# SIOG 231
# GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 9a

Geomagnetic Depth Sounding

2/6/2024

Recall the MT method, in which measurements of the electric and magnetic fields can be used to estimate a frequency dependent apparent resistivity which tells us about the electrical conductivity beneath the measuring site:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

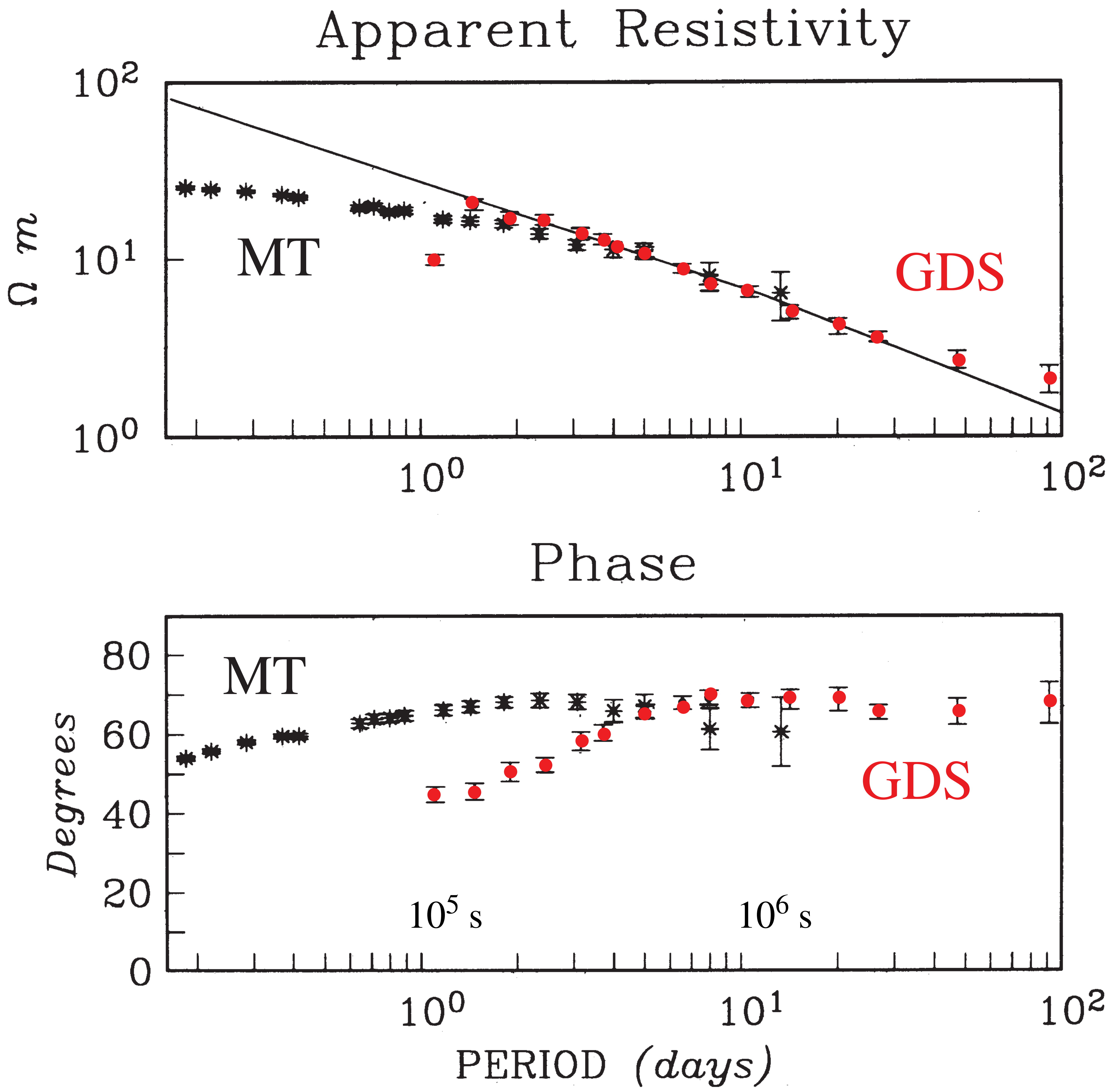
$$\rho_a = \frac{1}{\omega\mu_o}\left|\frac{E}{H}\right|^2 \qquad \phi = \tan^{-1}\left(\frac{E}{H}\right)$$

The MT method is not the ideal tool for looking deep inside Earth. If we write the MT equations as a function of period $T$, we see that the electric field falls off at the long periods we need to probe the deep Earth.

$$\rho_a = \frac{T\mu}{2\pi}\left|\frac{E}{B}\right|^2 \qquad E = B\sqrt{\frac{2\pi\rho_a}{T\mu}}$$

Loss of signal is exacerbated by the increase in conductivity with depth, which decreases resistivity, and lateral conductivity variations can distort $E$ and introduce uncertainty in resistivity.

The solution is the Geomagnetic Depth Sounding (GDS) method, which uses only the magnetic fields, and allows magnetic observatory data to generate MT-like responses out to a hundred day periods or more.




*Egbert and Booker (1992)*

We take the standard approach of assuming no magnetic sources in the atmosphere and writing the magnetic field is minus the gradient of a scalar potential:

$$\mathbf{B} = -\nabla\Phi(r, \theta, \phi)$$

We express the potential as spherical harmonics in associated Legendre polynomials with Schmidt normalized coefficients

