SIOG 231
GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 6

More on Spherical Harmonics

1/25/2024

Last Class

- Laplace's equation for geomagnetic scalar potential is valid in source free region - Earth's atmosphere
- Write a general solution to Laplace's equation in geocentric spherical coordinates
- Fully-normalized spherical harmonics with complex coefficients A_l^m and B_l^m allow us to easily do theory to separate internal and external field contributions
- But geomagnetists usually use real partially normalized Schmidt coefficients g_l^m and h_l^m (internal) and q_l^m, s_l^m (external).

Today's Class

- Why spherical harmonics?
- Specializing to the internal field
- Upward and Downward Continuation looked at 2 different ways
- Spatial power spectrum for the geomagnetic field

A Table of Spherical Harmonic Lore

	Property	Formula	Comments
1.	Laplacian in polar coordinates	$\nabla^2 = \frac{1}{r^2}\nabla_1^2 + \frac{1}{r}\frac{\partial^2 r}{\partial r^2}$	∇_1^2 is angular part of familiar Laplacian
2.	Eigenvalue	$\nabla_1^2 Y_l^m = -l(l+1)\,Y_l^m$, $l = 0, 1, 2, \cdots$	There are $2l+1$ linearly independent eigenfunctions per l
3.	Orthogonality	$\int d^2\hat{\mathbf{s}}\; Y_l^m(\hat{\mathbf{s}})\,Y_n^k(\hat{\mathbf{s}})^* = 0$, unless $l = n$ and $m = k$	True for every normalization
4.	Theoretician's normalization	$\int d^2\hat{\mathbf{s}}\;\lvert Y_l^m(\hat{\mathbf{s}})\rvert^2 = 1$	Other choices: 4π or $4\pi/(2l+1)$
5.	Completeness	$f(\hat{\mathbf{s}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}Y_l^m(\hat{\mathbf{s}})$	Works for any reasonable function f on $S(1)$
6.	Expansion coefficients	$c_{lm} = \int d^2\hat{\mathbf{s}}\; f(\hat{\mathbf{s}})\,Y_l^m(\hat{\mathbf{s}})^*$	Requires property 4
7.	Addition Theorem	$\frac{2l+1}{4\pi}P_l(\hat{\mathbf{s}}\cdot\hat{\mathbf{r}}) = \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{s}})\,Y_l^m(\hat{\mathbf{r}})^*$	Requires property 4
8.	Wavelength of Y_l^m	$\frac{2\pi}{l+½}$	Depends only on degree l, not on order m or $\hat{\mathbf{s}}$
9.	Appearance	Re Y_l^m vanishes on $2m$ meridians and $l-m$ parallels	Im Y_l^m the same but rotated about $\hat{\mathbf{z}}$
10.	Parseval's Theorem	$\int d^2\hat{\mathbf{s}}\;\lvert f(\hat{\mathbf{s}})\rvert^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\lvert c_{lm}\rvert^2$	Requires property 4. Get RMS value of f by dividing by 4π and taking square root
11.	Generating function	$\frac{1}{(1-2\mu r + r^2)^{½}} = \sum_{l=0}^{\infty} r^l\,P_l(\mu)$	Often used in conjunction with property 7
12.	Another orthogonality	$\int d^2\hat{\mathbf{s}}\;\nabla_1 Y_l^m(\hat{\mathbf{s}})\cdot\nabla_1 Y_n^k(\hat{\mathbf{s}})^* = l(l+1)\,\delta_{ln}\,\delta_{mk}$	Very useful but often omitted! Requires property 4.

Internal/External separation used 3,4,6,12

Why spherical harmonics?

Natural basis functions for representing any real-square integrable function on the sphere

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_l^m(\theta, \phi)$$

$$c_{lm} = \int d^2\hat{\mathbf{s}} f(\hat{\mathbf{s}}) Y_l^m(\hat{\mathbf{s}})$$

where on the unit sphere we can write $d^2\hat{\mathbf{s}} = \sin\theta d\theta d\phi$,
while for a sphere of radius r

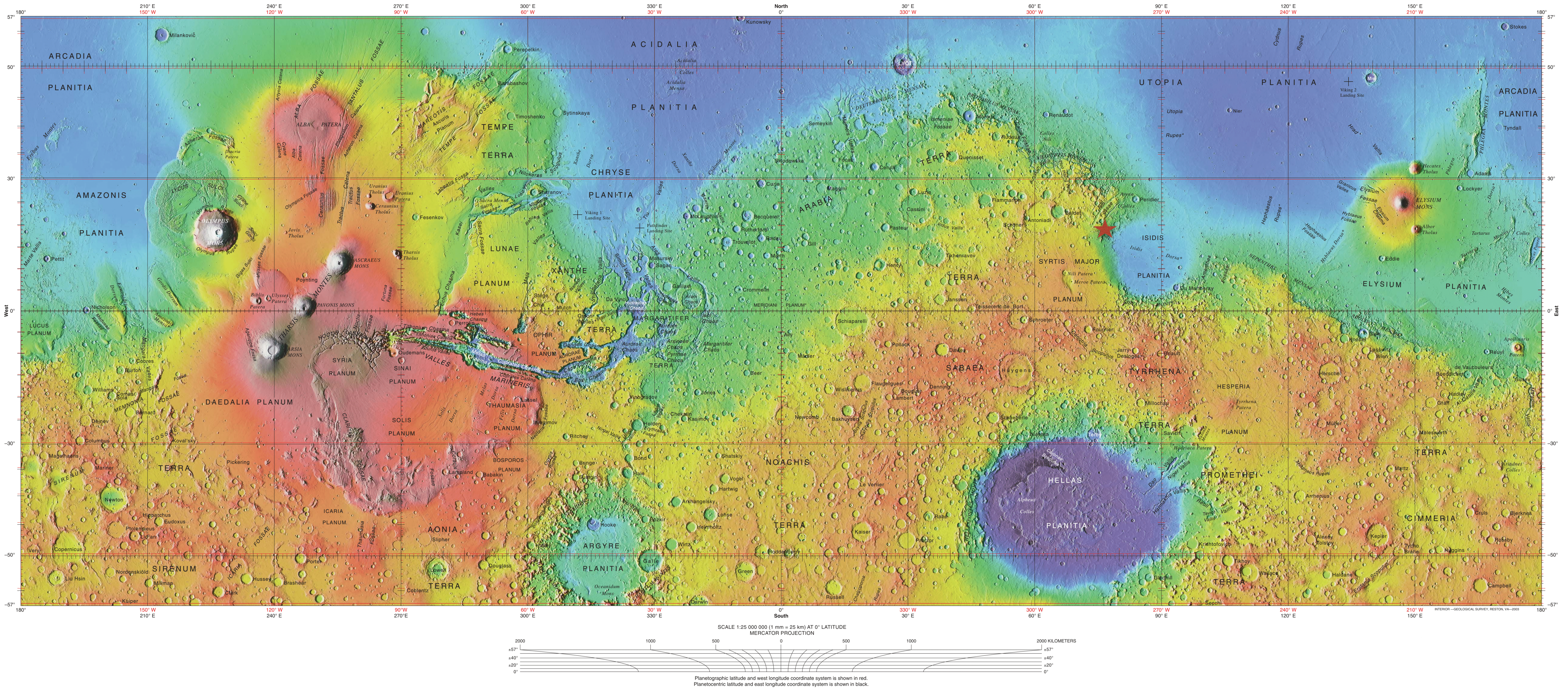
$$d^2\mathbf{s} = r^2 \sin\theta d\theta d\phi$$

If we know the coefficients in this expansion we can evaluate the function anywhere on the surface of the sphere. This makes them useful for all kinds of observables on planetary scale: topography, magnetic and gravitational field, displacement/ velocity in seismology, postglacial rebound, elastic flexure, …

What sort of functions might we represent with a Spherical Harmonic Expansion (SHE)?

