SIOG 231
GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 5

Gauss' Theory of the Main Field

1/23/2024

Today's Class

• Laplace's equation for geomagnetic scalar potential is valid in source free region - Earth's atmosphere

• Write a general solution to Laplace's equation in geocentric spherical coordinates

• Fully-normalized spherical harmonics with complex coefficients A_l^m and B_l^m allow us to easily do theory to separate internal and external field contributions

• But geomagnetists usually use real partially normalized Schmidt coefficients g_l^m and h_l^m (internal) and q_l^m, s_l^m (external).

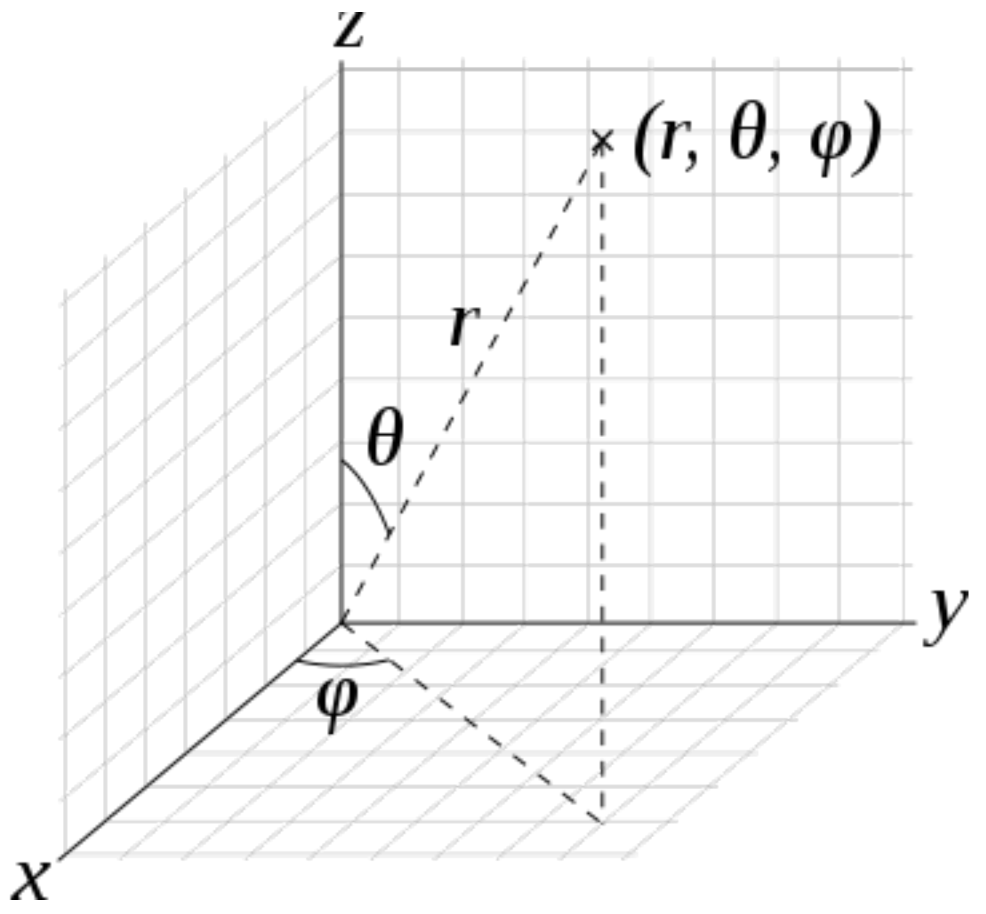
But first -

If you feel the need for some intuition about what the Laplacian is see the Khan academy tutorials

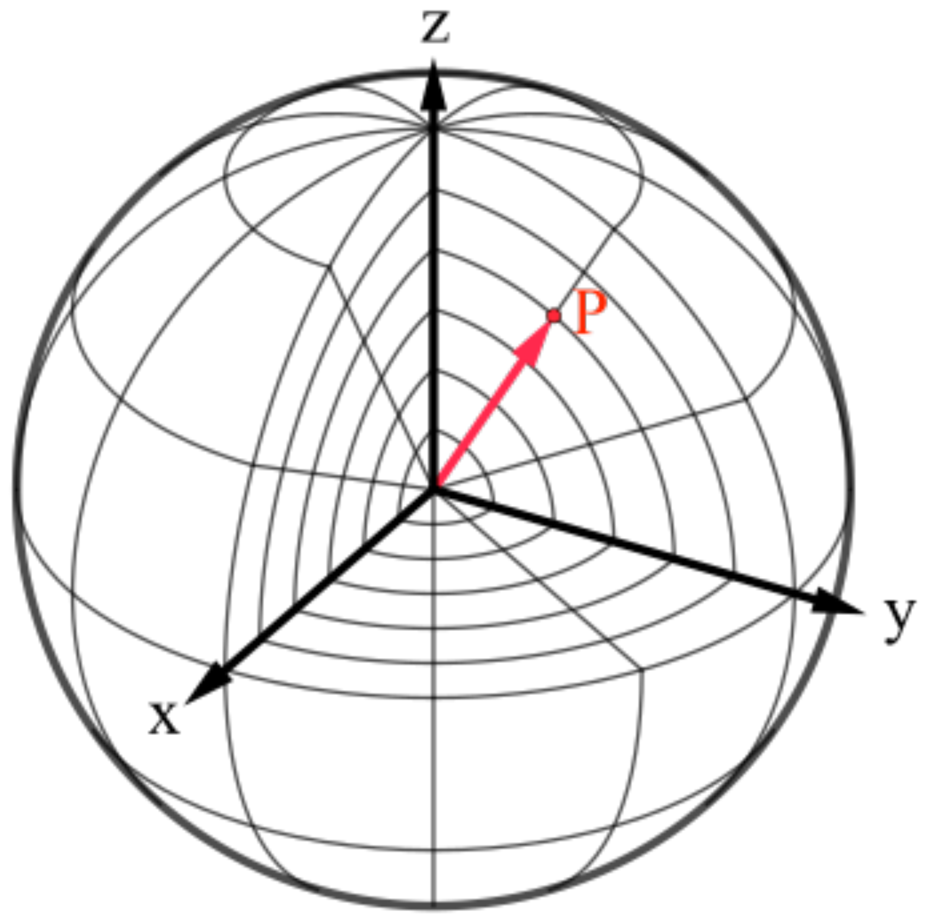
Note that we use the notation ∇^2 for the Laplacian operator but others sometimes use $\triangle$.

Next - a coordinate system

In a **spherical coordinate** system we specify location in terms of (r, θ, ϕ), where r is radius, θ is polar angle or geocentric colatitude, and ϕ is longitude. In a **geocentric** coordinate system the origin is Earth's center.



Spherical coordinates (r, θ, ϕ) as commonly used in *physics* (ISO 80000-2:2019 convention): radial distance r (distance to origin), polar angle θ (theta) (angle with respect to polar axis), and azimuthal angle ϕ (phi) (angle of rotation from the initial meridian plane). The symbol ϱ (rho) is often used instead of r.



A globe showing the radial distance, polar angle and azimuthal angle of a point P with respect to a unit sphere, in the physics convention. In this image, r equals 4/6, θ equals 30°, and ϕ equals 90°.