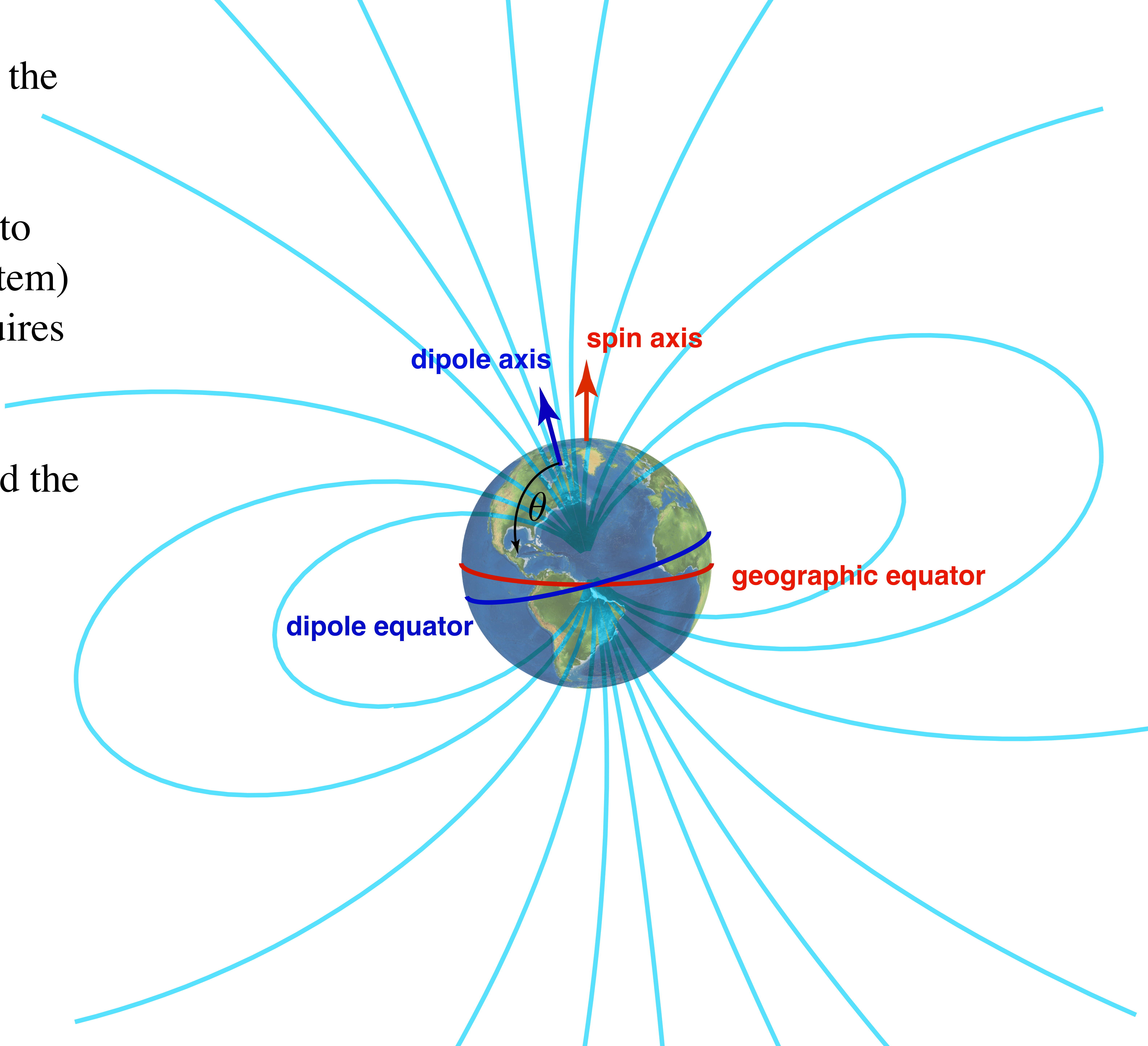
$$\Phi(r, \theta, \phi) \;=\; a_o \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left\{ i_l^m \left(\frac{a_o}{r}\right)^{l+1} \;+\; e_l^m \left(\frac{r}{a_o}\right)^{l} \right\} P_l^m(\cos\theta) e^{im\phi} \quad .$$

We changed our coordinate system to **geomagnetic coordinates**, defined by the best-fitting dipole, so now theta is geomagnetic colatitude, but $r$ and $a_o$ stay the same. Note that the Gauss coefficients are implicitly complex.

Geomagnetic coordinates are defined by the best-fitting magnetic dipole.

Because the dipole is tilted with respect to the spin axis (geographic coordinate system) describing it in spherical harmonics requires $g^1_1$ and $h^1_1$ terms along with $g^0_1$

In geomagnetic coordinates we only need the $g^0_1$ term.

