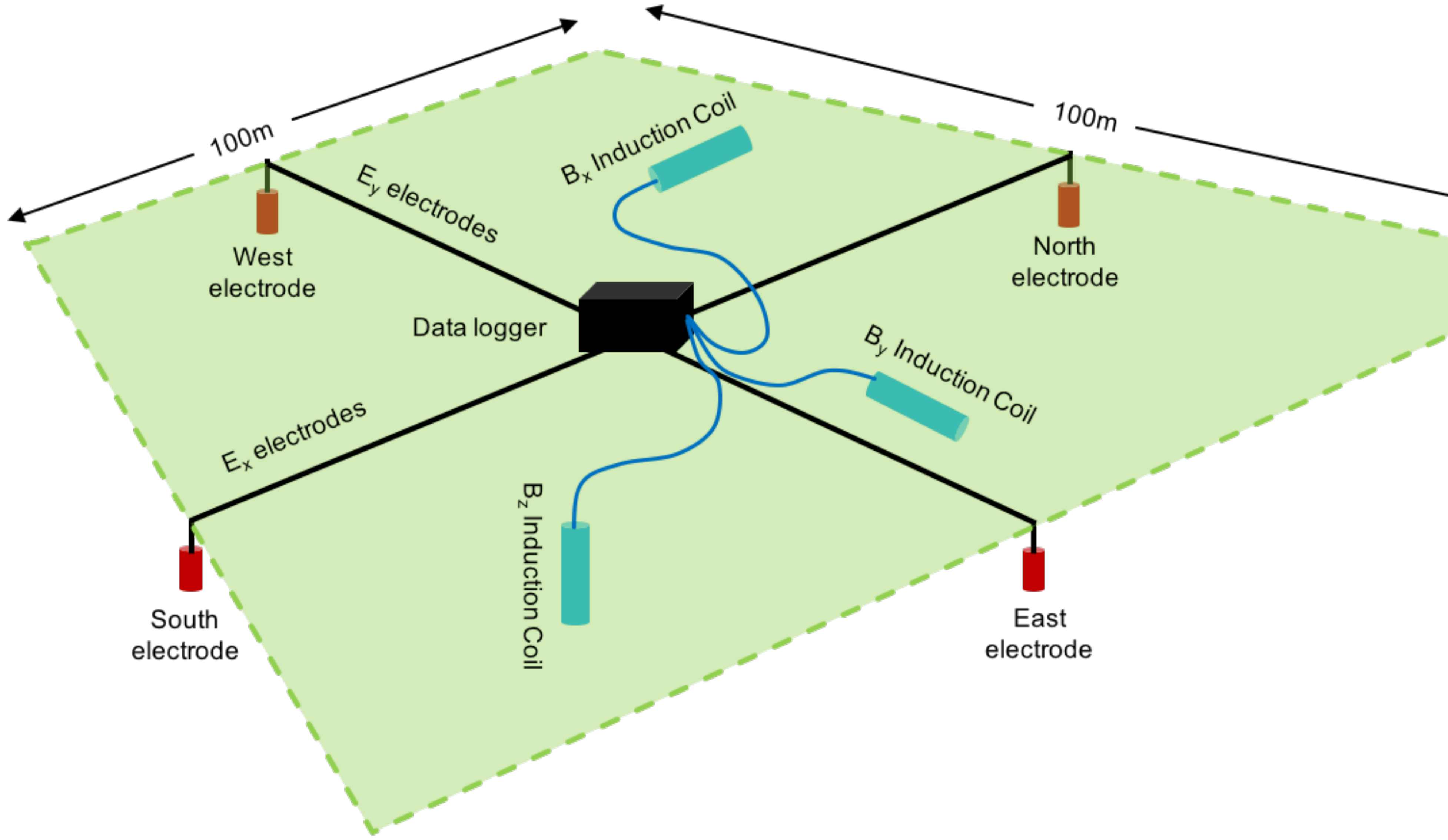
We just showed that $B(z) = B_o e^{i\omega t} e^{-z(1+i)/z_o}$ and recall $\nabla \times \mathbf{B} = \mu_o \sigma \mathbf{E}$