Geocentric spherical coordinates are comprised of the geocentric radial distance r, the colatitude θ, and the east longitude ϕ. A position vector $\mathbf{r}$ can be written in geographic Cartesian coordinates x, y, z as

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{pmatrix}$$

so that the spherical coordinates can be expressed in terms of x, y, z as:

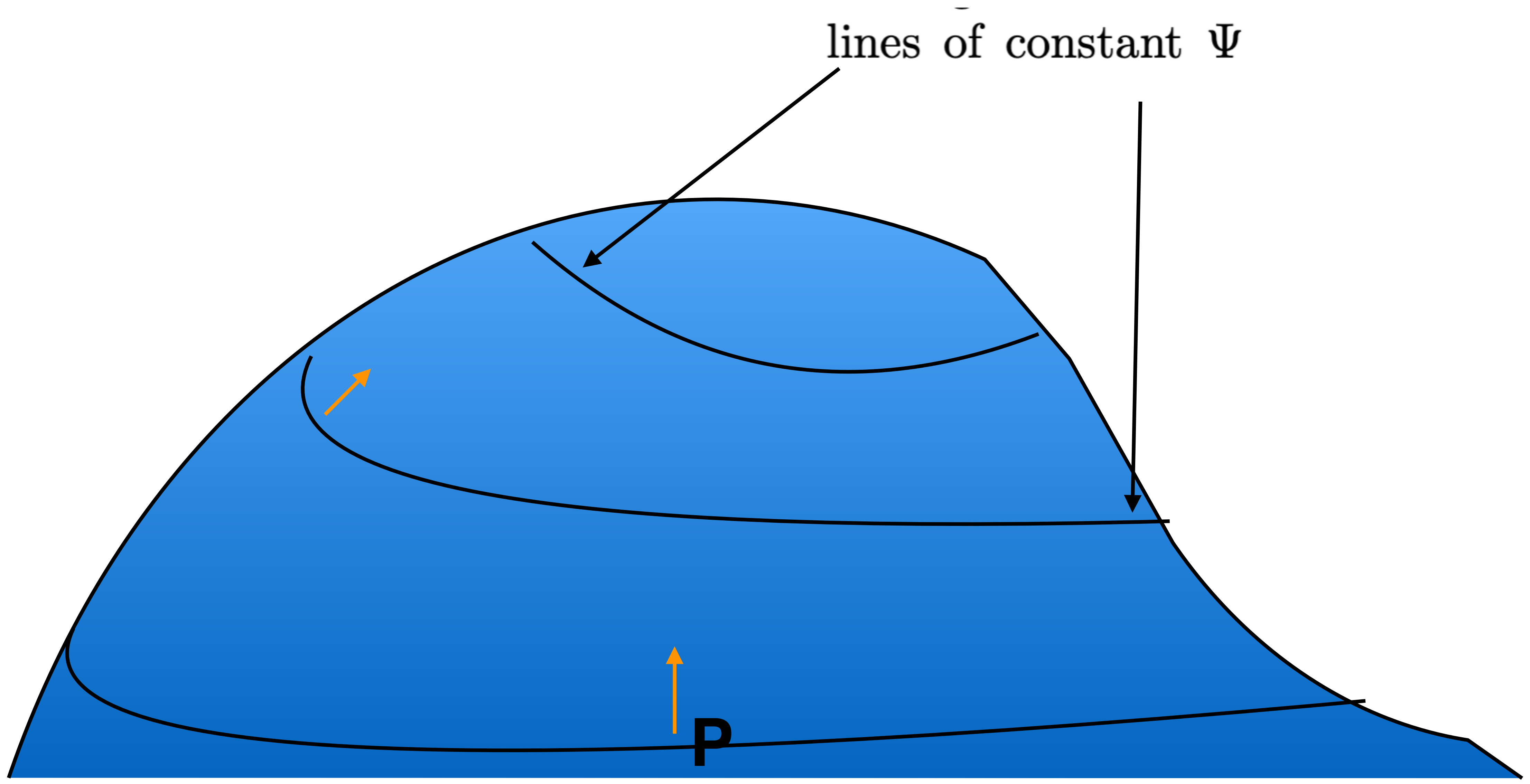
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$
$$\phi = \mathrm{atan2}(y, x).$$

For a whole lot about the many possible coordinate systems that are used in geomagnetic and geospace research see this paper by Laundal and Richmond

Coordinate Systems and Differential Operations

Gradient of a scalar field $\nabla\Psi$ is a vector that gives the direction and rate of change of the field. It is orthogonal to lines of constant Ψ, and this is the direction of maximum rate of change. If we want to know the derivative of Ψ in some other direction, e.g. $\hat{\mathbf{A}}$ that would be $\nabla\Psi \cdot \hat{\mathbf{A}}$.

Gradient in three Coordinate Systems:



Cartesian:

$$\nabla\Psi = \hat{\mathbf{x}}\frac{\partial\Psi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Psi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Psi}{\partial z} = \partial_j \Psi_i$$

Cylindrical:

$$\nabla\Psi = \hat{\mathbf{r}}\frac{\partial\Psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial\Psi}{\partial z}$$

Spherical:

$$\nabla\Psi = \hat{\mathbf{r}}\frac{\partial\Psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}$$

Coordinate Systems and Differential Operations - cont

Divergence of a vector field $\nabla \cdot \mathbf{A}$ is a scalar that represents the 3-dim spatial derivative of a vector field. If $\mathbf{A}$ represents fluid flow $\nabla \cdot \mathbf{A}$ is net outward flow per unit volume surrounding point (x, y, z).

Divergence in three Coordinate Systems:

Cartesian:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \partial_i A_i$$

Cylindrical:

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

Spherical:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Coordinate Systems and Differential Operations - cont

Laplacian of a scalar field $\nabla^2\Psi = 0$ is a second order differential equation. $\nabla^2\Psi = \nabla \cdot (\nabla\Psi)$.

Laplacian in three Coordinate Systems:

Cartesian:

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = \partial_i(\partial_i\Psi)$$

Cylindrical:

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\Psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\partial^2\Psi}{\partial z^2}$$

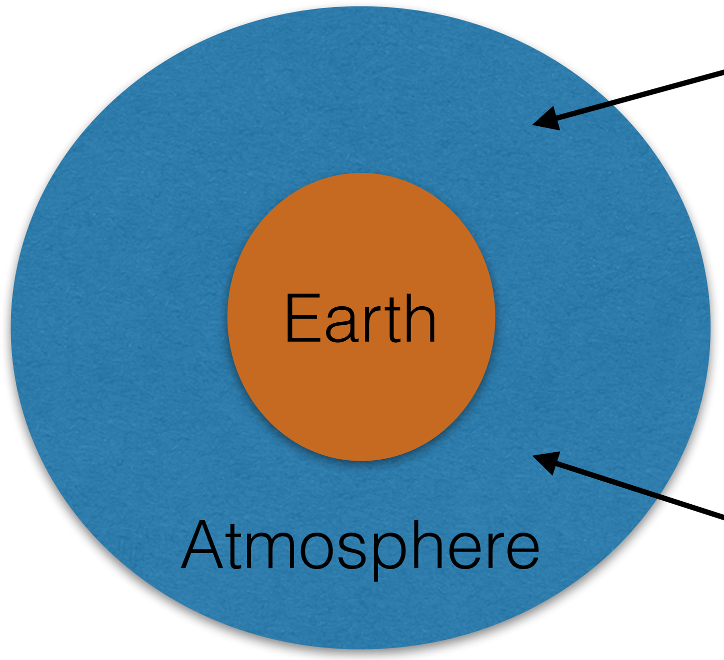
Spherical:

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

Recall the static approximation to the geomagnetic field from the end of Lecture 2

- Magnetic field representation in terms of scalar potential

Approximation: Earth's atmosphere is an insulator with no electrical currents



$$\nabla \times \boldsymbol{B} = 0$$ **since J=0**

$$\boldsymbol{B} = -\nabla \Psi$$ **See vector identity number I2**

Since $\nabla \cdot \boldsymbol{B} = 0$ **no monopoles**

$$\boxed{\nabla^2 \Psi = 0}$$ Laplace's Equation

A Bit of Theory of Harmonic Functions - there is a lot!