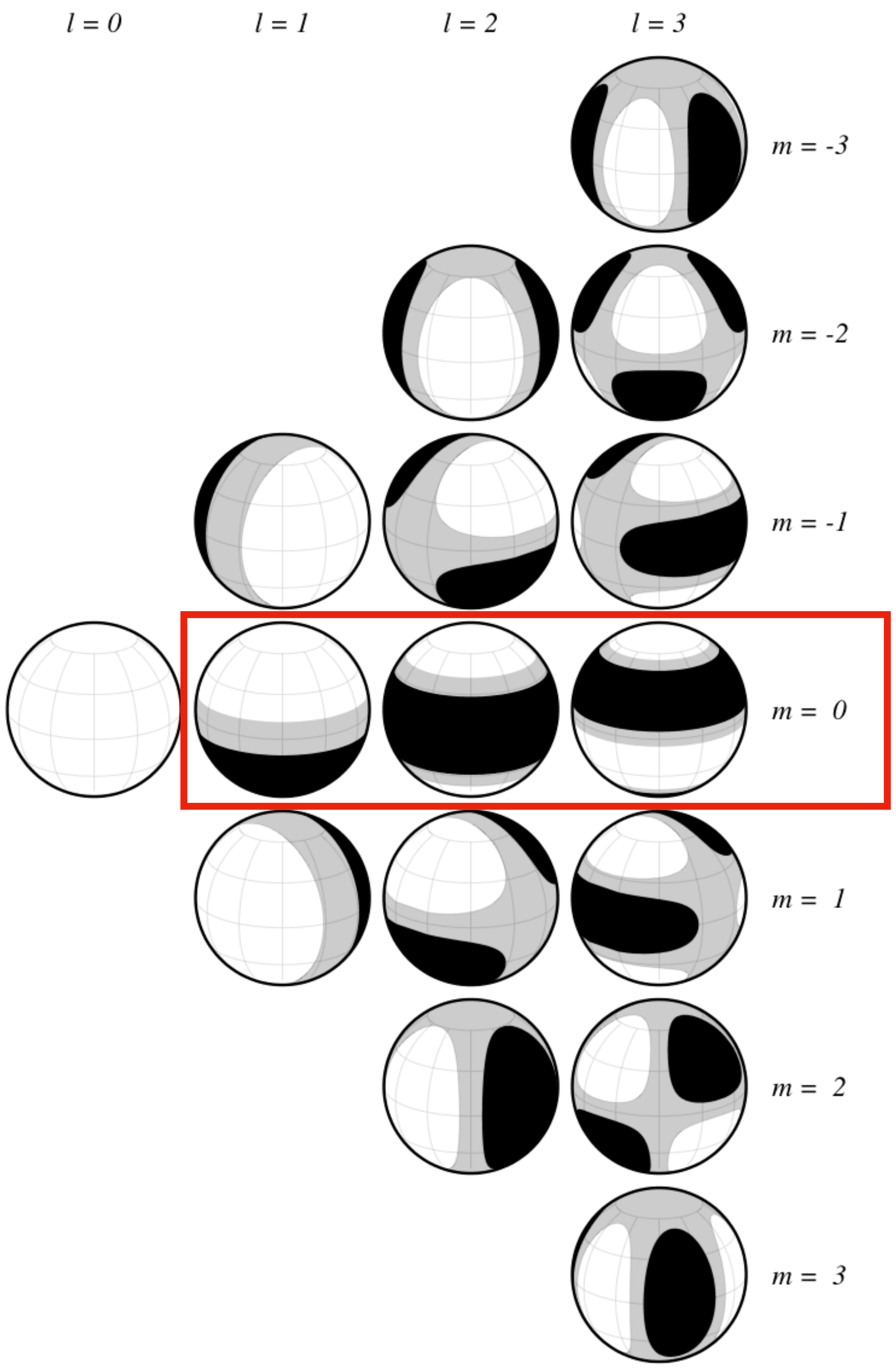


Let's assume that in geomagnetic coordinates, the non-axisymmetric terms can be neglected.

$$\Phi = a_o \sum_{l=1}^{\infty} \left\{ i_l \left( \frac{a_o}{r} \right)^{l+1} + e_l \left( \frac{r}{a_o} \right)^{l} \right\} P_l^0(cos\theta) \quad .$$

By making m = 0 the exponential term goes away and the coefficients become real. The coefficients can be static, but for EM induction we can consider variations with either time or frequency.

For GDS using magnetic observatory data, we convert time series measurements to frequency, putting the coefficients back into the complex plane.



Similar to MT, we can define an electromagnetic response as a ratio, here internal to external fields as a function of frequency.

$$Q_l(\omega) = \frac{i_l(\omega)}{e_l(\omega)}$$

With enough data, we could fit the coefficients, but there are reasons to use a single observatory:

- not all observatories produce equal quality data, and the distribution is uneven
- the time the observatories have been operating varies
- even though the theory assumes radial symmetry, variations in conductivity can be estimated

Since we have ignored the non-zonal components, the horizontal field $H$ always points towards magnetic north or south, the vertical field $Z$ remains the same, and for an observatory at Earth's surface $r = a_o$. This allows us to take the appropriate derivatives to recover $H$ and $Z$.

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)$$

$$H = \left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)_{r=a_o} = \left(\frac{a_o}{r}\sum_l^{\infty}\left\{i_l\left(\frac{a_o}{r}\right)^{l+1} + e_l\left(\frac{r}{a_o}\right)^{l}\right\}\frac{\partial P_l^0(\cos\theta)}{\partial\theta}\right)_{r=a_o}$$

$$= \sum_l A_{H,l}\frac{\partial P_l^0(\cos\theta)}{\partial\theta}$$

$$Z = \left(\frac{\partial\Phi}{\partial r}\right)_{r=a_o} = \left(a_o\sum_l^{\infty}\{-i_l a_o^{l+1}(l+1)r^{-l-2} + e_l a_o^{-l}(l)r^{l-1}\}P_l^0(cos\theta)\right)_{r=a_o}$$

$$= \sum_l A_{Z,l}P_l^0(\cos\theta)$$

(Note that H and Z are in the opposite directions to $B_\theta$ and $B_r$ .) New expansion coefficients:

$$A_{H,l} = i_l + e_l \qquad A_{Z,l} = le_l - (l+1)i_l$$

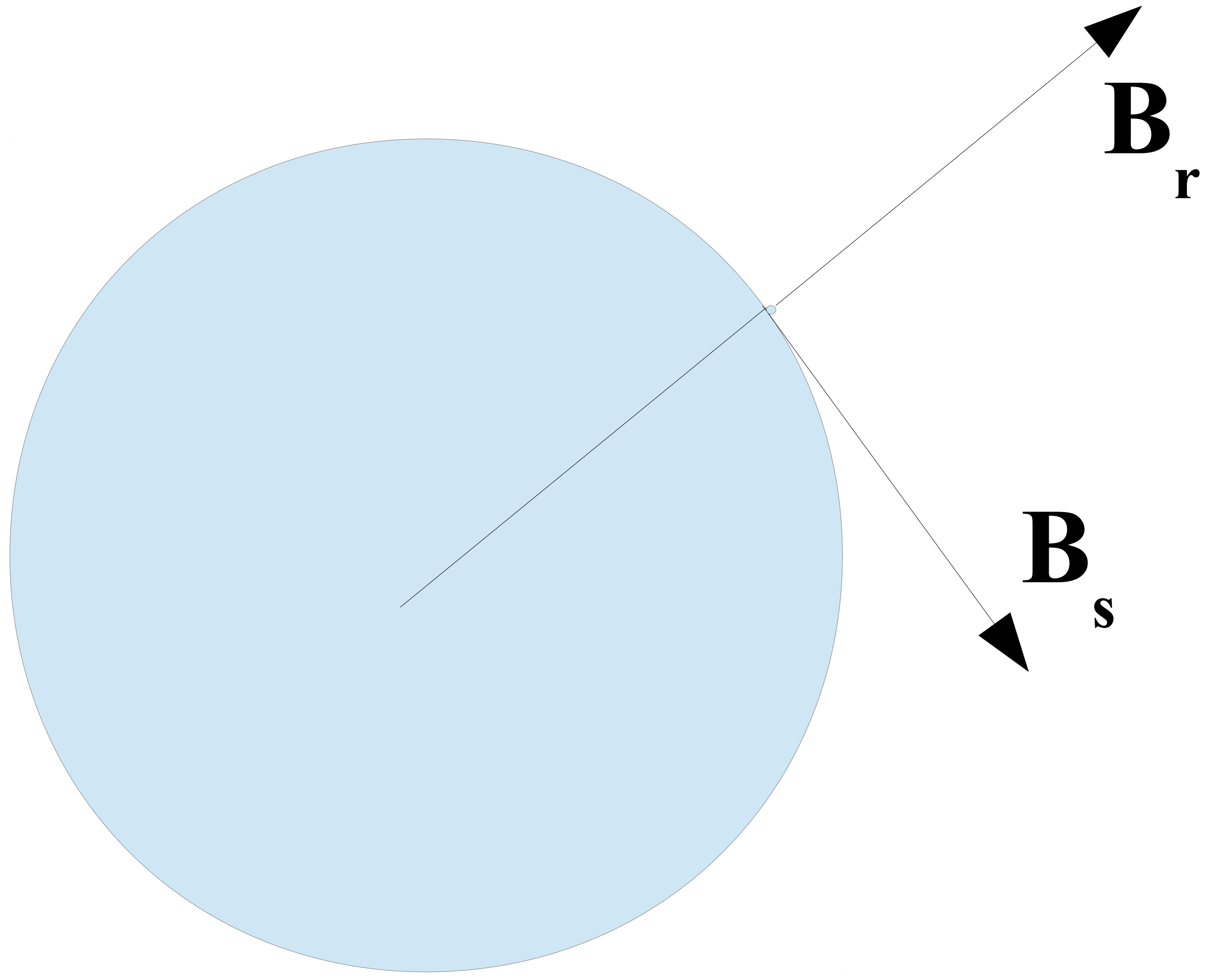
$$A_{H,l} = i_l + e_l \qquad\qquad A_{Z,l} = l e_l - (l+1) i_l$$