Mars Topography - USGS

★ Jezero Crater



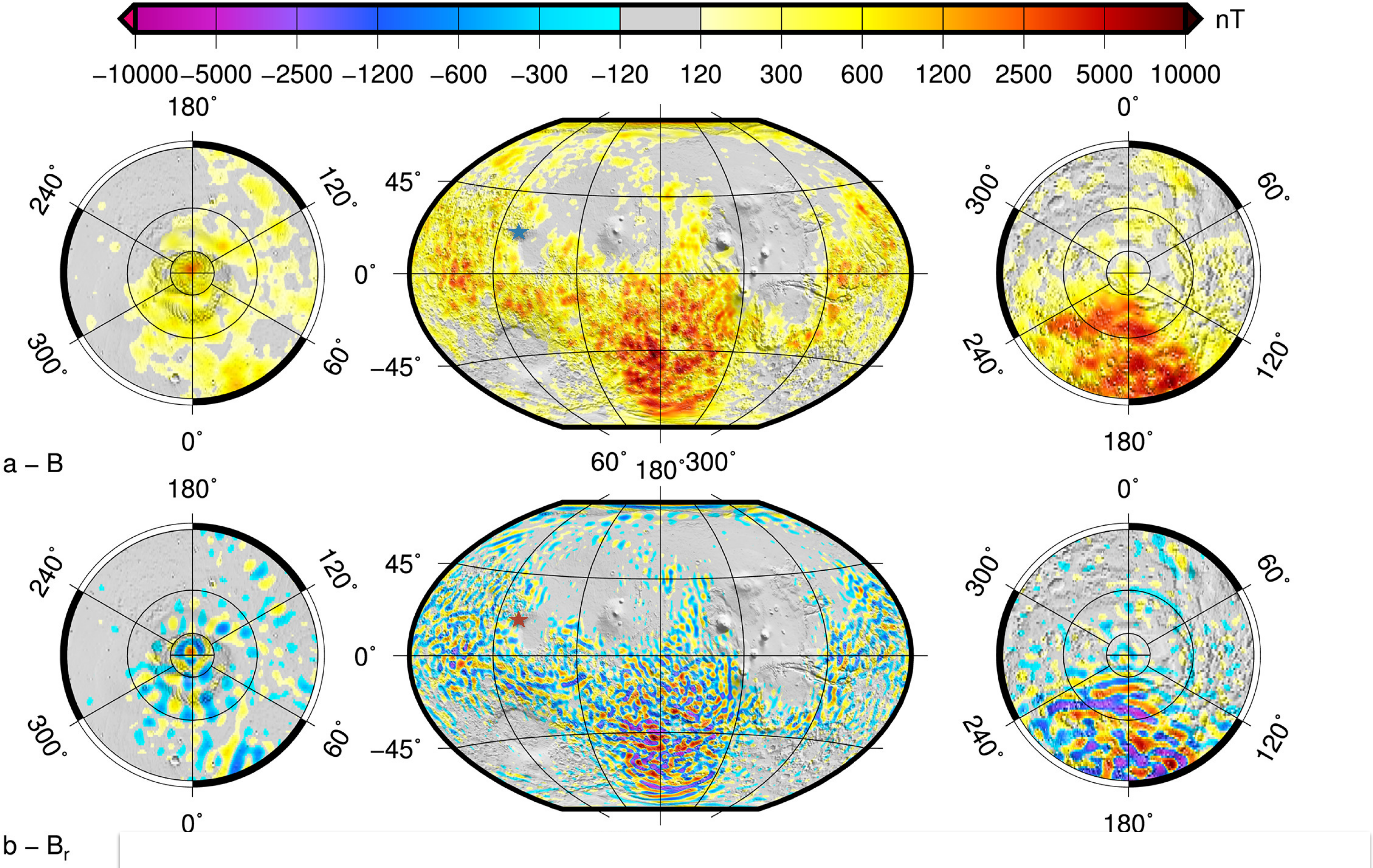
Topographic Map of Mars

Magnetic field at Mars Surface, Langlais et al., 2019

https://doi.org/10.1029/2018JE005854

Scalar Field

Radial Field



Upward and Downward Continuation -I

- The solution to Laplace's equation for the scalar potential Ψ given in terms of spherical harmonics is valid through the region D where $\nabla^2\Psi = 0$.

- To find B we take the gradient of Ψ.

- Then just need to specify the Gauss coefficients and (r, θ, ϕ) to evaluate the field anywhere in D.

- Increasing r is called **upward continuation**. Decreasing r is **downward continuation.**

Let's ignore external field contributions and just consider the internal field

Upward and Downward continuation

- Suppose that we have a collection of observations on one surface, but would like to infer something about the source at some other altitude or radius;

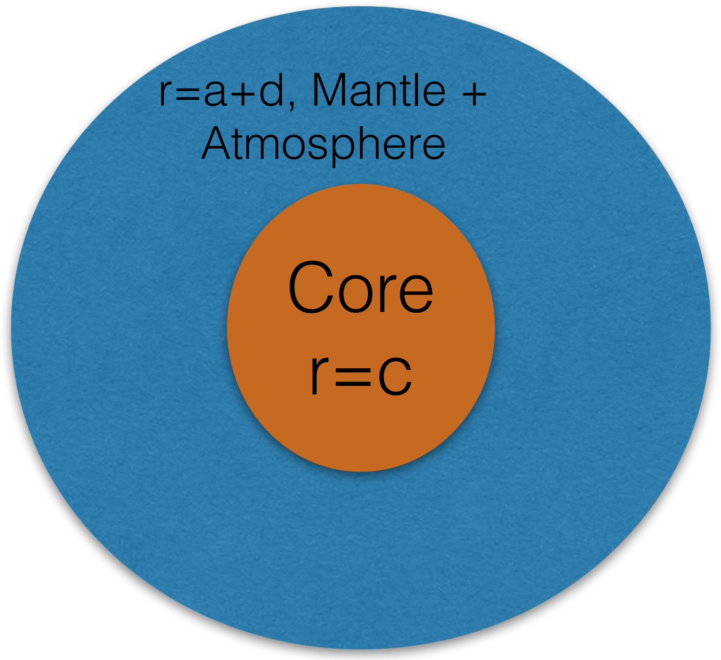
- Assumption: Earth's mantle is an insulator and there are no magnetic sources within it (approximations commonly adopted when studying the magnetic field at the core) then we can write the magnetic field ***B*** as the gradient of a scalar potential within that region too:

$$\boldsymbol{B} = -\nabla\Psi \qquad \text{and} \qquad \nabla^2\Psi = 0$$

- Finding B_r further away from the sources is known as ***upward continuation***.
- Finding B_r values towards the sources is known as ***downward continuation***.