But if **B** is in the *x* direction the only non-zero component of the curl is $\partial B_x / \partial z$ in the y-component:

no *Bz* or *By*

$$\nabla \times \mathbf{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}, \boxed{\frac{\partial B_x}{\partial z}} - \frac{\partial B_z}{\partial x}, \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right)$$

Bx uniform across surface



so

$$E_y = \frac{1}{\mu_o \sigma_o} \frac{dB_x}{dz} = -\frac{1+i}{\mu_o \sigma_o z_o} B_x = -\frac{k}{\mu_o \sigma_o} B_x$$

and similarly

$$E_x = \frac{1}{\mu_o \sigma_o} \frac{-dB_y}{dz} = \frac{1+i}{\mu_o \sigma_o z_o} B_y = \frac{k}{\mu_o \sigma_o} B_y$$

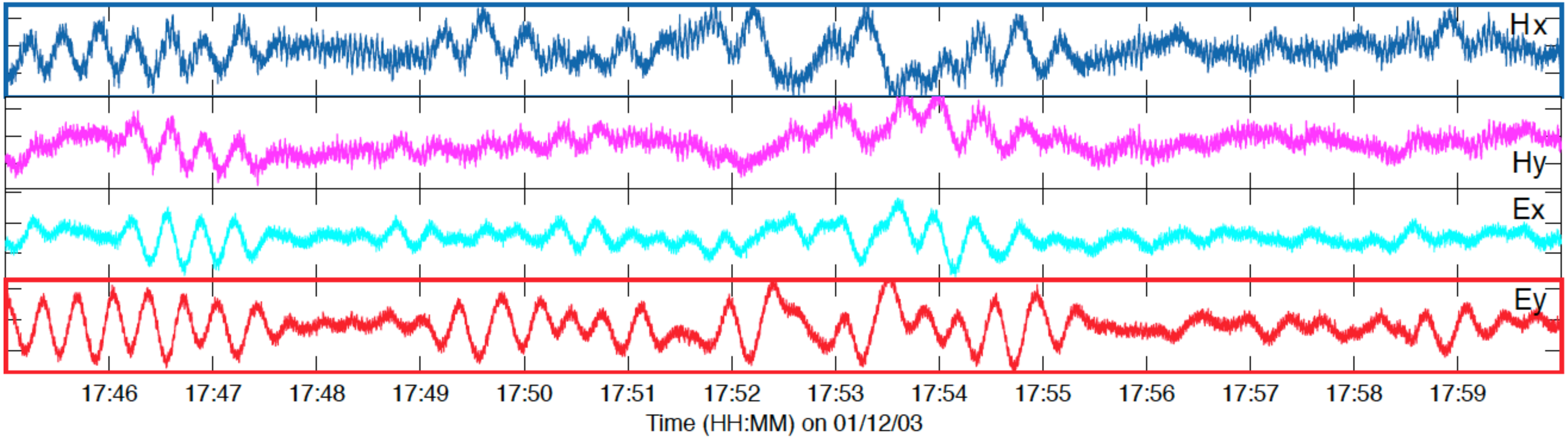
This is valid for any depth z, but we are only interested in the surface where $z = 0$ and $B_x = B_o e^{i\omega t}$

We have that

$$E_y = -\frac{k}{\mu_o \sigma_o} B_x \qquad E_x = \frac{k}{\mu_o \sigma_o} B_y \qquad \text{where} \qquad k = \sqrt{i\omega\mu_o\sigma_o}$$

These equations tell us that there is an induced electric field that is linearly proportional to the external magnetic field. The constant of proportionality depends on conductivity and frequency. *Ey* is anti-correlated with *Bx*, and *Ex* is correlated with *By* (but both with a 45° phase shift).