$\Psi(r, \theta, \phi)$ is harmonic in *D*, a bounded region of real space if $\nabla^2\Psi = 0$. Then

- Ψ is infinitely differentiable at all points in *D* - note we are talking about spatial derivatives.
- Maximum and minimum values of Ψ always occur on the boundaries of *D* - not inside.
- For any spherical surface within *D* the average value of Ψ over the surface equals its value at the center of the sphere.
- There is a uniqueness theorem for the Neumann Boundary value problem: if you know $\partial\Psi/\partial n$ everywhere on the surface of a compact body *B,* and Ψ falls off like *1/r* as *r* goes to infinity*,* then this knowledge is sufficient to determine Ψ ***uniquely*** everywhere outside B.

GAUSS' THEORY OF THE MAIN FIELD

- Gauss' Separation of Harmonic Fields into Parts of Internal and External Origin

$$\boldsymbol{B} = -\nabla \Psi \qquad \nabla^2 \Psi = 0$$

$$\boxed{\Psi(r,\theta,\phi) = a \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \Big[\underbrace{A_l^m \left(\frac{r}{a}\right)^l}_{\text{External}} + \underbrace{B_l^m \left(\frac{a}{r}\right)^{l+1}}_{\text{Internal}} \Big] Y_l^m(\theta,\phi)}$$

a=constant reference radius (e.g. Earth's surface= 6371 km)

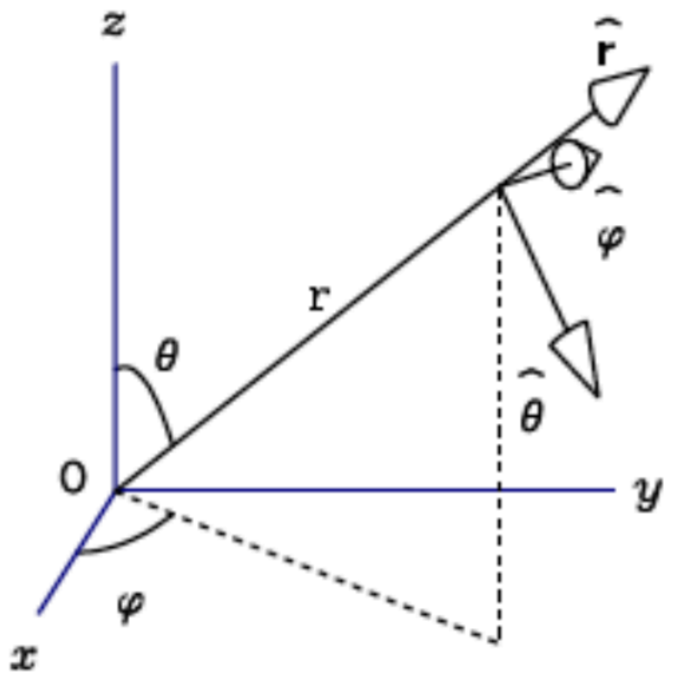
- Gauss (in 1832) demonstrated the predominance of the internal part

- The expansion begin at l = 1, excluding the monopoles as a source

Solution of Laplace’s Equation

- This can be found by the standard technique of separation of variables and is given in the form of an infinite sum of spherical harmonic functions and radial polynomials.
- What are the spherical harmonics? They are eigenfunctions of the surface Laplacian

$$\nabla_1 = \hat{\theta}\,\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{\sin\theta}\,\frac{\partial}{\partial\phi} \tag{1}$$



$$= r\nabla - \mathbf{r}\,\frac{\partial}{\partial r}\ . \tag{2}$$

The subscript one is to remind us the operator acts over the unit sphere, $S(1)$. The first definition shows how to compute the surface gradient in a spherical polar coordinate system; the second assumes the function is defined in all of space and just subtracts out the radial part. The second definition shows the operator is independent of coordinate orientation and also that nothing funny happens at the poles. The ordinary Laplacian operator in $\mathbb{R}^3$ is

$$\nabla^2 = \frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_i} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{3}$$

The full Laplacian can be written as the sum of the radial part and the surface Laplacian

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\nabla_1^2 \tag{4}$$

where ∇_1^2 is the *surface Laplacian*, sometimes also called the *Beltrami operator*; relative to a coordinate system

$$\nabla_1^2 = \frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\ . \tag{5}$$

We can think of ∇_1^2 as the ordinary Laplacian, with the radial part subtracted and scaled by r^2 to make it unitless: from (4)

$$\nabla_1^2 = r^2\nabla^2 - r\,\frac{\partial^2}{\partial r^2} r\ . \tag{6}$$

Note there is also a dimensional form of the surface gradient

$$\nabla_S = r^{-1}\nabla_1$$

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a=constant reference radius (e.g. Earth's surface= 6371 km)

- Gauss (in 1832) demonstrated the predominance of the internal part

- The expansion begin at l = 1, excluding the monopoles as a source

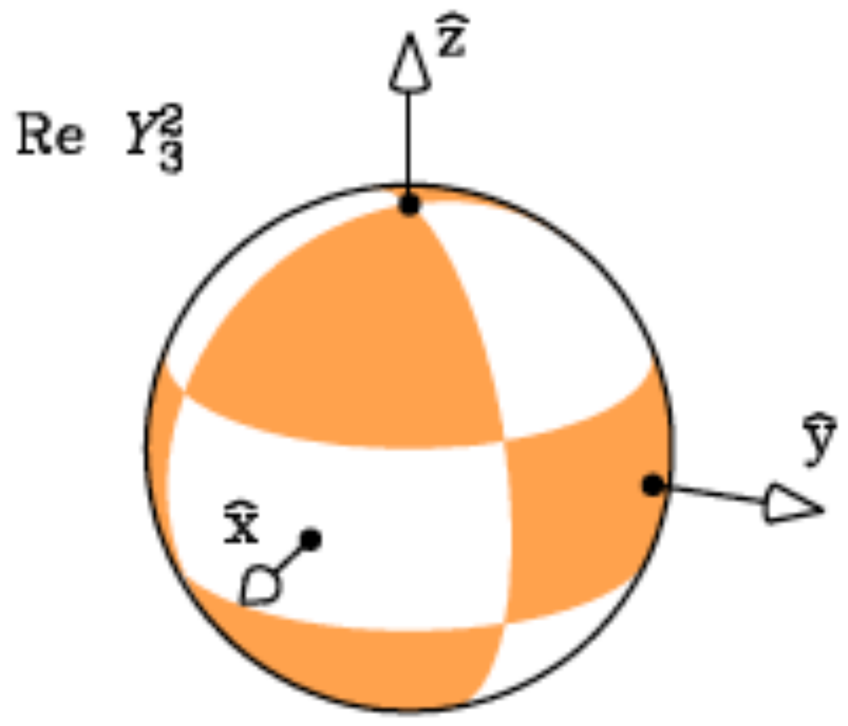
Spherical harmonic functions are defined on the surface of a sphere

$$Y_l^m(\theta,\phi) = N_{lm} e^{im\phi} P_l^m(cos\theta)$$

l is called the degree and m is the order of Y_l^m . Note $e^{im\phi} = \cos m\phi + i \sin m\phi$

For fully normalized spherical harmonics the normalization constant is

$$N_{lm} = (-1)^m [\frac{2l+1}{4\pi}]^{\frac{1}{2}} [\frac{(l-m)!}{(l+m)!}]$$



and the *Associated Legendre Function* is

$$P_l^m(\mu) = \frac{1}{2^l l!}(1-\mu^2)^{\frac{m}{2}} \frac{\partial^{l+m}}{\partial \mu^{l+m}}(\mu^2-1)^l$$

What do the P_l^ms look like?

note that $\mu = \cos\theta$ and $s = \sin\theta$

$s = \sin\theta = (1-\mu^2)^{1/2}$.