Specializing to the internal part of Earth's field we write the potential in the region $c < r < a + d$, with $c = 3485km$ the radius at the Core-Mantle Boundary (CMB), $a = 6371km$ the mean surface radius, and d the height of the atmosphere.

$$\Psi(r,\theta,\phi) = a\sum_{l=1}^{\infty}\sum_{m=-l}^{l} B_l^m\left(\frac{a}{r}\right)^{l+1} Y_l^m(\theta,\phi)$$



The magnetic field is given by the negative gradient $B = -\nabla\Psi$

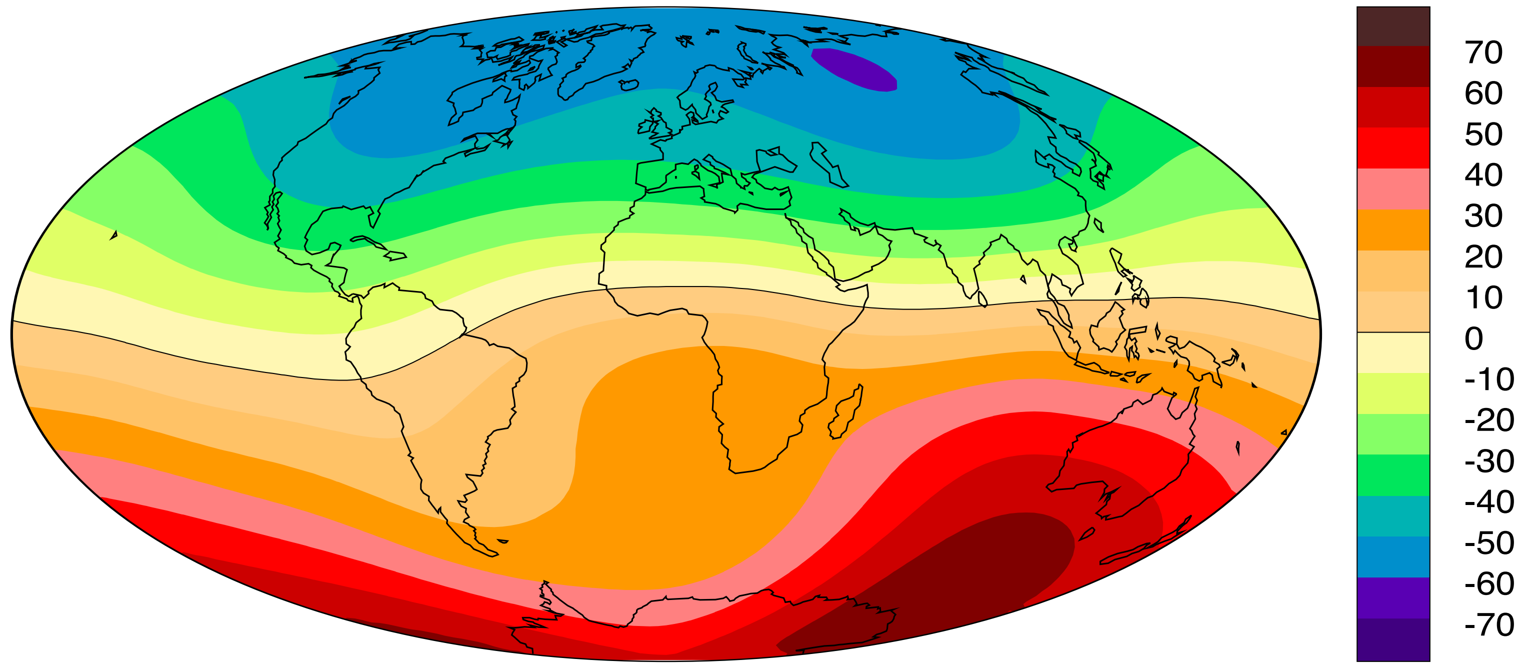
$$\nabla\Psi = \hat{\mathbf{r}}\frac{\partial\Psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}$$

For example the radial component of the field by taking $\partial/\partial r$

$$B_r(r,\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}(l+1)B_l^m\left(\frac{a}{r}\right)^{l+2} Y_l^m(\theta,\phi) \qquad c < r < a + d$$