This is just the same as the internal-external separation shown in Lecture 5:

$$\boldsymbol{B} = \hat{\boldsymbol{r}} B_r + \boldsymbol{B}_s \qquad\qquad \nabla = \hat{\boldsymbol{r}} \partial_r + \frac{1}{r} \nabla_1$$

$$B_r = -\partial_r \Psi|_{r=a} = -\sum_{l,m} \left[ l A_l^m - (l+1) B_l^m \right] Y_l^m(\theta, \phi)$$

$$\boldsymbol{B}_s = -\frac{1}{r} \nabla_1 \Psi = -\sum_{l,m} \left[ A_l^m + B_l^m \right] \nabla_1 Y_l^m$$




- We can always recover the internal and external coefficients separately from our knowledge of ***B*** on $r = a$, combining the following two equations:

$$l A_l^m - (l+1) B_l^m = -\int B_r \left( Y_l^m \right)^* d^2 \hat{\boldsymbol{r}}$$

$$A_{l'}^{m'} + B_{l'}^{m'} = -\frac{\int \boldsymbol{B}_s \cdot (\nabla_1 Y_{l'}^{m'})^* d^2 \hat{\boldsymbol{r}}}{(l'(l'+1))}$$

Two equations in two unknowns:

$$A_{H,l} = i_l + e_l \qquad\qquad A_{Z,l} = le_l - (l+1)i_l$$

Solve:

$$i_l = \frac{lA_{H,l} - A_{Z,l}}{2l+1} \qquad\qquad e_l = \frac{A_{Z,l} + (l+1)A_{H,l}}{2l+1}$$

Now our response function is

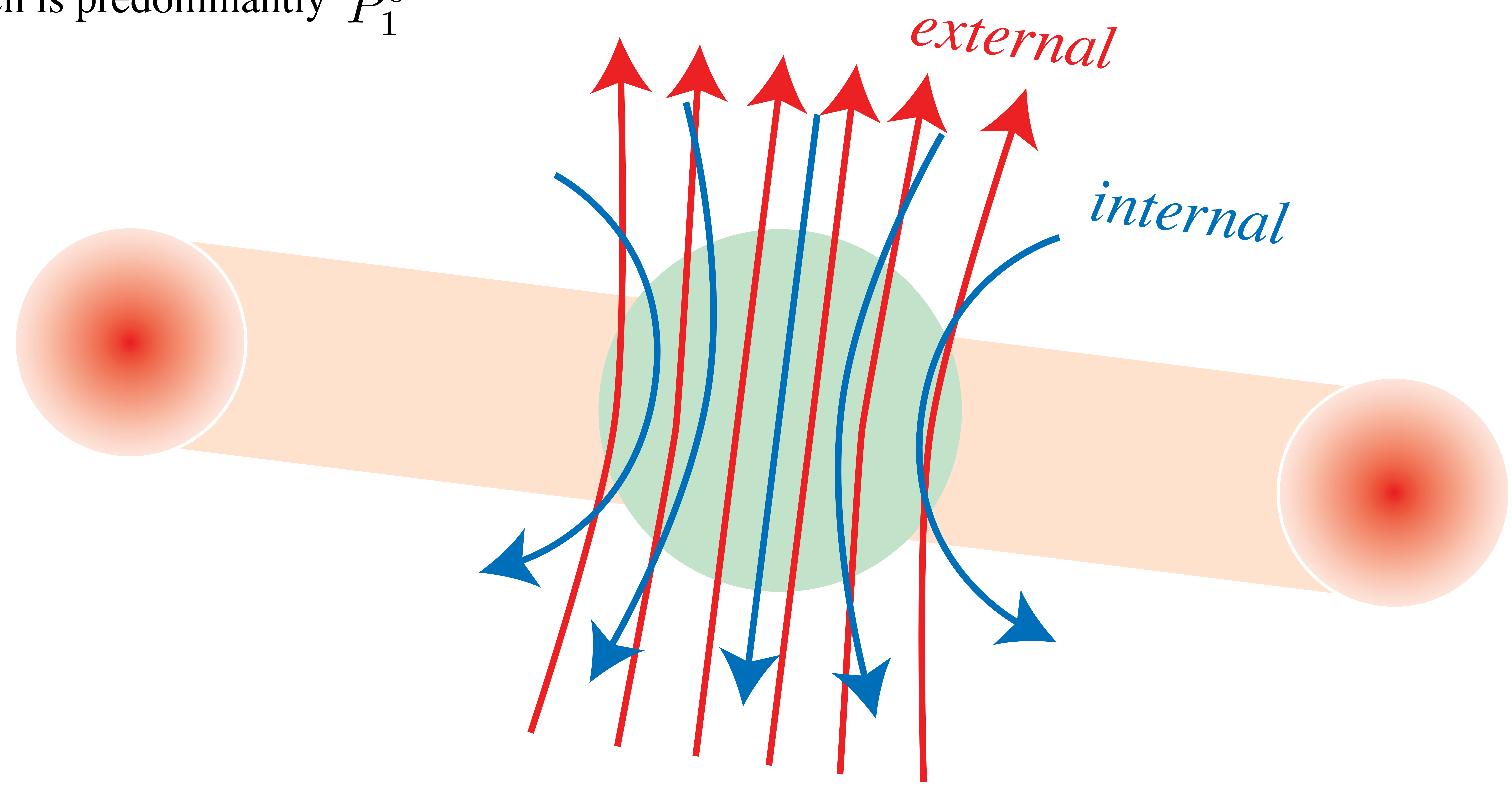
$$Q_l = \frac{i_l}{e_l} = \frac{l - A_{Z,l}/A_{H,l}}{l+1+A_{Z,l}/A_{H,l}} = \frac{l - W_l}{l+1+W_l}$$

introducing yet another response $$W_l = \frac{A_{Z,l}}{A_{H,l}} \quad .$$

The inductive scale length, a measure of conductivity and depth, has units of length and is given by

$$c_l = \frac{a_o W_l}{l(l+1)}$$ (Remember: all this is a function of frequency)