$$P_0(\mu) = 1$$

$$P_1(\mu) = \mu; \qquad P_1^1(\mu) = s$$

$$P_2(\mu) = (3\mu^2 - 1)/2; \quad P_2^1(\mu) = 3\mu s; \quad P_2^2(\mu) = 3s^2$$

$$P_3(\mu) = \mu(5\mu^2 - 3)/2$$

$$P_3^1(\mu) = 3s(5\mu^2 - 1)/2$$

$$P_3^2(\mu) = 15s^2\mu; \quad P_3^3(\mu) = 15s^3$$

$$P_4(\mu) = (35\mu^4 - 30\mu^2 + 3)/8$$

$$P_4^1(\mu) = 5s\mu(7\mu^2 - 3)/2; \quad P_4^2(\mu) = 15s^2(7\mu^2 - 1)/2$$

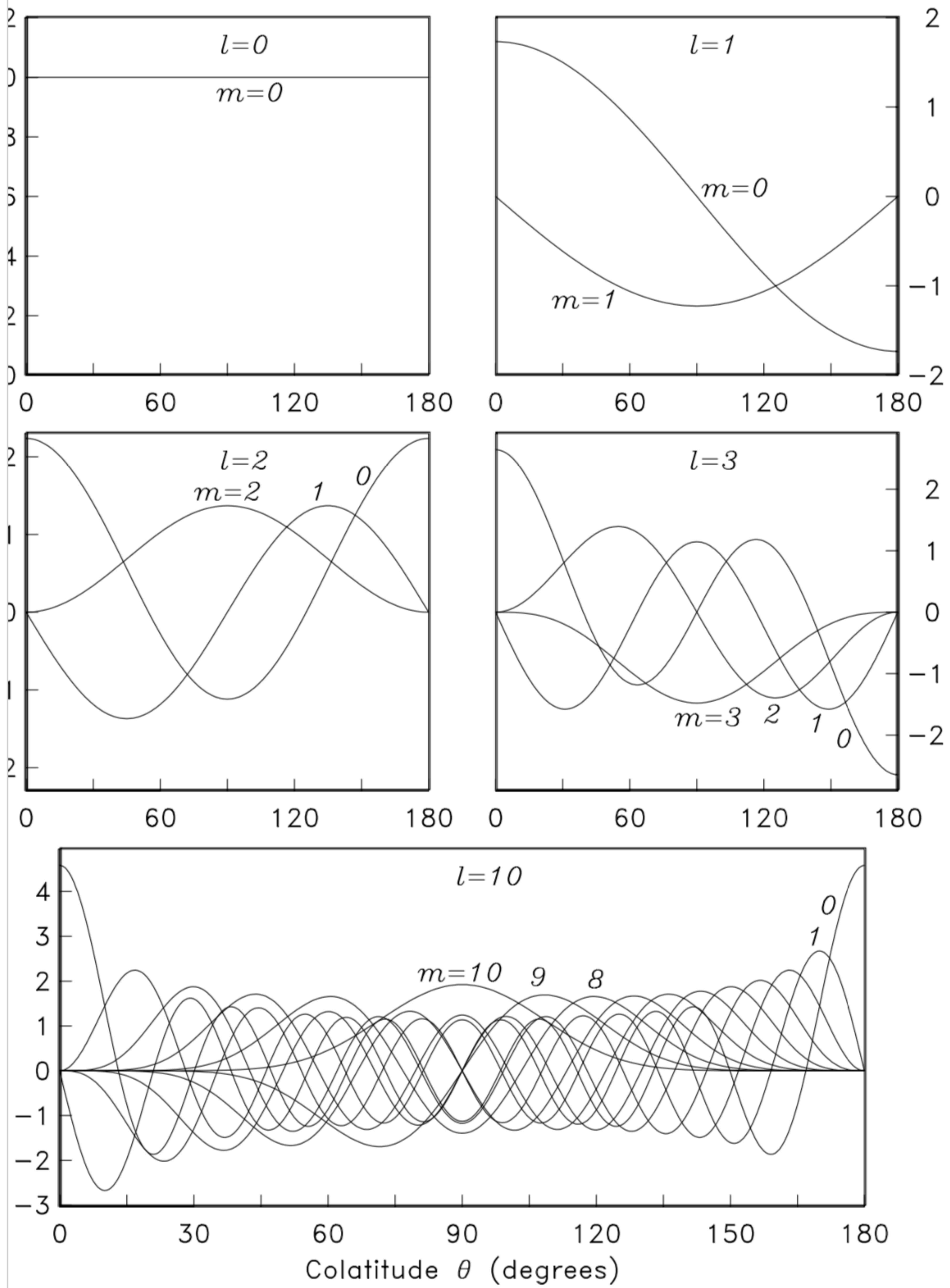
$$P_4^3(\mu) = 105s^3\mu; \quad P_4^4(\mu) = 105s^4$$

$$P_5(\mu) = \mu(63\mu^4 - 70\mu^2 + 15)/8$$

$$P_5^1(\mu) = 15s(21\mu^4 - 14\mu^2 + 1)/8$$

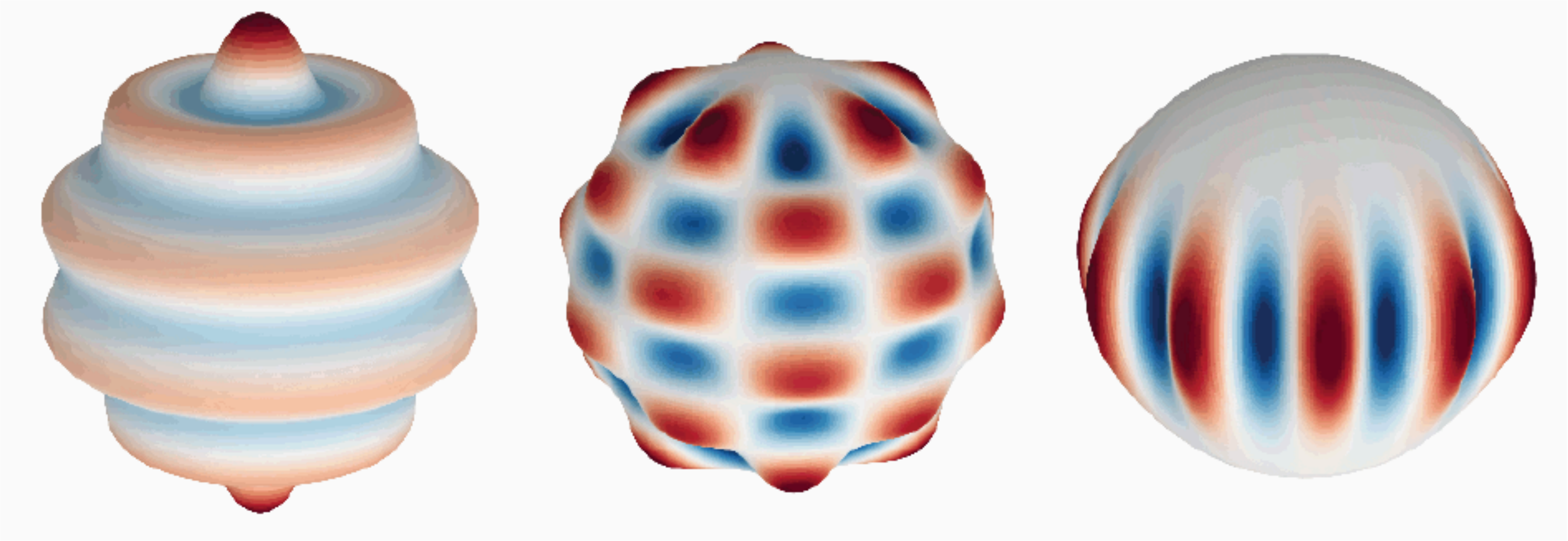
$$P_5^2(\mu) = 105s^2\mu(3\mu^2 - 1)/2; \quad P_5^3(\mu) = 105s^3(9\mu^2 - 1)/2$$