Site t03 from GoM 2003: 15 minutes at 32 Hz sampling



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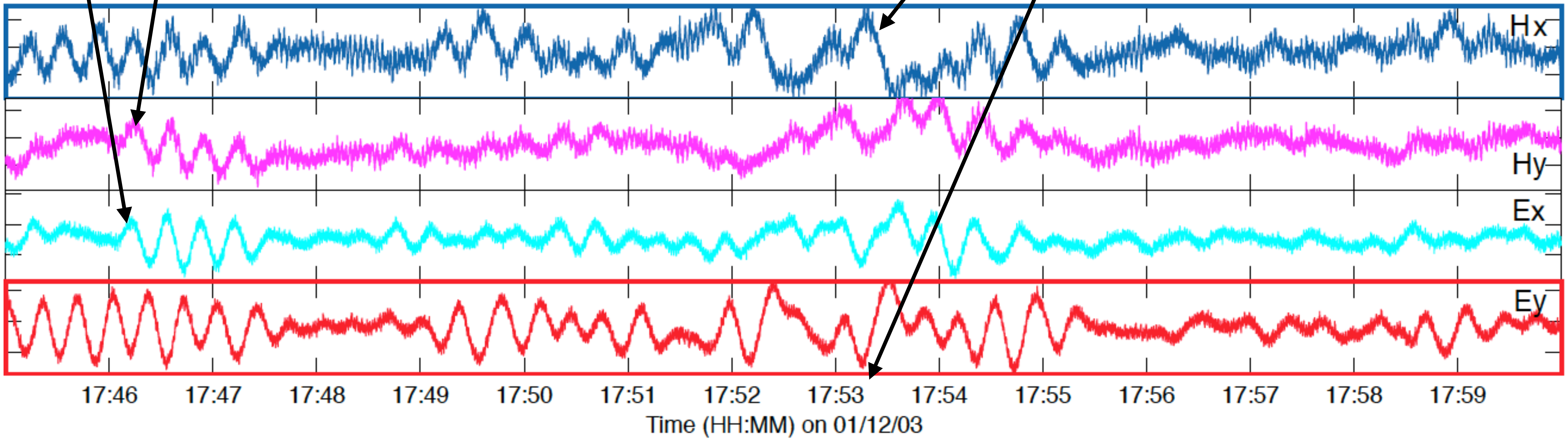
Ex is correlated with *By*

Ey is anti-correlated with *Bx*



The Magnetotelluric Method continued:

We can take the ratio of the electric to magnetic field at any particular frequency to obtain the half-space resistivity:

$$\left|\frac{E_y}{B_x}\right|^2 = \left(\frac{k}{\mu_o\sigma_o}\right)^2 = \frac{\omega\mu_o\sigma_o}{(\mu_o\sigma_o)^2} = \frac{\omega}{\mu_o\sigma_o}$$

$$\rho = \frac{\mu_o}{\omega}\left|\frac{E_y}{B_x}\right|^2 \qquad\qquad \phi = \tan^{-1}\left(\frac{E}{B}\right)$$

This is the MT equation made famous in Cagniard's 1953 paper.

$$\rho = 0.2\,T\left(\frac{E}{H}\right)^2.$$

This is all still only true for a half-space, but we can call this apparent resistivity regardless of how complicated the structure is.

The two components of **E** are related to the two components of **B** through the impedance matrix **Z**