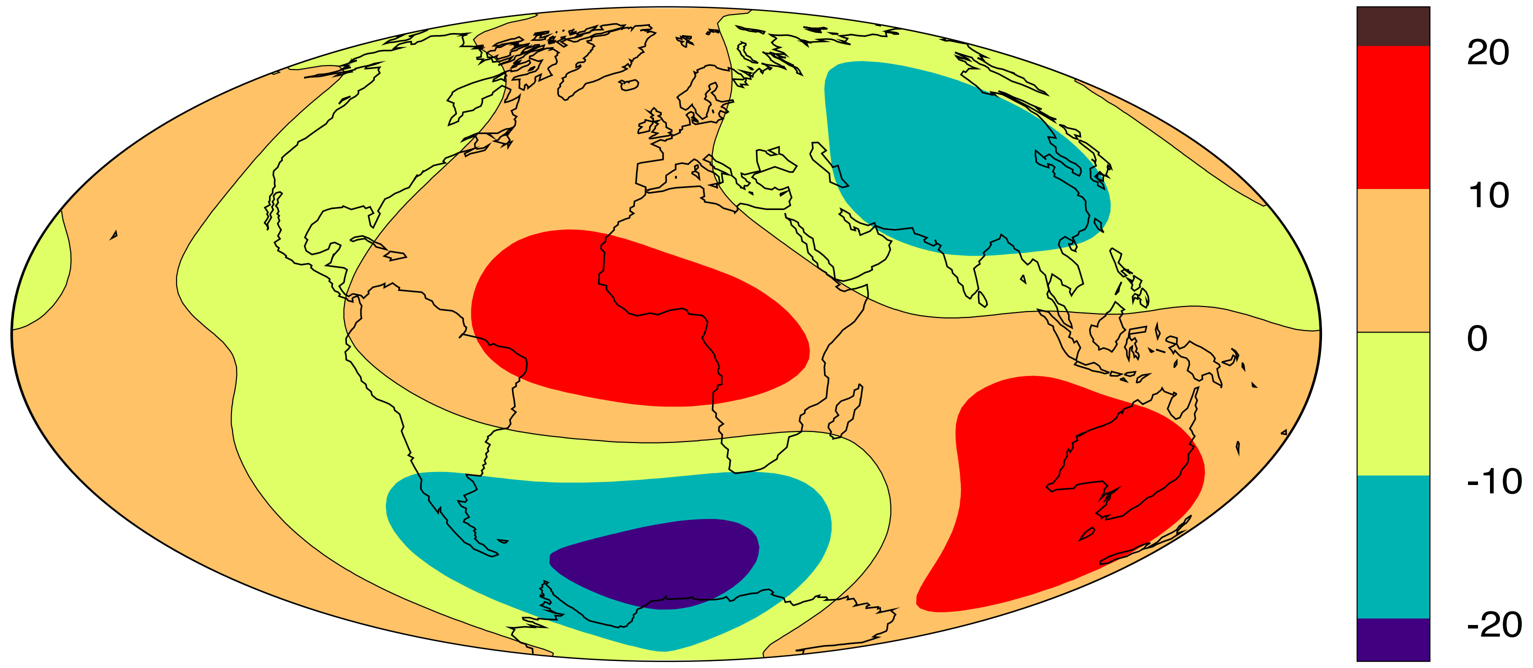
B_r for the IGRF in 2020

Radial field at r=a



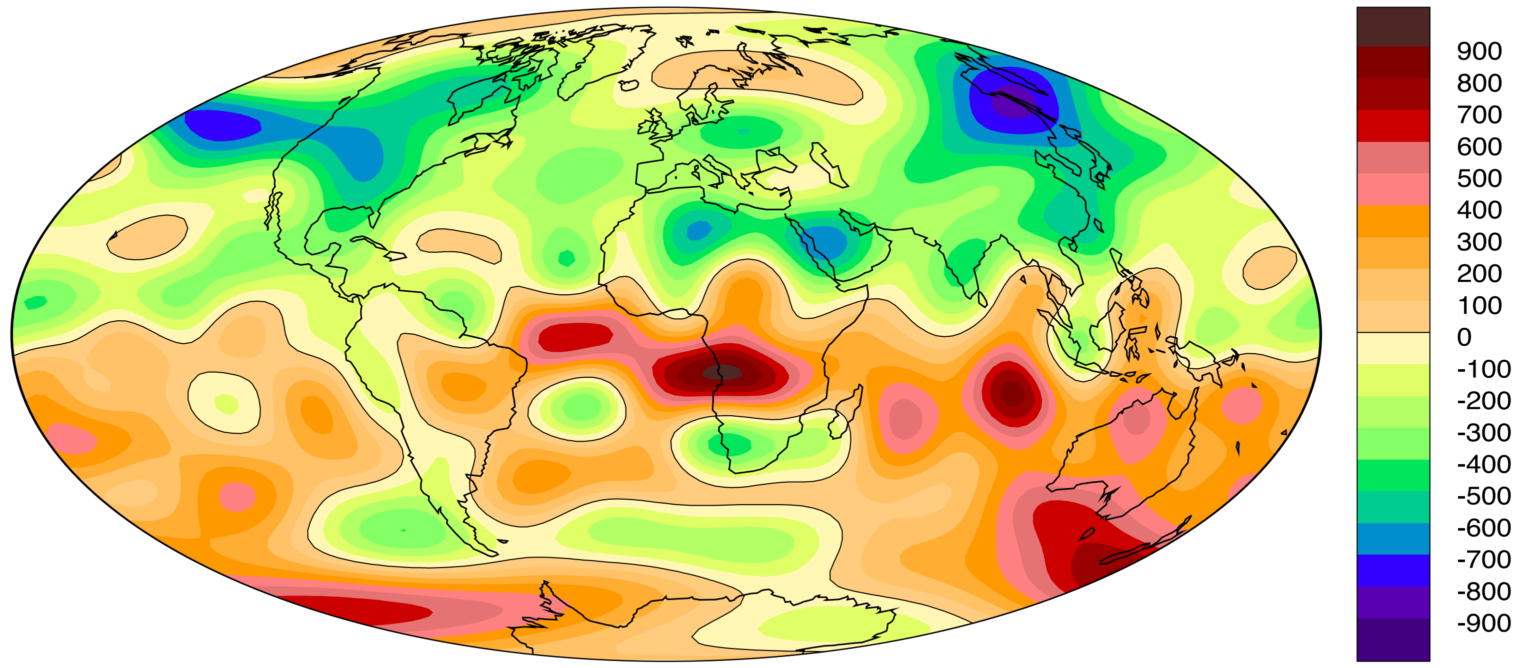
IGRF 2020 Radial field Br at r=a

Non-dipole radial field at r=a



IGRF 2020 Non-Dipole Radial field Br at r=a

Downward continued radial field at r=c



IGRF 2020 Radial field Br at r=c

Upward and Downward Continuation -II

In Homework 1 you showed that an alternative to Ψ for the magnetic potential is $\Omega = rB_r$. It will have different Gauss coefficients in the spherical harmonic representation - call them β_l^m. We can find them by the same method as before.

We perform the integrals on the known field over *S(a)*.

$$\nabla^2(rB_r) = \nabla^2(\Omega) = 0$$

$$\Omega(r, \theta, \phi) = rB_r(r, \theta, \phi) = a \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \beta_l^m \left(\frac{a}{r}\right)^{l+1} Y_l^m(\theta, \phi)$$

$$\beta_l^m = \int_{S(a)} B_r(a, \theta' \phi') Y_l^m(\theta', \phi')^* \frac{d^2 r'}{a^2}$$

A Bit of Theory of Harmonic Functions - there is a lot!

$\Psi(r, \theta, \phi)$ is harmonic in D, a bounded region of real space if $\nabla^2 \Psi = 0$. Then

- Ψ is infinitely differentiable at all points in D - note we are talking about spatial derivatives.
- Maximum and minimum values of Ψ always occur on the boundaries of D - not inside.
- For any spherical surface within D the average value of Ψ over the surface equals its value at the center of the sphere.
- There is a uniqueness theorem for the Neumann Boundary value problem: if you know $\partial\Psi/\partial n$ everywhere on the surface of a compact body *B,* and Ψ falls off like *1/r* as *r* goes to infinity, then this knowledge is sufficient to determine Ψ ***uniquely*** everywhere outside B.

Green's Function Formulation

- Using the Neumann BVP we can write the field at Earth's surface in terms of point sources of B_r at the core-mantle boundary (CMB).

- The Green's function gives the form for translating a unit of B_r at radius c to a larger value of the radius (upward continuation). To get the whole field we need to integrate over the sphere $S(c)$

Upward Continuation

Suppose we know B_r everywhere on the surface of the core $r = c$.
Integrating against the Green's function $G(\mathbf{r}, \mathbf{r}') = (c/r)^2 K(c/r, \mathbf{r} \cdot \mathbf{r}')$ gives the value of $B_r(r, \hat{\mathbf{r}})$ at a larger radius r. We can get the functional form for K using information from the table of spherical harmonic lore. Details are omitted here.