$$P_5^4(\mu) = 945s^4\mu; \quad P_5^5(\mu) = 945s^5.$$



Example Y_l^ms for l=10

zonal, m=0 tesseral sectoral, l=m



zonal $l_{10}m_0$ tesseral $l_{10}m_5$ sectoral $l_{10}m_{10}$

A Table of Spherical Harmonic Lore

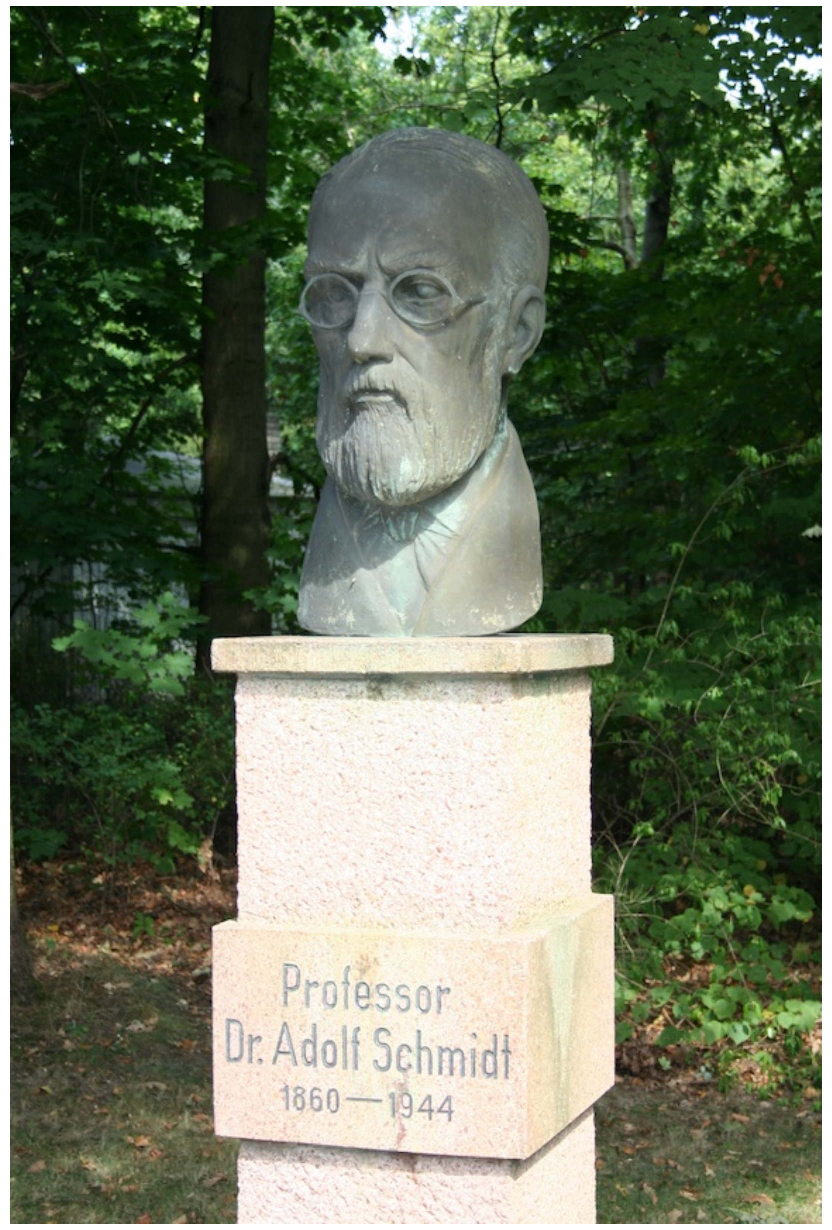
	Property	Formula	Comments
1.	Laplacian in polar coordinates	$\nabla^2 = \frac{1}{r^2}\nabla_1^2 + \frac{1}{r}\frac{\partial^2 r}{\partial r^2}$	∇_1^2 is angular part of familiar Laplacian
2.	Eigenvalue	$\nabla_1^2 Y_l^m = -l(l+1) Y_l^m$, $l = 0, 1, 2, \cdots$	There are $2l+1$ linearly independent eigenfunctions per l
3.	Orthogonality	$\int d^2\hat{\mathbf{s}}\ Y_l^m(\hat{\mathbf{s}})\, Y_n^k(\hat{\mathbf{s}})^* = 0$, unless $l = n$ and $m = k$	True for every normalization
4.	Theoretician's normalization	$\int d^2\hat{\mathbf{s}}\ \lvert Y_l^m(\hat{\mathbf{s}})\rvert^2 = 1$	Other choices: 4π or $4\pi/(2l+1)$
5.	Completeness	$f(\hat{\mathbf{s}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} Y_l^m(\hat{\mathbf{s}})$	Works for any reasonable function f on $S(1)$
6.	Expansion coefficients	$c_{lm} = \int d^2\hat{\mathbf{s}}\ f(\hat{\mathbf{s}})\, Y_l^m(\hat{\mathbf{s}})^*$	Requires property 4
7.	Addition Theorem	$\frac{2l+1}{4\pi} P_l(\hat{\mathbf{s}}\cdot\hat{\mathbf{r}}) = \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{s}})\, Y_l^m(\hat{\mathbf{r}})^*$	Requires property 4
8.	Wavelength of Y_l^m	$\frac{2\pi}{l+½}$	Depends only on degree l, not on order m or $\hat{\mathbf{s}}$
9.	Appearance	Re Y_l^m vanishes on $2m$ meridians and $l-m$ parallels	Im Y_l^m the same but rotated about $\hat{\mathbf{z}}$
10.	Parseval's Theorem	$\int d^2\hat{\mathbf{s}}\ \lvert f(\hat{\mathbf{s}})\rvert^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \lvert c_{lm}\rvert^2$	Requires property 4. Get RMS value of f by dividing by 4π and taking square root
11.	Generating function	$\frac{1}{(1-2\mu r + r^2)^{½}} = \sum_{l=0}^{\infty} r^l P_l(\mu)$	Often used in conjunction with property 7
12.	Another orthogonality	$\int d^2\hat{\mathbf{s}}\ \nabla_1 Y_l^m(\hat{\mathbf{s}})\cdot\nabla_1 Y_n^k(\hat{\mathbf{s}})^* = l(l+1)\,\delta_{ln}\,\delta_{mk}$	Very useful but often omitted! Requires property 4.

Internal/External separation uses 3,4,6,12

Fully normalized spherical harmonics (property 4) are most convenient for theory - but most geomagnetists use Schmidt normalization. This changes N_l^m in slide 15 and uses a real representation for longitudinal variation.