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix}$$

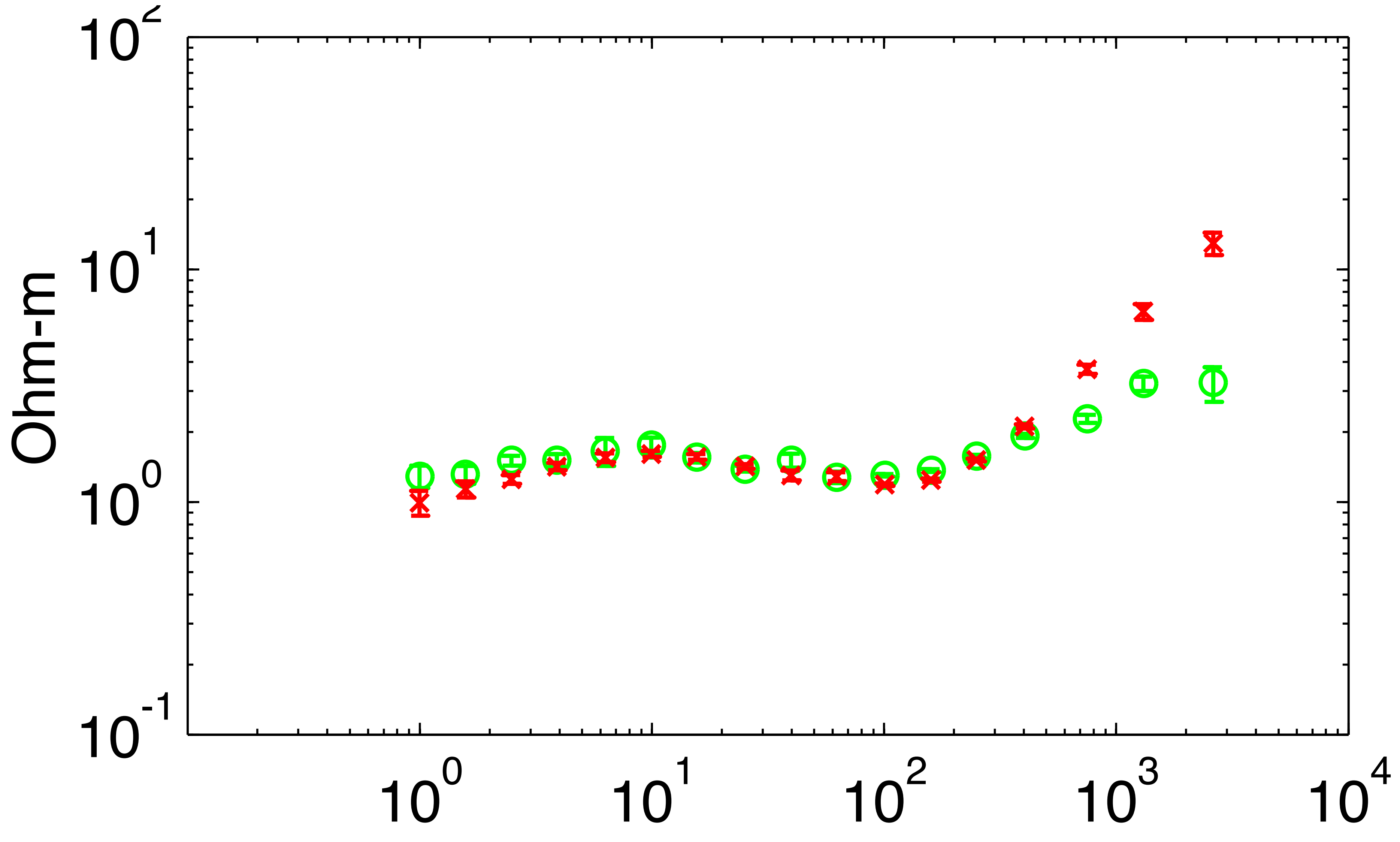
Note that all terms are complex numbers as a function of frequency