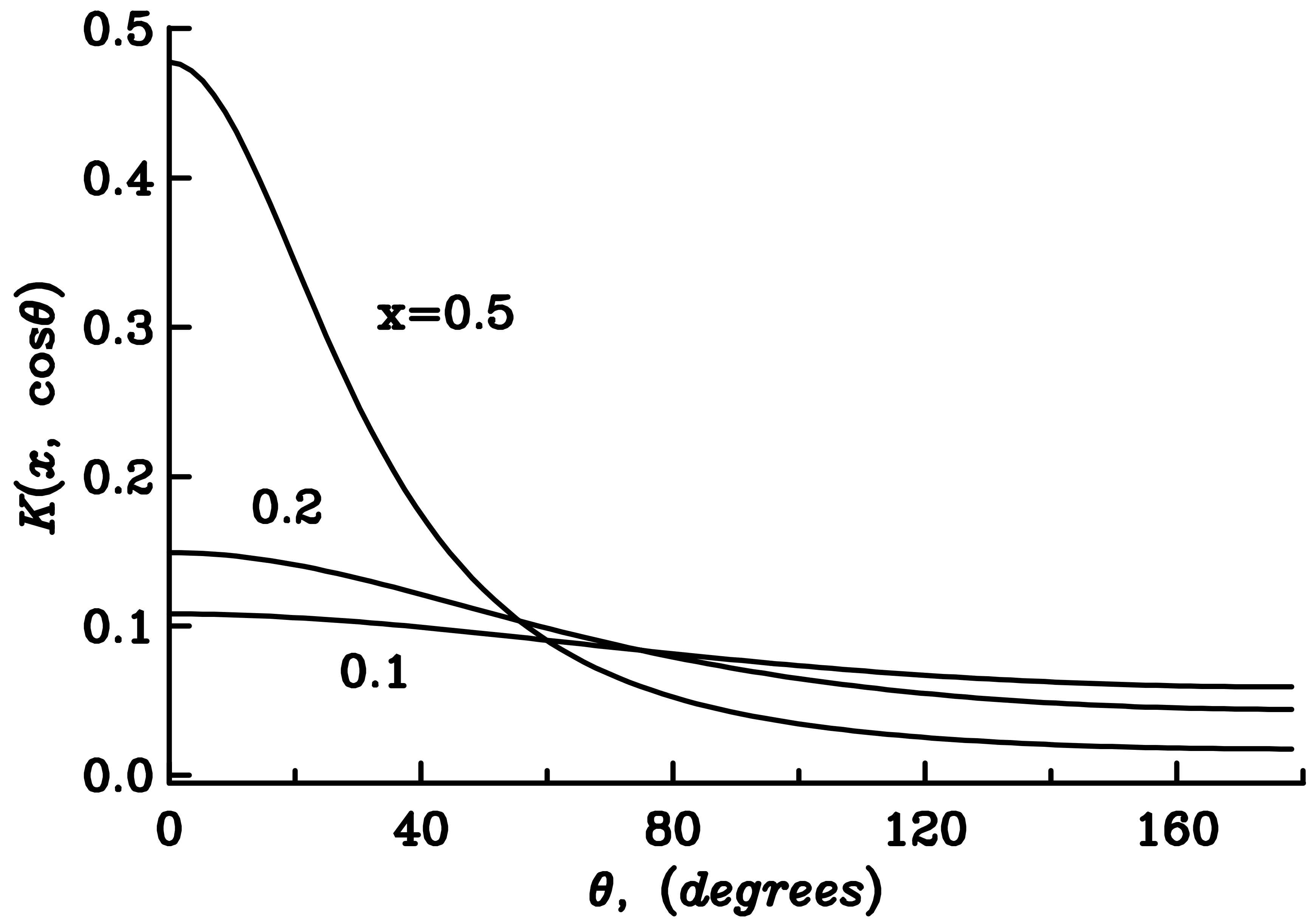


Figure 3.2.1

$$B_r(r, \hat{\mathbf{r}}) = \int_{S(c)} (c/r)^2 K(c/r,\ \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')\, B_r(c, \hat{\mathbf{r}}')\, d^2\hat{\mathbf{r}}'$$

$$K(x, \cos\theta) = \frac{1}{4\pi} \frac{1 - x^2}{(1 + x^2 - 2x\cos\theta)^{3/2}}$$

$$x = c/r \quad \text{and} \quad \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos\theta$$

Why is the Green's Function formulation useful?

- Imagine you have a localized source of radial magnetic field, $B_r(c)$, at the core-mantle boundary (CMB). Let's assume it's a spike-like delta function.

- What does that look like at Earth's surface? The Green's function describes its contribution to the field at Earth's surface, $B_r(a)$, which is spread over a broad region, averaging the input at $r=c$, and reducing its amplitude.

- Obviously this applies not just to $B_r(a)$ but to all the geomagnetic elements at Earth's surface, D, I, F, etc.

And example of an intensity spike at the CMB (r=c) and Earth's surface (r=a)

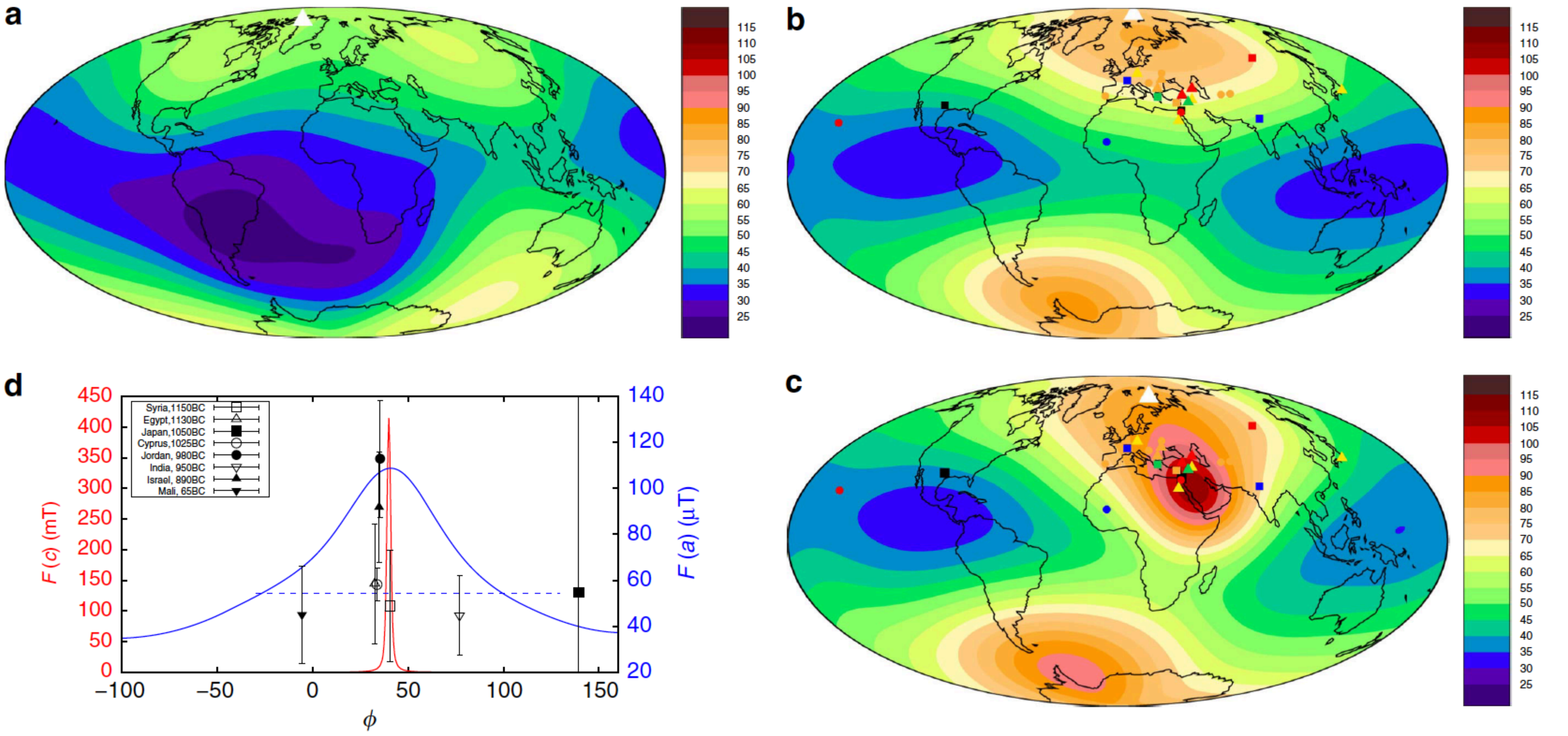


Figure 4 | The Levantine geomagnetic spike. Contours of field intensity, F (μT), at Earth's surface ($r=a$) from the CHAOS-4 model[2] at 2010 (**a**) the CALS10k.2 global field model at 1000 BC (**b**) and CALS10k.2 at 1000 BC plus a superposed best-fitting spike at 20° N, 40° E with amplitude $A=400$ mT and s.d. of $\sigma=1°$ at the CMB (**c**). Symbols show paleointensities for samples dated at 1150–1050 BC (triangles), 1049–950 BC (squares) and 949–850 BC (circles). Symbol colours are blue (40–50 μT), green (51–60 μT), yellow (61–70 μT), orange (71–90 μT), red (91–115 μT) and black (>115 μT). White triangles in **a**–**c** mark the north pole of the dipole field. (**d**) Longitudinal cross-section through the spike in **c**, at Earth's surface (blue, right ordinate) and the CMB ($r=c$, red, left ordinate). The horizontal dashed line marks the width at half maximum $\delta_2(a)$. Available data within 20° ± 15° N are shown corrected to 20° N using the formula for an axial dipole field, $F \propto (1+3\cos^2\theta)^{0.5}$, where θ is colatitude. Error bars correspond to the uncertainties in Fig. 3b. Open and closed symbols cannot be simultaneously matched by the model.