So far we have allowed all zonal terms $l$, but most of the signal is coming from the ring current which is predominantly $P_1^0$




Induced fields are $P_1^0$ too.

For the simplified $P_1^0$ geometry W just becomes the ratio of vertical to horizontal field with trigonometric terms:

$$W = \frac{A_{Z,1}}{A_{H,1}} = \frac{Z/P_1^0(\cos\theta)}{H/\frac{\partial}{\partial\theta}P_1^0(\cos\theta)}$$

$$Z = \sum_l A_{Z,l} P_l^0(\cos\theta)$$

$$H = \sum_l A_{H,l} \frac{\partial P_l^0(\cos\theta)}{\partial\theta}$$

Recalling that $P_1^0(\cos\theta) = \cos\theta$

$$W = \frac{Z\sin\theta}{H\cos\theta} = \frac{Z(\omega)}{H(\omega)}\tan\theta$$

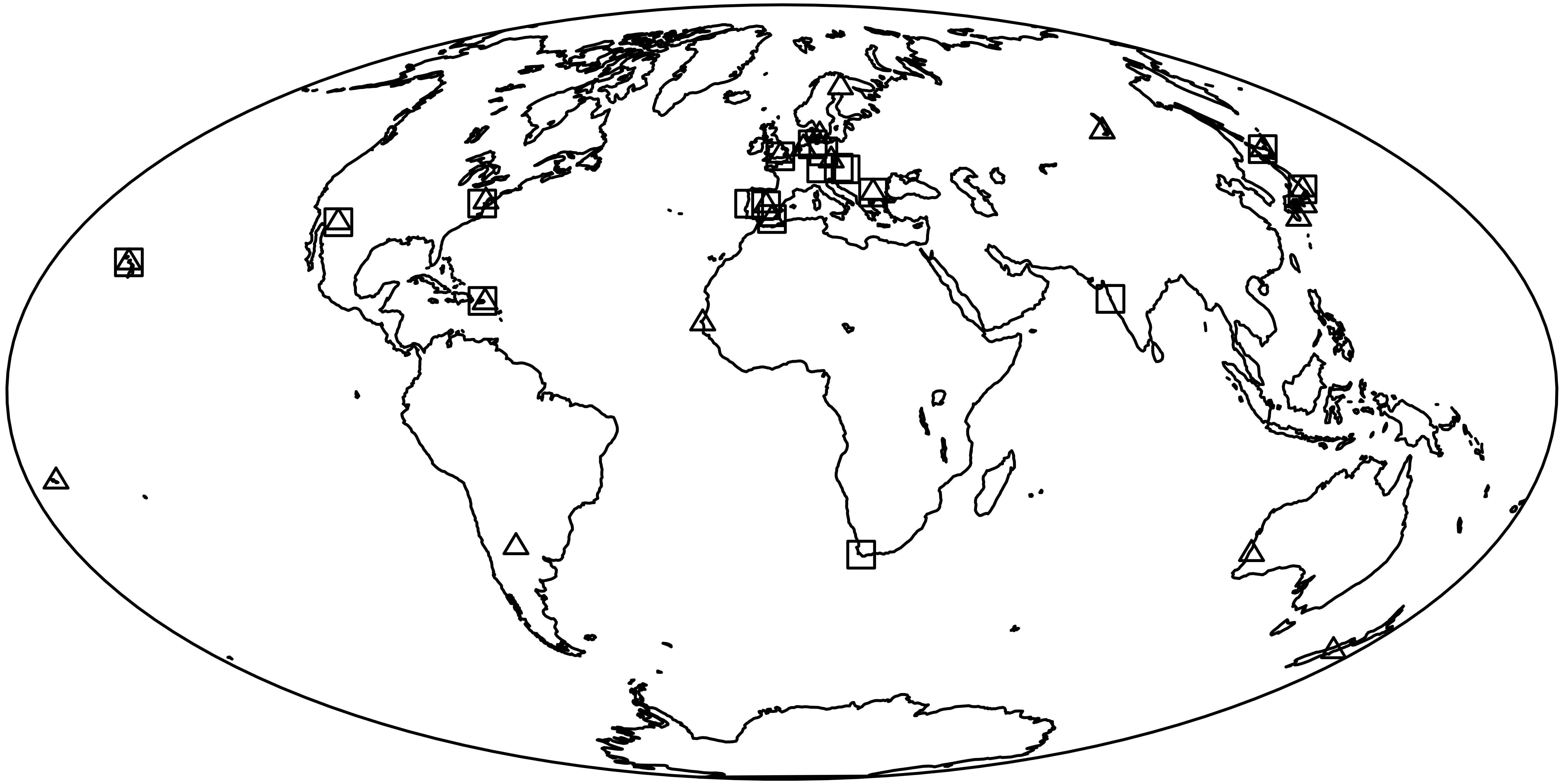
(Blows up at theta = [0, 90, 180])