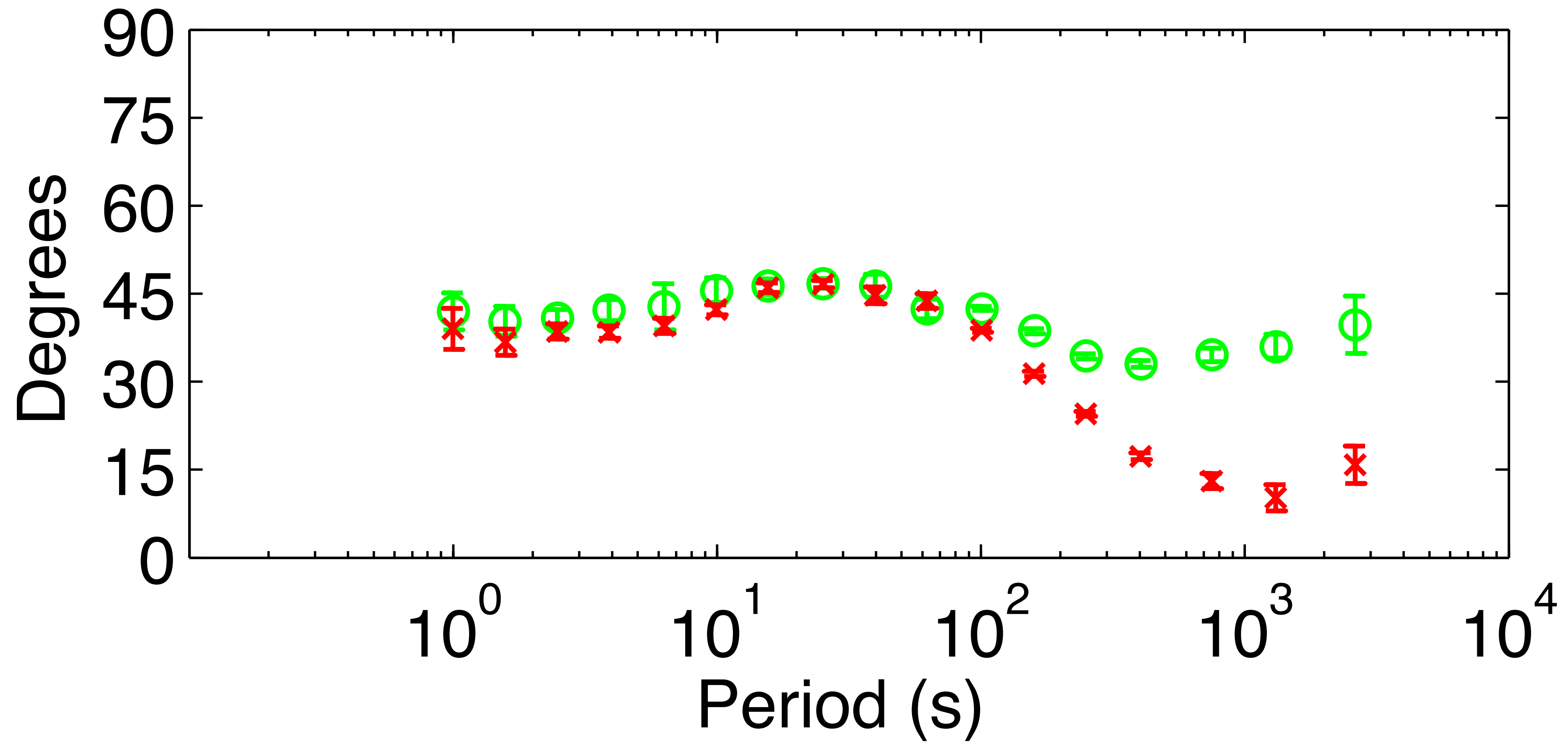
except that most MT people like to use the magnetizing field **H** to define impedance. This slightly changes the apparent resistivity formula (of course, the phase remains the same).