Upward continuation

- A fundamental property of the integral is that:

$$|B_r(r,\hat{\boldsymbol{r}})| \leqslant \int_{S(a)} |(a/r)^2 K(a/r, \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}')| |B_r(a,\hat{\boldsymbol{r}}')|_{max}\, d^2\hat{r}'$$

$$|B_r(r,\hat{\boldsymbol{r}})| < |B_r(a,\hat{\boldsymbol{r}}')|_{max}, \qquad r > a$$

- The magnitude of B_r on a sphere of radius $r > a$, is always less than the maximum magnitude on the sphere of radius a;

- The maximum value falls off like r^{-2} .Technically upward continuation is bounded linear mapping;

- The upward continuation is stable. This means that a small error in the field on the inner sphere remains small on the outer one.

Upward and Downward continuation

$$\Omega_l^m(r) = \beta_l^m \left(\frac{a}{r} \right)^{l+1}$$

- In upward continuation: short wavelength energy disappears from the field preferentially as we go to spheres of larger radius;

- In downward continuation: the shorter wavelength components of the field are magnified relative to the longer wavelength ones;

- Mapping from *S(a)* to *S(r)* when $r < a$ is an example of an unstable process. Roughly this means small errors in the field may grow when the field is downward continued.

Downward continuation to the core

- The spherical harmonic expression for the Z component of the geomagnetic field due to internal sources is:

$$Z = -\sum_{l=1}^{L} \sum_{m=0}^{l} (l+1) \left(\frac{a}{r}\right)^{l+2} [g_l^m \cos(m\phi) + h_l^m \sin(m\phi)] P_l^m(\theta)$$

- Assuming the mantle contains no magnetic field sources (is to first approximation an insulator) then we can simple change r from r = 6371 km to c = 3481 km.

- This involves each spherical harmonic at Earth's surface being multiplied by a factor

$$(l+1)\left(\frac{a}{c}\right)^{l+2}$$

- Since a >> c, harmonics with larger l are amplified more on downward continuation

Errors in Upward and Downward continuation

$$\left|e_{max}\right|_{r=a} < \eta, \qquad \eta > 0$$

$$\left|e_{max}\right|_{r=r} < \eta \left(\frac{a}{r}\right)^{2}$$

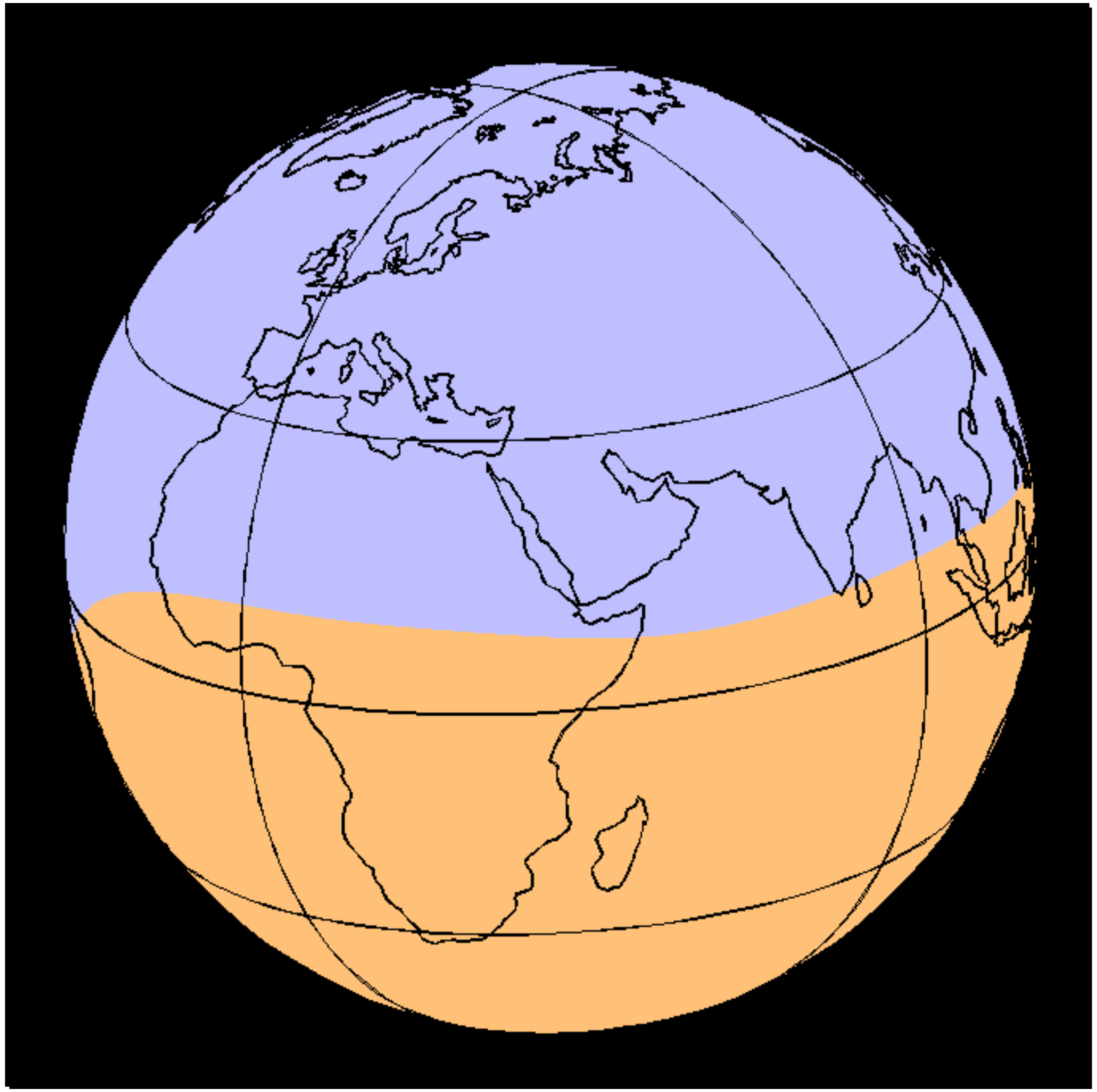
Upward

$$\left|e_{max}\right|_{r=a} = \eta, \qquad r < a, \eta > 0$$

$$\left|e_{max}\right|_{r=r} = \eta \left(\frac{a}{r}\right)^{l+1}$$

Downward

Downward continuation to the core



movie (credit: C. Finlay)

The IGRF is the International Geomagnetic Reference Field

- In geomagnetism, basis functions (spherical harmonics) are normalized so that:

$$\int_{S(1)} \left(Y_l^m\right)^2 d^2\hat{\boldsymbol{r}} = \frac{4\pi}{2l+1}$$

$$\Psi(r,\theta,\phi) = a\sum_{l=1}^{\infty}\left(\frac{a}{r}\right)^{l+1}\sum_{m=0}^{l} N_{lm}\left(g_l^m\cos(m\phi) + h_l^m\sin(m\phi)\right)P_l^m(\cos\theta)$$

- The normalization constant is: ***Schmidt normalization***

$$N_{lm} = 1, \qquad m = 0$$

$$N_{lm} = \sqrt{\frac{(l-m)!}{(l+m)!}}, \qquad m > 0$$

Geomagnetic Conventions for Spherical Harmonics

In geomagnetism most researchers eschew the complex spherical harmonic representation for the field, replacing $e^{im\phi}$ with real sines and cosines and adopt Schmidt partial normalization for the spherical harmonics. The basis functions are renormalized so that with Y_l^m as we defined them earlier

$$\int_{S(1)} |Y_l^m|^2 d^2\hat{\mathbf{r}} = \frac{4\pi}{(2l+1)}$$

then

$$N_{lm} = 1, \quad m = 0 \qquad\qquad N_{lm} = \sqrt{\frac{(l-m)!}{(l+m)!}}, \quad m > 0 \quad .$$