Schmidt Normalization

GAUSS' THEORY OF THE MAIN FIELD



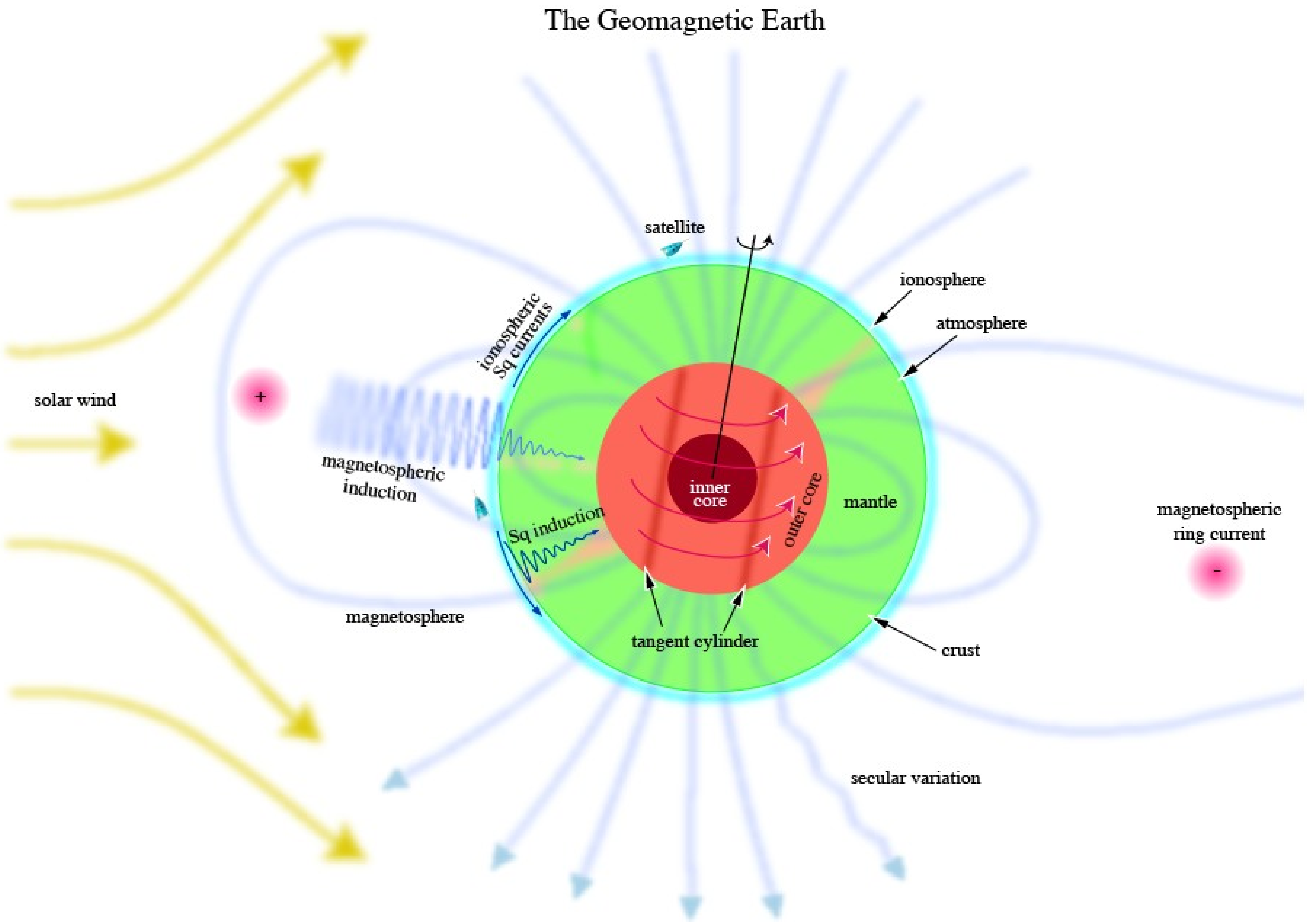
Separating internal and external fields

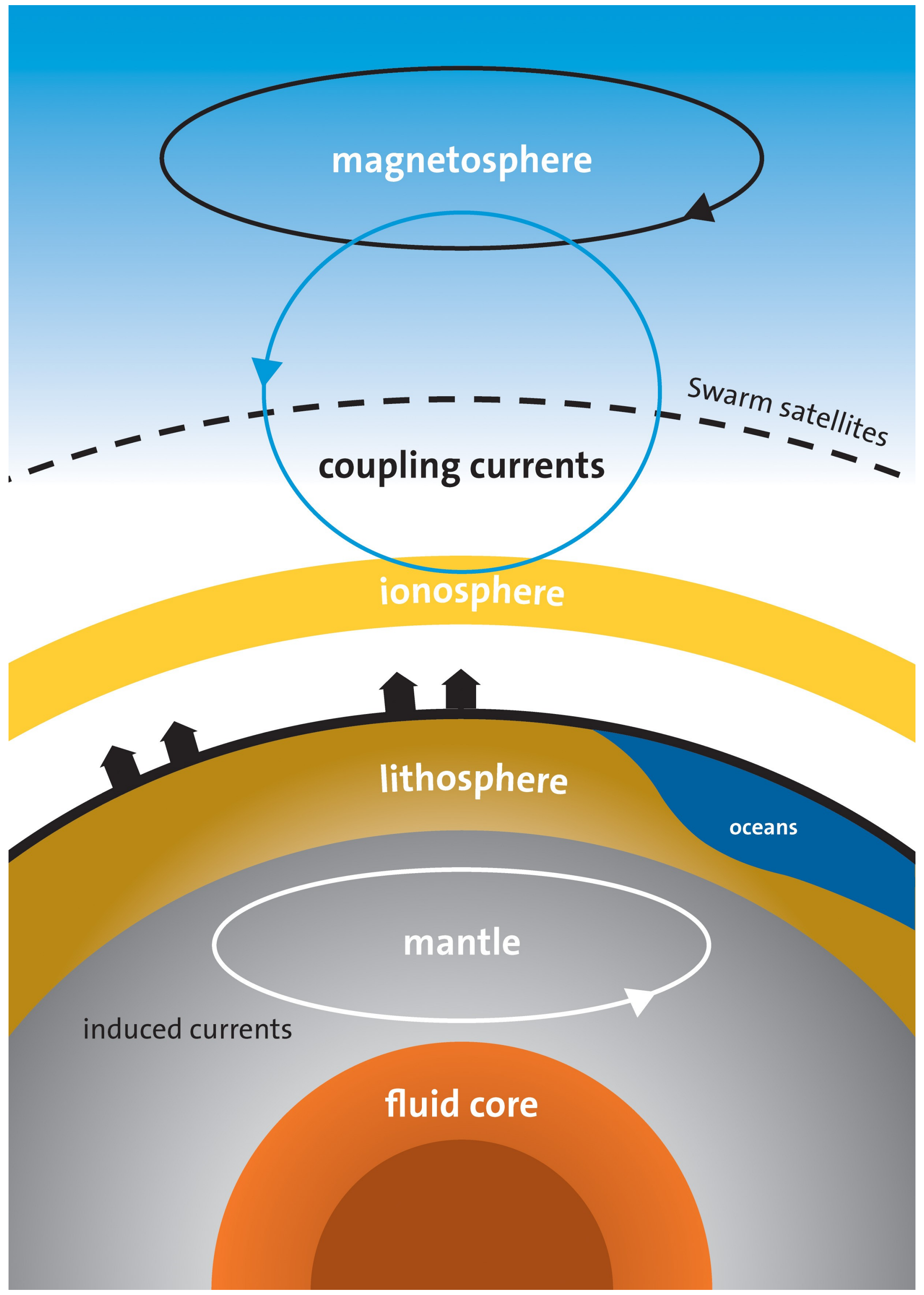
$$\Psi = a \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left[\left(\frac{a}{r}\right)^{l+1} \left(g_l^m \cos(m\phi) + h_l^m \sin(m\phi)\right) P_l^m(\theta) + \left(\frac{r}{a}\right)^{l} \left(q_l^m \cos(m\phi) + s_l^m \sin(m\phi)\right) P_l^m(\theta) \right]$$

Internal sources External sources

- Note we have 2 spherical harmonic expansions in the general solution to Laplace's equation in spherical geometry.
- Fields associated with internal and external sources have different radial dependence.
- This allows internal (e.g. core) and external (e.g. magnetospheric) sources to be efficiently separated, provided observations at different altitudes are available.
- Except during geomagnetic storms the internal field consists of around 97% of the total field observed at Earth's surface.