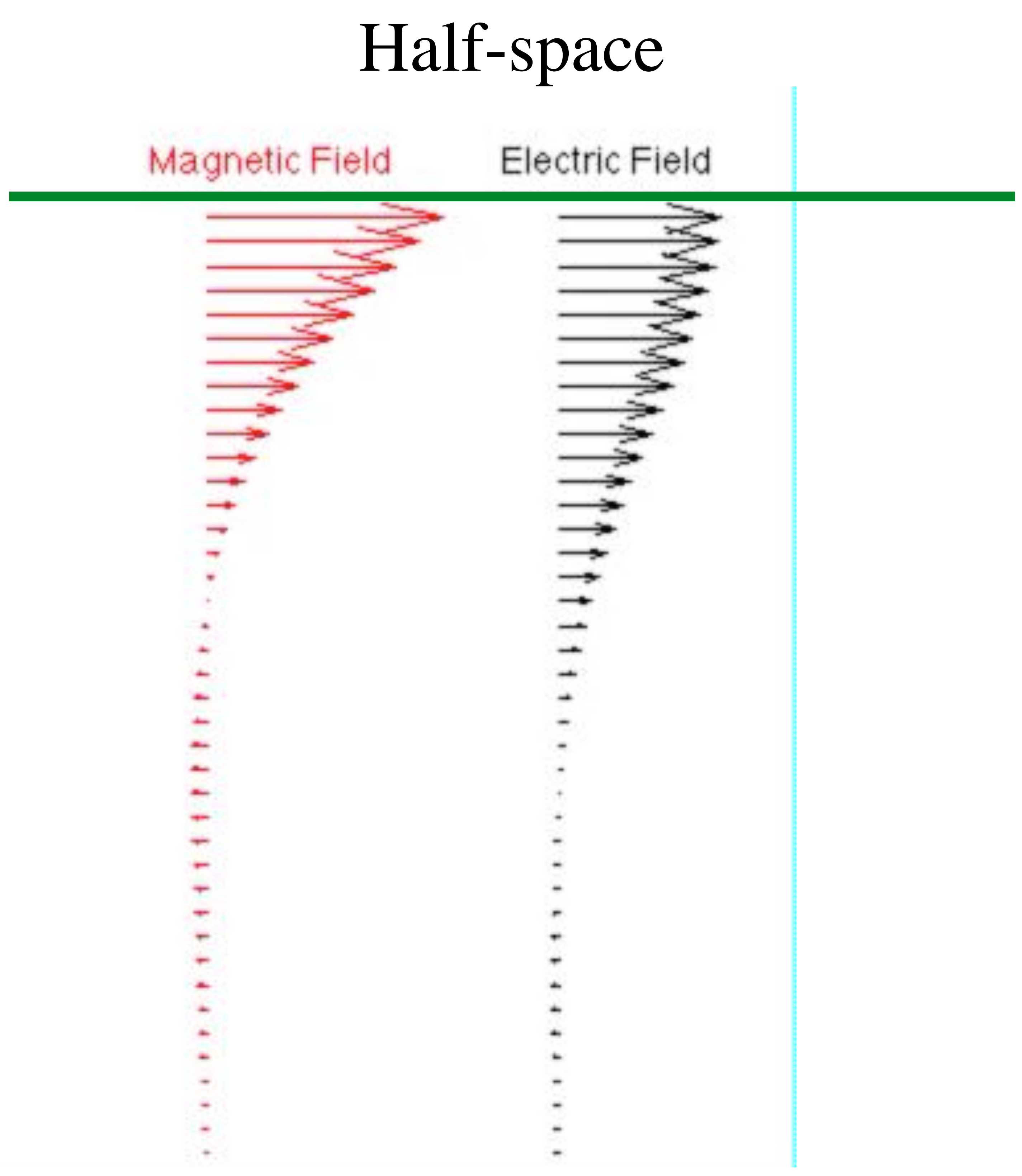
$$\rho_a = \frac{1}{\omega\mu_o}\left|\frac{E}{H}\right|^2 \qquad \phi = \tan^{-1}\left(\frac{E}{H}\right)$$

For global studies many use Weidelt's c:

$$c(\omega) = -\frac{E_y}{i\omega B_x} = -\frac{E(0)}{E'(0)} \qquad \left(= -\frac{Z}{i\omega}\right)$$







Here is what the MT fields look like in a uniform conductor. The fields decay exponentially with a scale length given by the **skin depth**