In Schmidt quasi-normalized form the geomagnetic potential with both internal and external sources is written

$$\Psi(r,\theta,\phi) = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left[\left(\frac{a}{r}\right)^{l+1} \left(g_l^m \cos(m\phi) + h_l^m \sin(m\phi)\right) \tilde{P_l^m}(\theta) + \left(\frac{r}{a}\right)^{l} \left(q_l^m \cos(m\phi) + s_l^m \sin(m\phi)\right) \tilde{P_l^m}(\theta) \right]$$

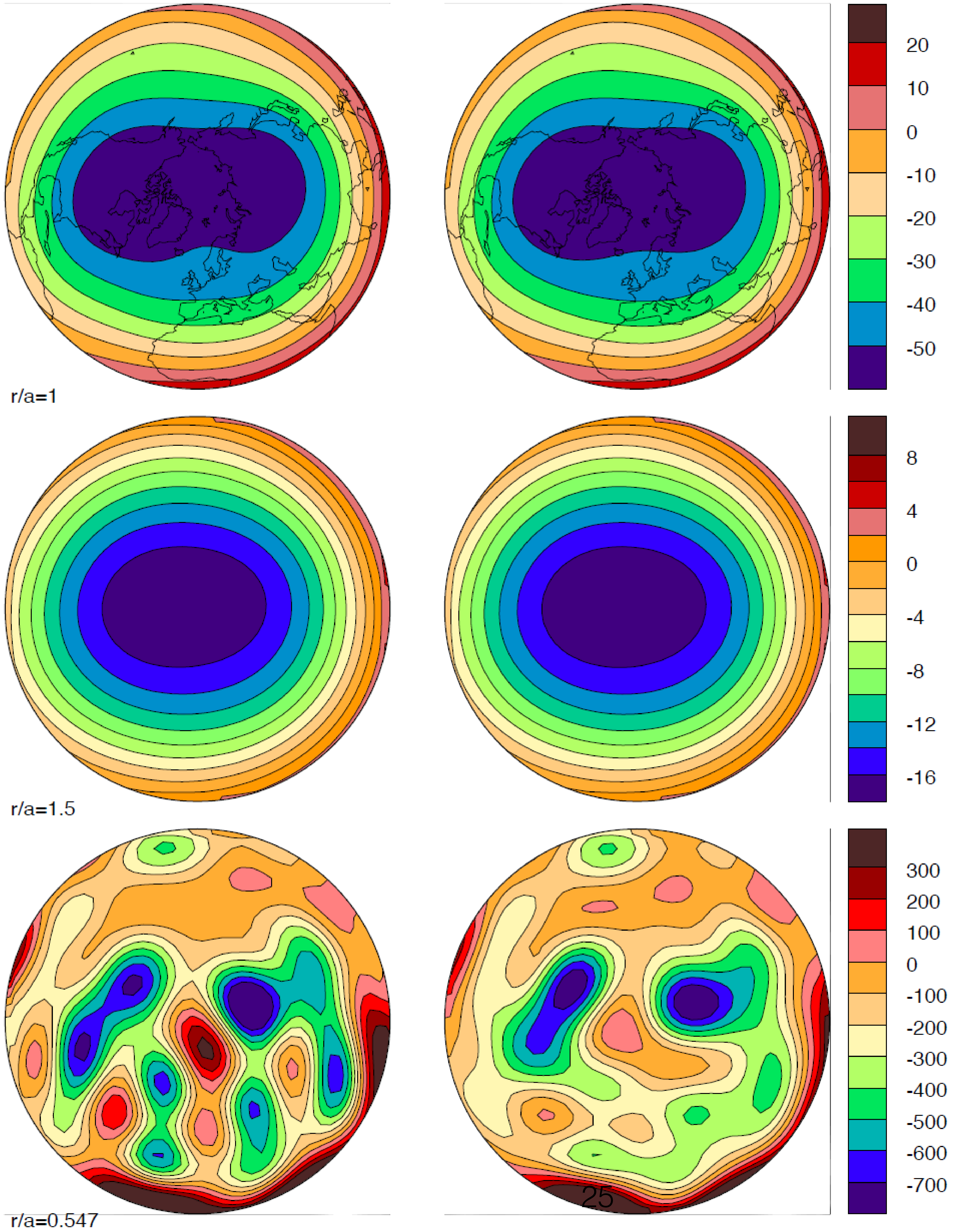
Internal

External

Here the $\tilde{P_l^m}(\theta)$ has implicitly absorbed the normalization coefficient N_l^m. Going forward we will assume Schmidt normalization with the notation P_l^m.

Radial component of the IGRF1980 magnetic field

with added noise without noise



Earth Surface, r=a

r=1.5a

CMB, r=0.547a

Figure 3.3 notes

Gauss Coefficients for IGRF-2000

l	m	g_l^m	h_l^m	l	m	g_l^m	h_l^m	l	m	g_l^m	h_l^m
1	0	-29615	0	6	2	74	64	9	0	5	0
1	1	-1728	5186	6	3	-161	65	9	1	9	-20
2	0	-2267	0	6	4	-5	-61	9	2	3	13
2	1	3072	-2478	6	5	17	1	9	3	-8	12
2	2	1672	-458	6	6	-91	44	9	4	6	-6
3	0	1341	0	7	0	79	0	9	5	-9	-8
3	1	-2290	-227	7	1	-74	-65	9	6	-2	9
3	2	1253	296	7	2	0	-24	9	7	9	4
3	3	715	-492	7	3	33	6	9	8	-4	-8
4	0	935	0	7	4	9	24	9	9	-8	5
4	1	787	272	7	5	7	15	10	0	-2	0
4	2	251	-232	7	6	8	-25	10	1	-6	1
4	3	-405	119	7	7	-2	-6	10	2	2	0
4	4	110	-304	8	0	25	0	10	3	-3	4
5	0	-217	0	8	1	6	12	10	4	0	5
5	1	351	44	8	2	-9	-22	10	5	4	-6
5	2	222	172	8	3	-8	8	10	6	1	-1
5	3	-131	-134	8	4	-17	-21	10	7	2	-3
5	4	-169	-40	8	5	9	15	10	8	4	0
5	5	-12	107	8	6	7	9	10	9	0	-2
6	0	72	0	8	7	-8	-16	10	10	-1	-8
6	1	68	-17	8	8	-7	-3				

These are in nT.

The Geomagnetic Spectrum

$$R_l = \frac{(2l+1)(l+1)}{4\pi} \sum_{m=-l}^{l} |b_l^m|^2 = (l+1) \sum_{m=0}^{l} \left[(g_l^m)^2 + (h_l^m)^2 \right]$$