Sources of Earth's magnetic field





(credit: ESA)

Earth's magnetic field sources

Fully Normalized

GAUSS' THEORY OF THE MAIN FIELD

- Gauss' Separation of Harmonic Fields into Parts of Internal and External Origin

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$$\boxed{\Psi(r,\theta,\phi) = a\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Bigg[\underbrace{A_l^m\left(\frac{r}{a}\right)^l}_{\text{External}} + \underbrace{B_l^m\left(\frac{a}{r}\right)^{l+1}}_{\text{Internal}}\Bigg]Y_l^m(\theta,\phi)}$$

a=constant reference radius (e.g. Earth's surface= 6371 km)

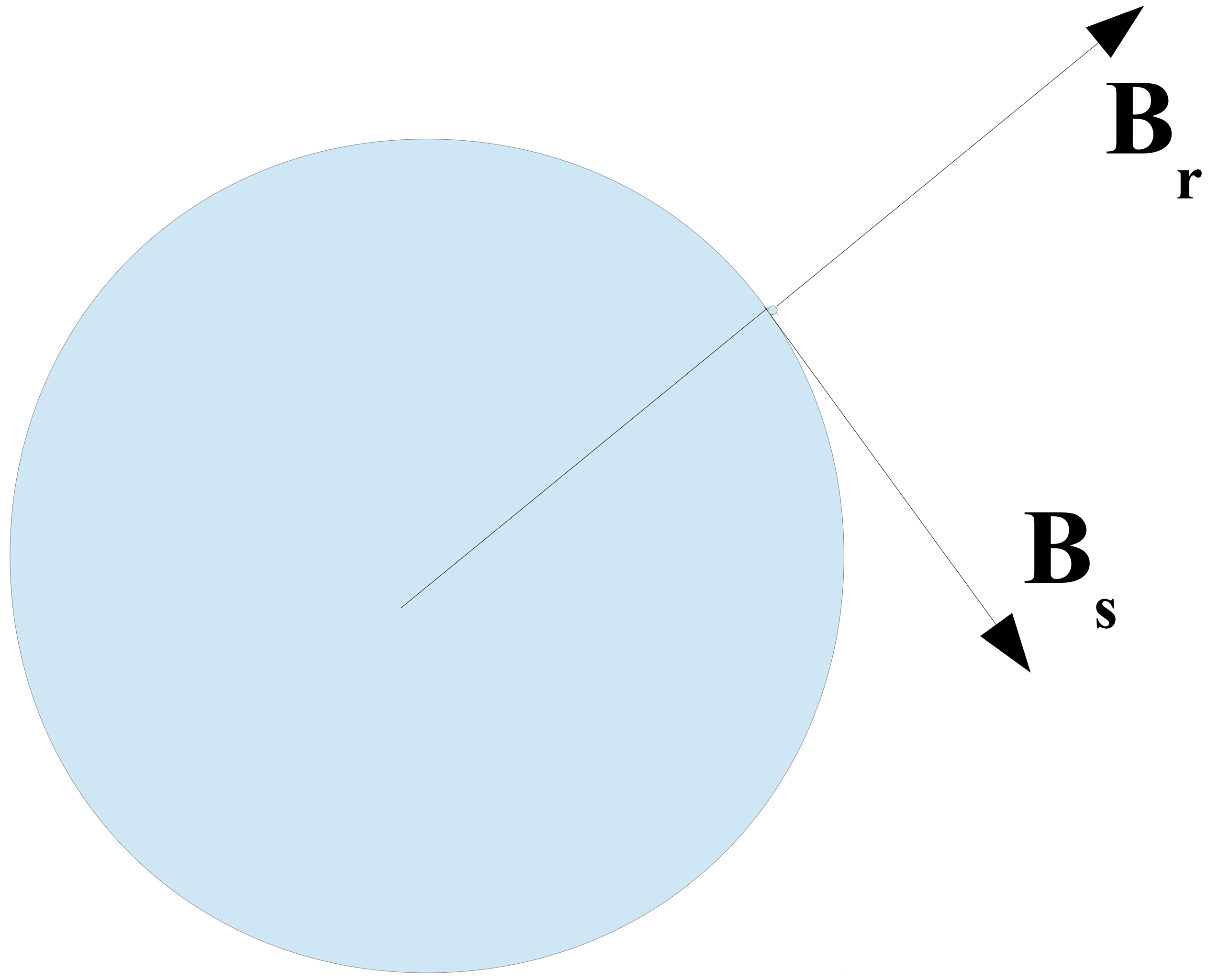
- Gauss (in 1832) demonstrated the predominance of the internal part

- The expansion begin at $l = 1$, excluding the monopoles as a source

How did Gauss separate internal and external parts of the field?
First recall that we measure ***B*** not Ψ.

- Assumption: ***B*** is known everywhere on the surface of the sphere $r = a$:

$$\boldsymbol{B} = \hat{\boldsymbol{r}} B_r + \boldsymbol{B}_s \qquad \nabla = \hat{\boldsymbol{r}} \partial_r + \frac{1}{r} \nabla_1$$



$$B_r = -\partial_r \Psi |_{r=a} = -\sum_{l,m} \left[l A_l^m - (l+1) B_l^m \right] Y_l^m(\theta, \phi)$$

$$\boldsymbol{B}_s = -\frac{1}{r} \nabla_1 \Psi = -\sum_{l,m} \left[A_l^m + B_l^m \right] \nabla_1 Y_l^m$$

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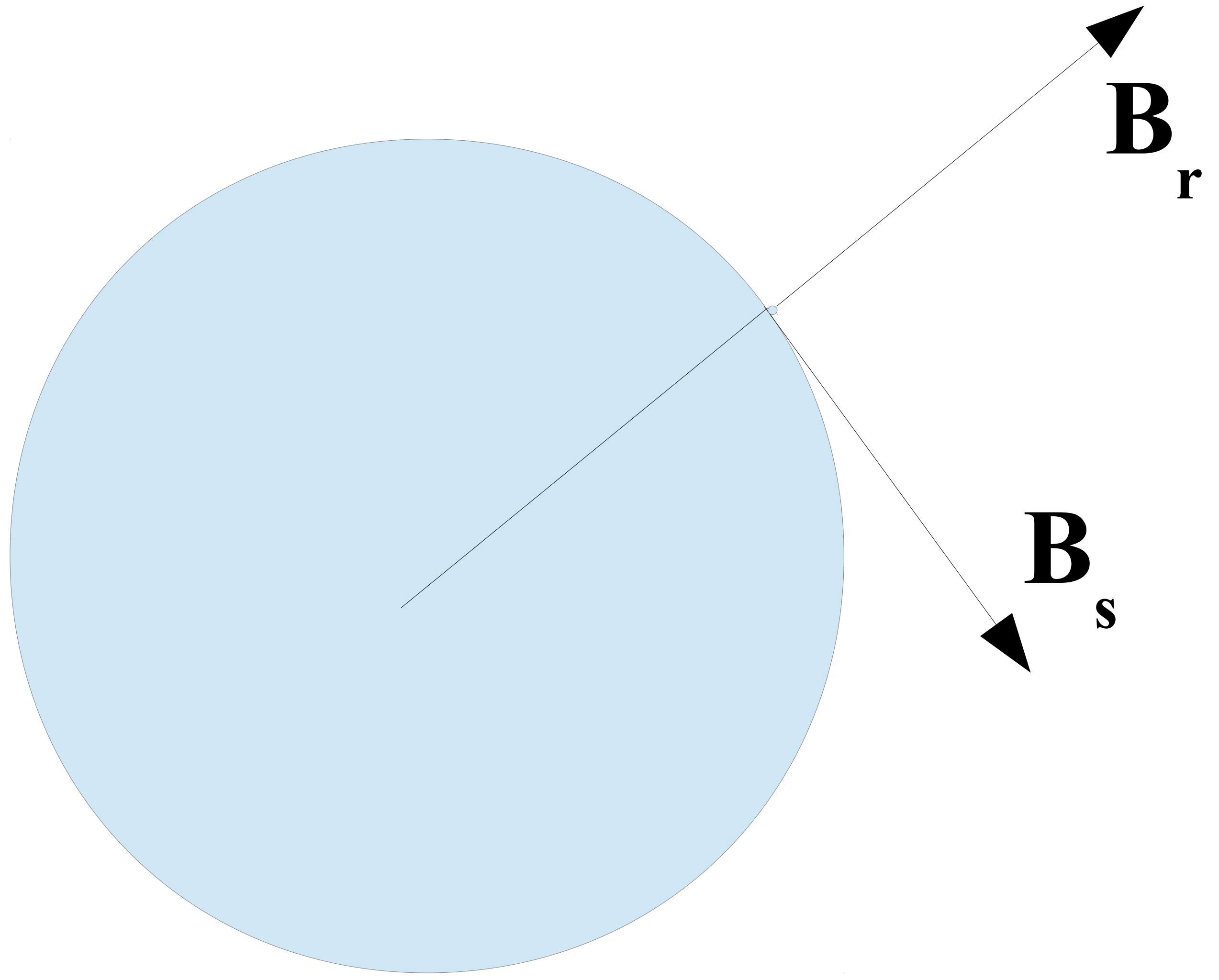