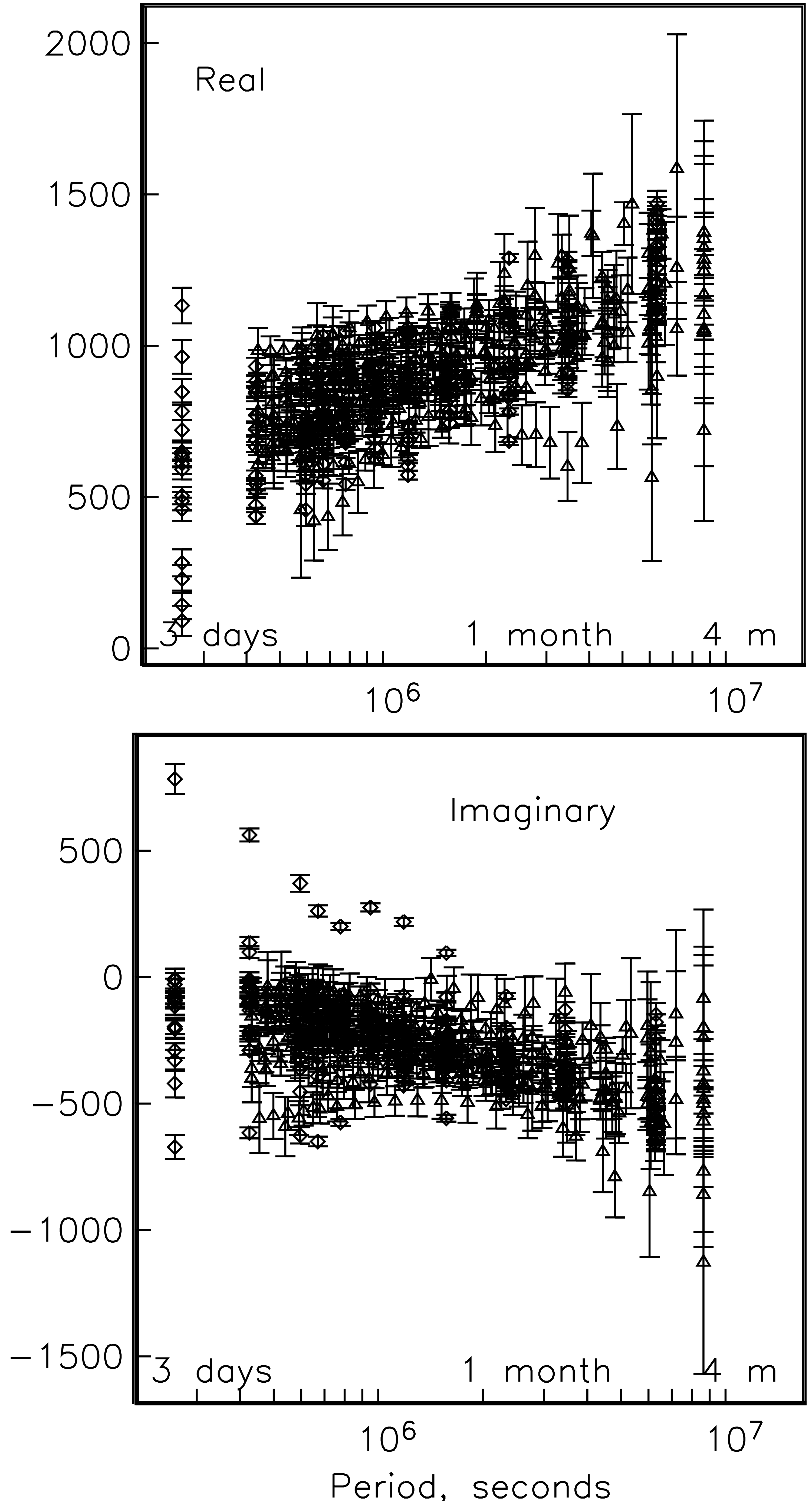
$$c = a_o W/2 \qquad \rho_a = \omega\mu_o|c|^2 \qquad \phi = \arg(c) \quad .$$

A workflow:

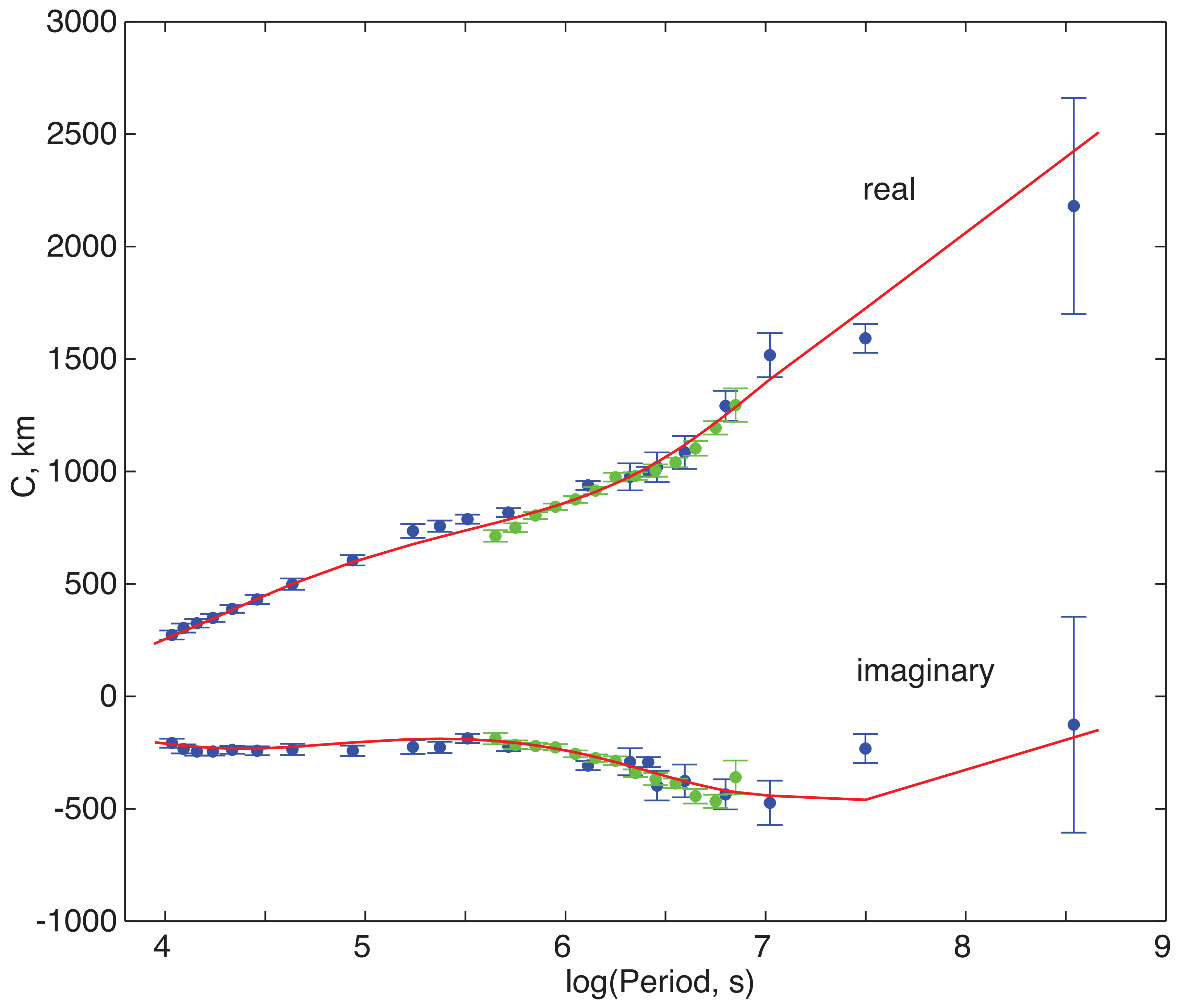
- Take horizontal and vertical magnetic field records from an observatory
- Fourier transform them to frequency
- Take the ratio at each frequency and scale by tangent of magnetic colatitude
- Scale by $a_o/2$