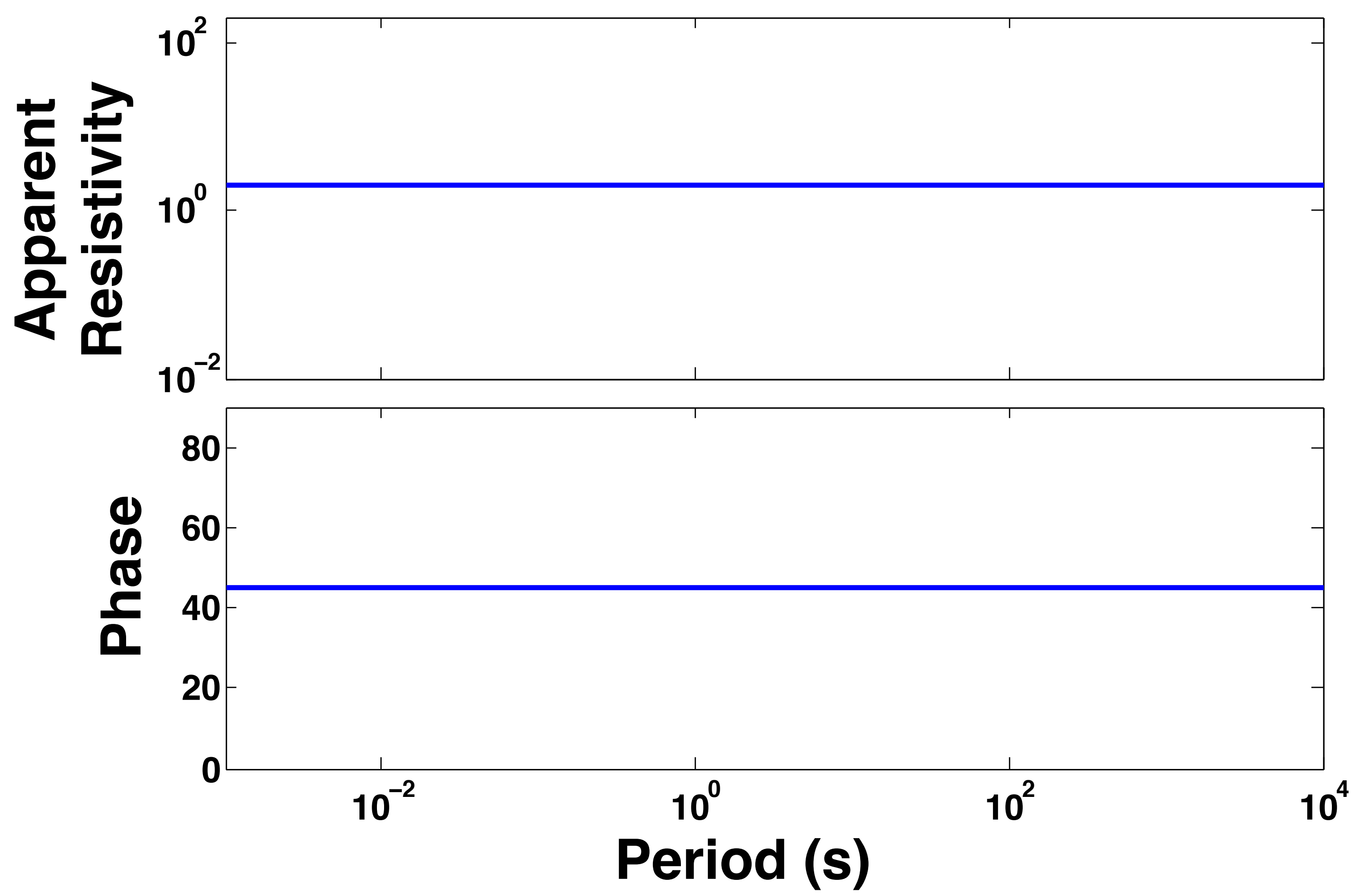
$$z_o \approx \frac{500 \text{ m}}{\sqrt{\sigma f}}$$

The induced electric field is 45° out of phase with the primary magnetic field.