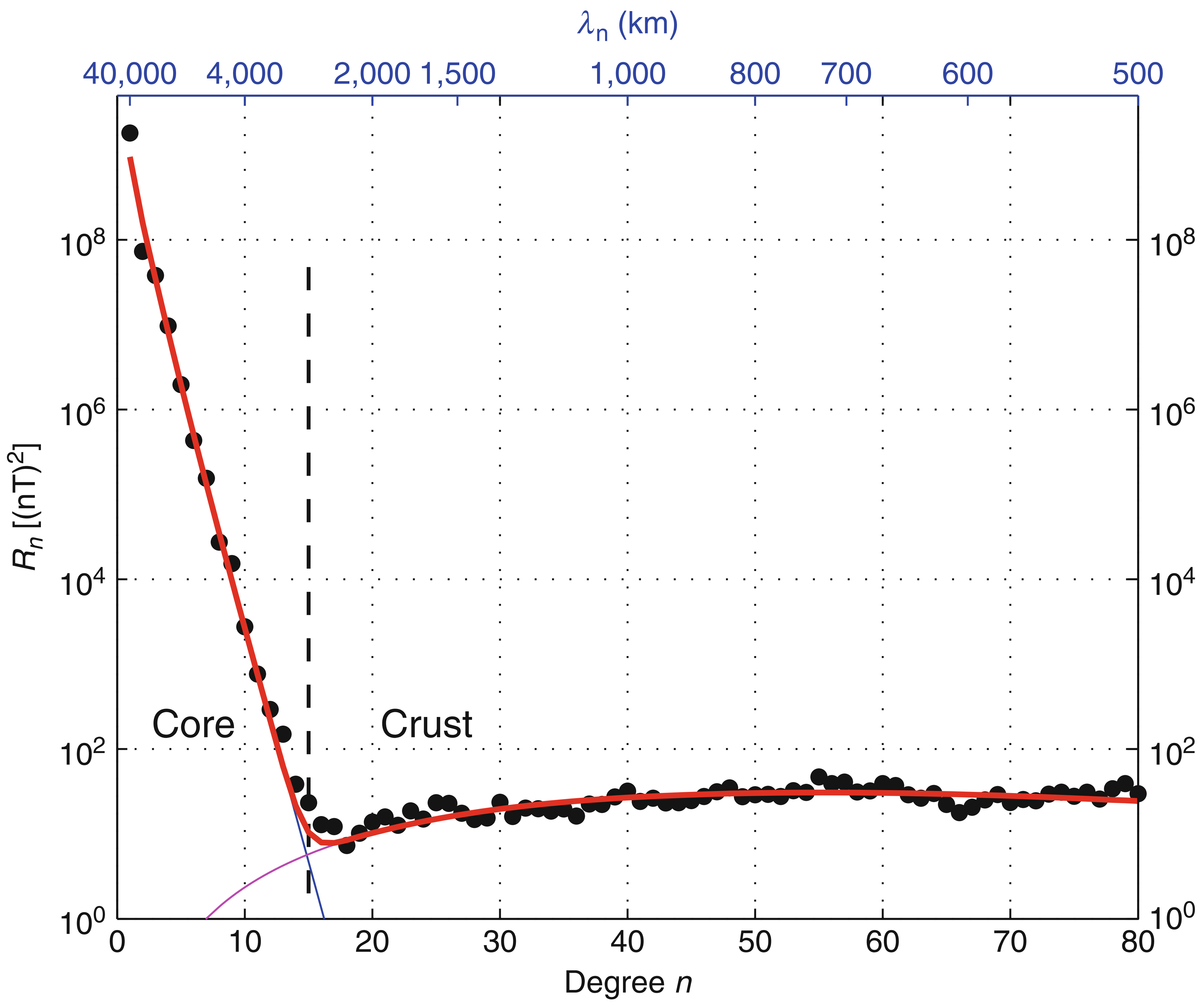


Fig. 3.4.6.1

Spatial power spectrum of the geomagnetic field at the Earth's surface. Black dots represent the spectrum of a recent field model (Olsen et al. 2009; Maus et al. 2008). Also shown are theoretical spectra (Voorhies et al., 2002) for the core (*blue*) and crustal (*magenta*) part of the field, as well as their superposition (*red curve*)

R_l is a spatial power spectrum and lets us divide magnetic field according to its spatial wavelength. Recall SH properties 8 & 10.

8.	Wavelength of Y_l^m	$\frac{2\pi}{l+½}$	Depends only on degree l, not on order m or $\hat{\mathbf{s}}$
10.	Parseval's Theorem	$\int d^2\hat{\mathbf{s}}\ \lvert f(\hat{\mathbf{s}})\rvert^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \lvert c_{lm}\rvert^2$	Requires property 4. Get RMS value of f by dividing by 4π and taking square root

What happens to the spectrum when the field is downward continued to the CMB?

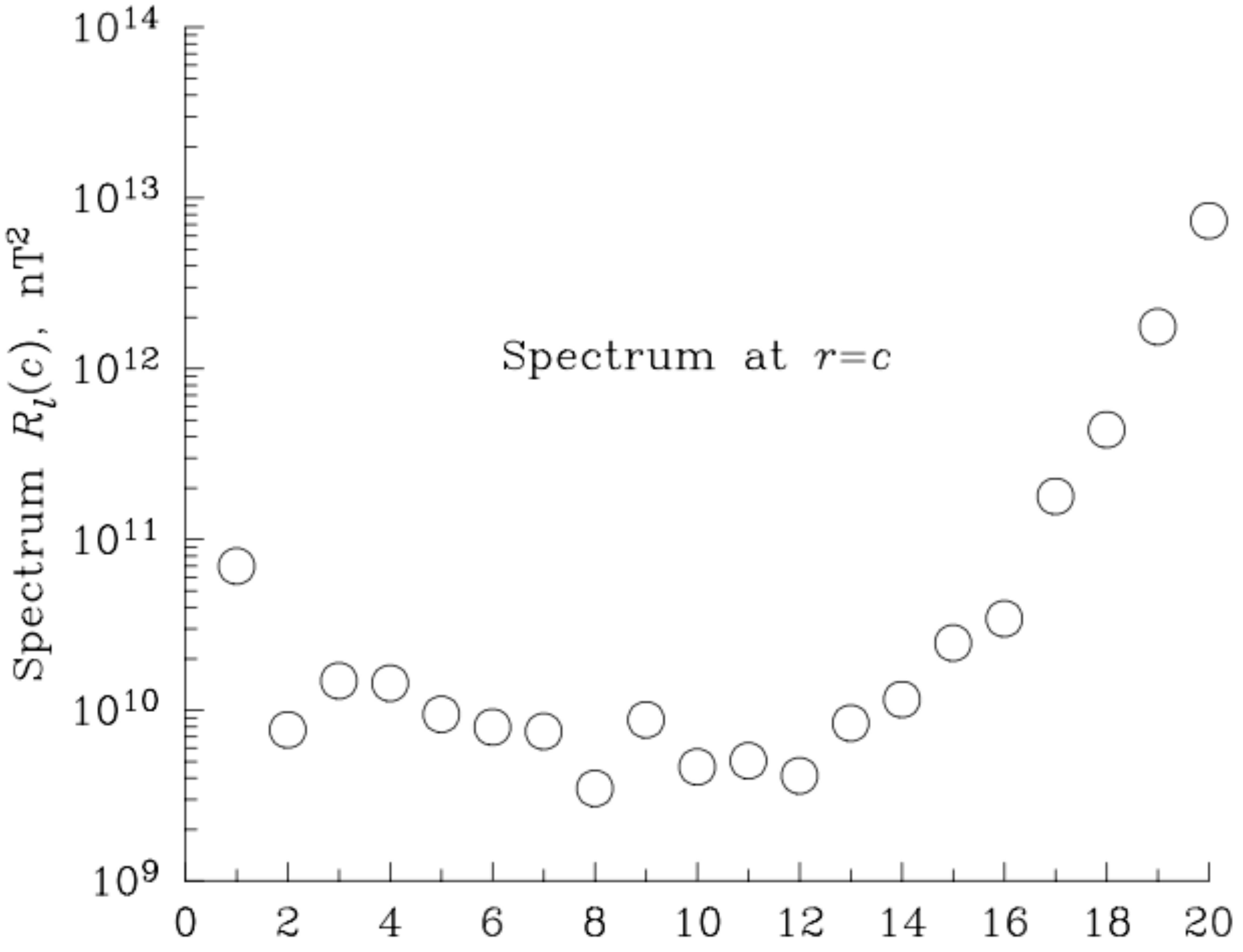


Figure 3.6.4.2 Lowes spectrum evaluated at the core-mantle boundary.