$$B_r = -\partial_r \Psi|_{r=a} = -\sum_{l,m} \left[l A_l^m - (l+1) B_l^m \right] Y_l^m(\theta, \phi)$$

$$\boldsymbol{B}_s = -\frac{1}{r} \nabla_1 \Psi = -\sum_{l,m} \left[A_l^m + B_l^m \right] \nabla_1 Y_l^m$$

We want to find the A_l^m and B_l^m for all l and m

$$B_r = -\partial_r \Psi|_{r=a} = -\sum_{l,m} \left[l A_l^m - (l+1) B_l^m \right] Y_l^m(\theta, \phi)$$

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We use properties 3, 4, 6, and 12 from the table of SH lore.

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$$\boldsymbol{B}_s = -\frac{1}{r} \nabla_1 \Psi = -\sum_{l,m} \left[A_l^m + B_l^m \right] \nabla_1 Y_l^m$$

By using properties 3, 4, 6, and12 from the table of lore

- We can always recover the internal and external coefficients separately from our knowledge of ***B*** on $r = a$, combining the following two equations:

$$\boxed{\begin{aligned} l A_l^m - (l+1) B_l^m &= -\int B_r (Y_l^m)^* d^2 \hat{\boldsymbol{r}} \\ A_l^m + B_l^m &= -\frac{\int \boldsymbol{B}_s \cdot (\nabla_1 Y_l^m)^* d^2 \hat{\boldsymbol{r}}}{(l(l+1))} \end{aligned}}$$

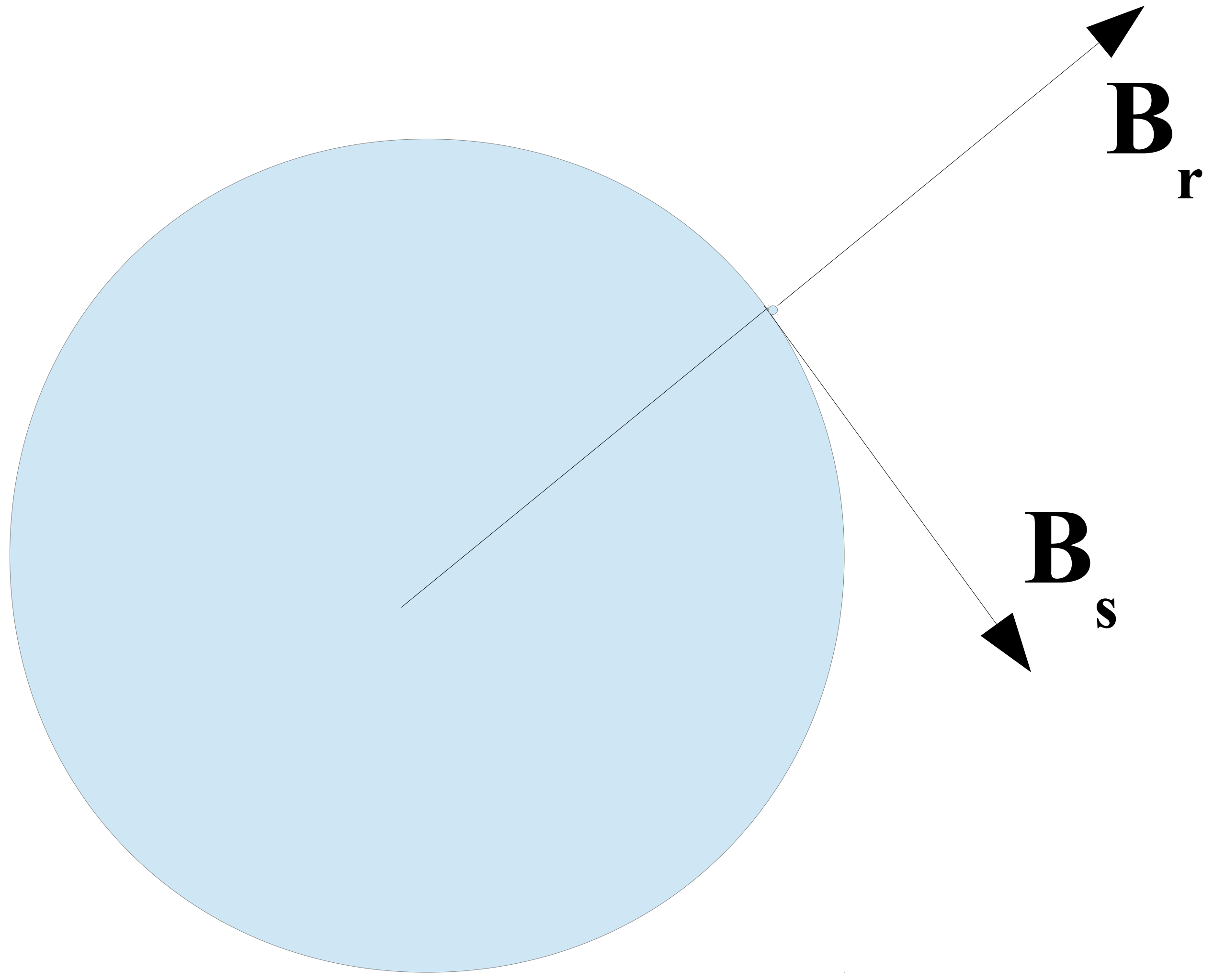
Summarizing: Gauss' Separation of Harmonic Fields into Parts of Internal and External Origin

- Assumption: ***B*** is known everywhere on the surface of the sphere $r = a$:

$$\boldsymbol{B} = \hat{\boldsymbol{r}}\, B_r + \boldsymbol{B}_s \qquad \nabla = \hat{\boldsymbol{r}}\, \partial_r + \frac{1}{r} \nabla_1$$

$$B_r = -\partial_r \Psi|_{r=a} = -\sum_{l,m} \left[l A_l^m - (l+1) B_l^m \right] Y_l^m(\theta, \phi)$$

$$\boldsymbol{B}_s = -\frac{1}{r} \nabla_1 \Psi = -\sum_{l,m} \left[A_l^m + B_l^m \right] \nabla_1 Y_l^m$$



- We can always recover the internal and external coefficients separately from our knowledge of ***B*** on $r = a$, combining the following two equations:

$$\boxed{\begin{aligned} l A_l^m - (l+1) B_l^m &= -\int B_r (Y_l^m)^* d^2 \hat{\boldsymbol{r}} \\ A_l^m + B_l^m &= -\frac{\int \boldsymbol{B}_s \cdot (\nabla_1 Y_l^m)^* d^2 \hat{\boldsymbol{r}}}{(l(l+1))} \end{aligned}}$$