We can compute a half-space equivalent electrical resistivity (apparent resistivity) at each frequency:

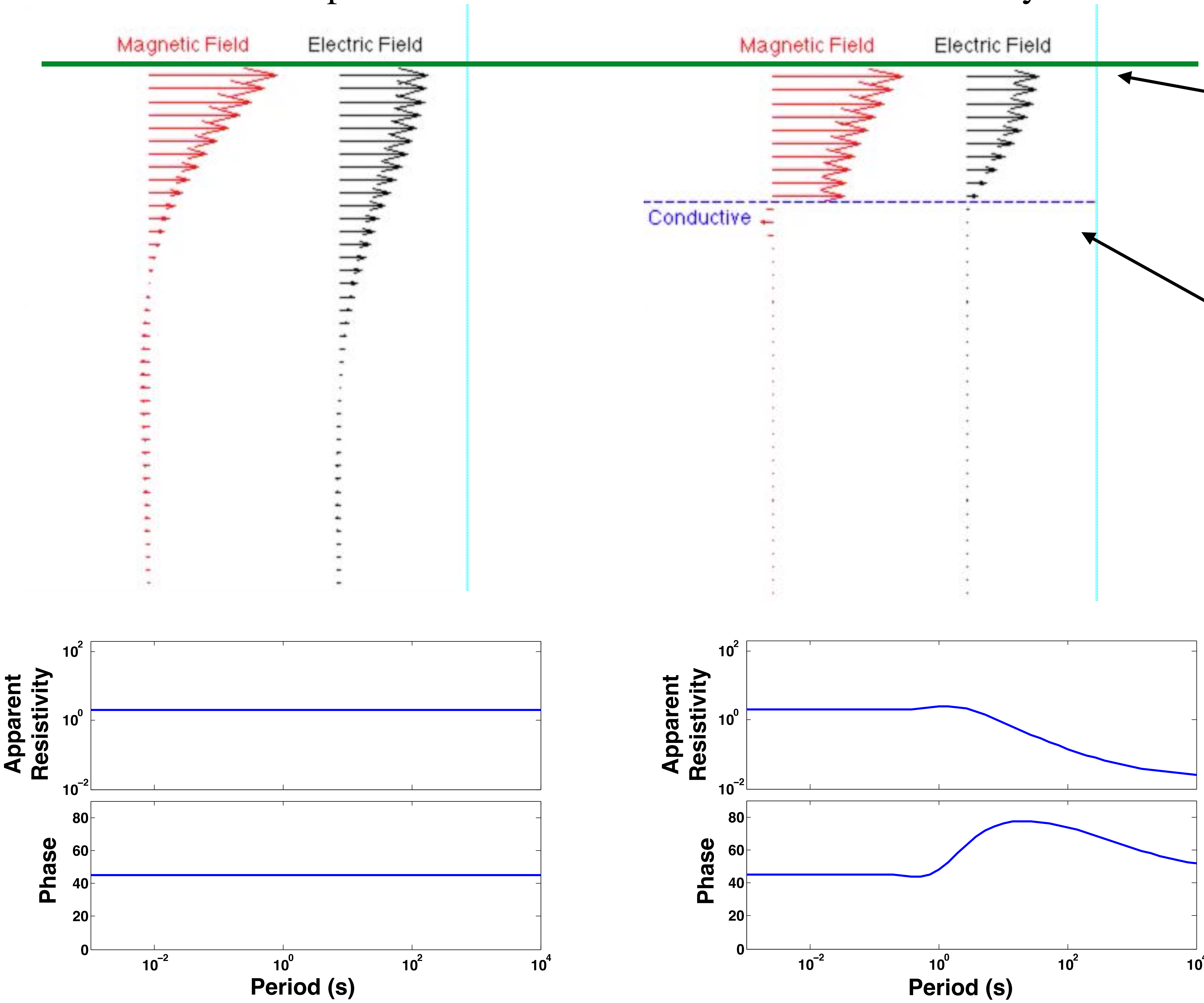
$$\rho_{\mathrm{a}}(\omega) = \frac{\mu_o}{\omega} \left| \frac{E(\omega)}{B(\omega)} \right|^2$$

We can also compute the phase difference between E and B. These become the MT sounding curves.



Half-space

Conductive Layer



E is smaller

We can add a conductive layer at depth and things change at the surface

Half-space | Conductive Layer | Resistive Layer