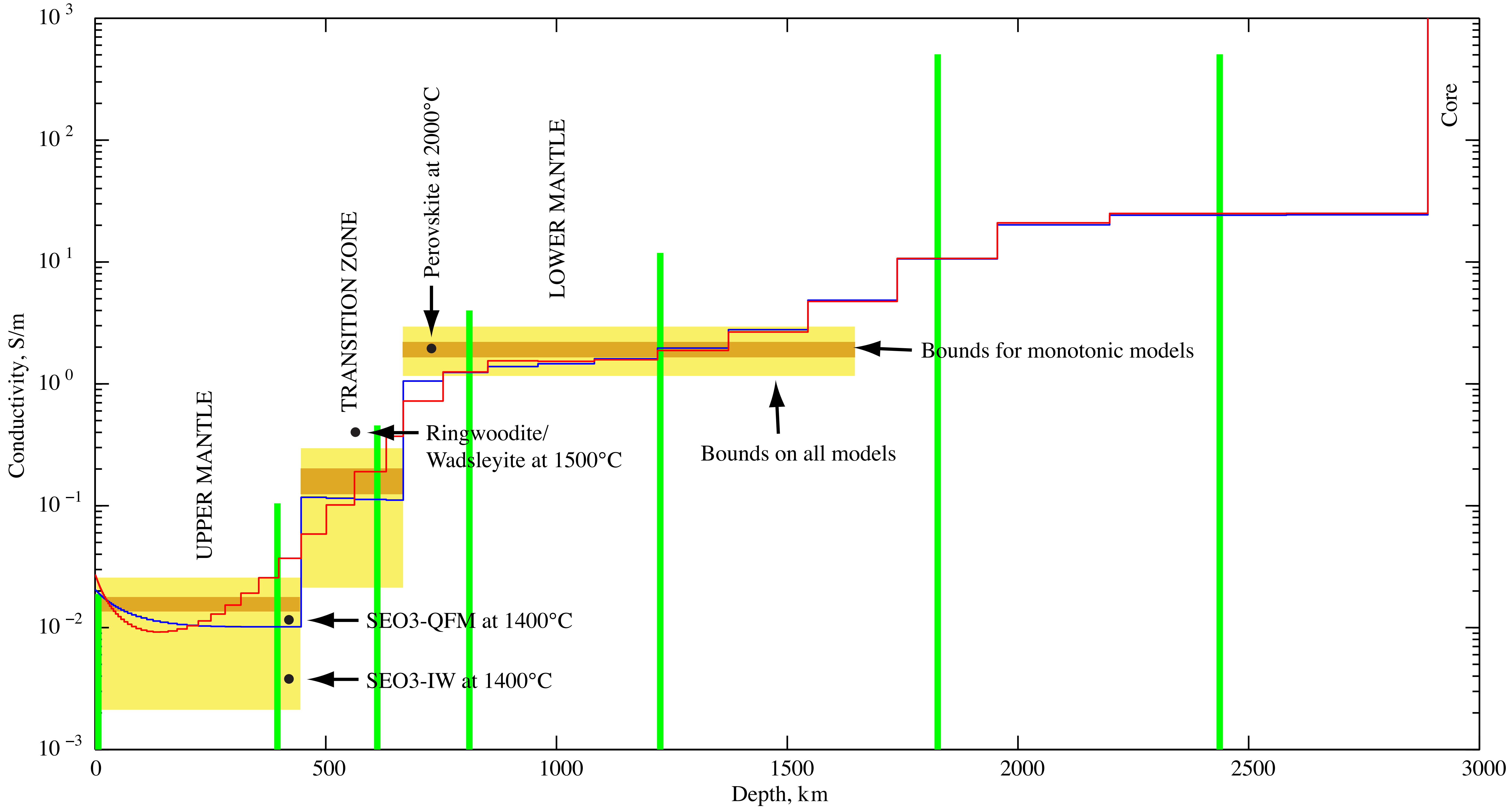
Some observatory GDS responses:



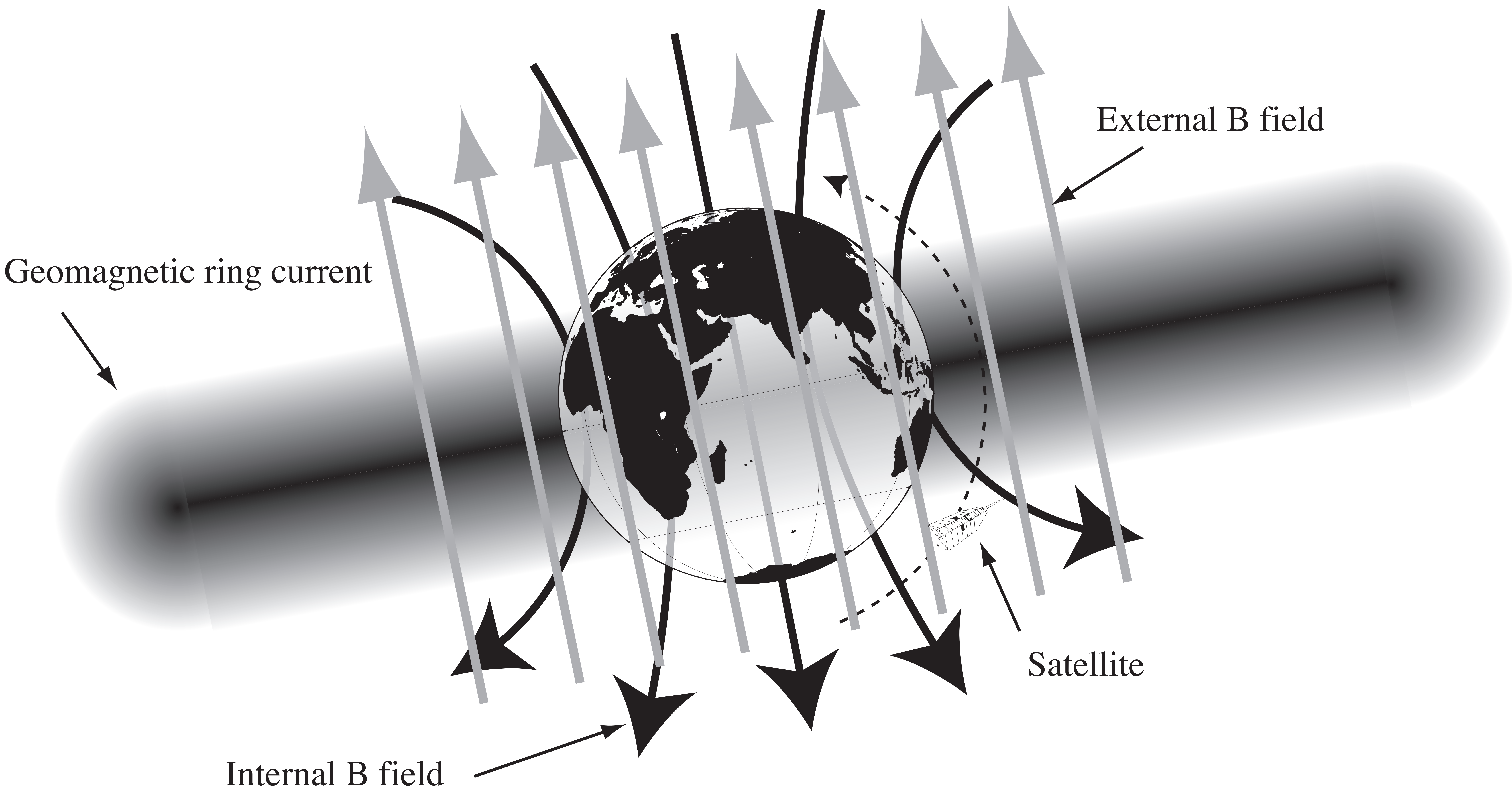


# Probing the deep Earth:

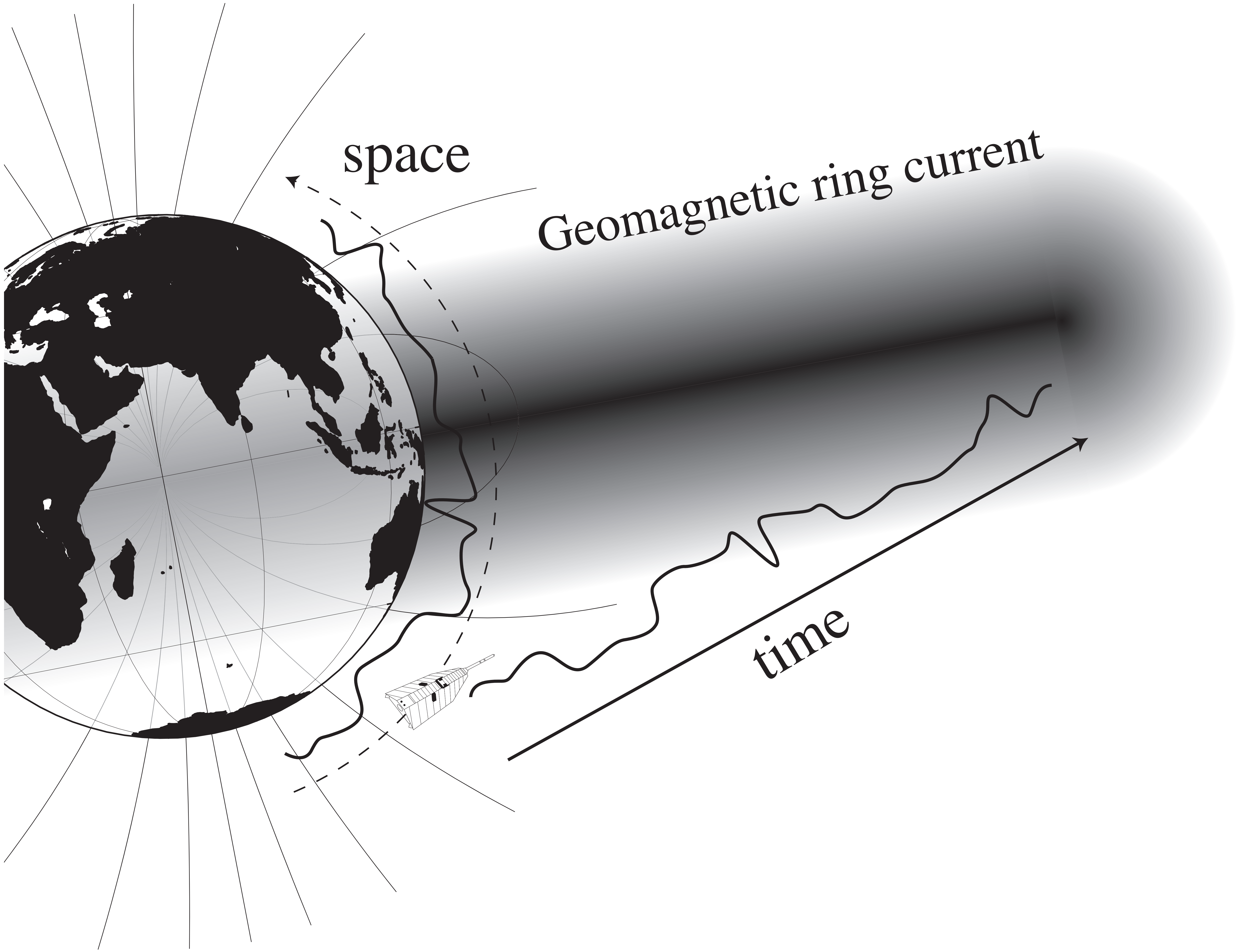


Using magnetic satellites:



Need to separate time and space

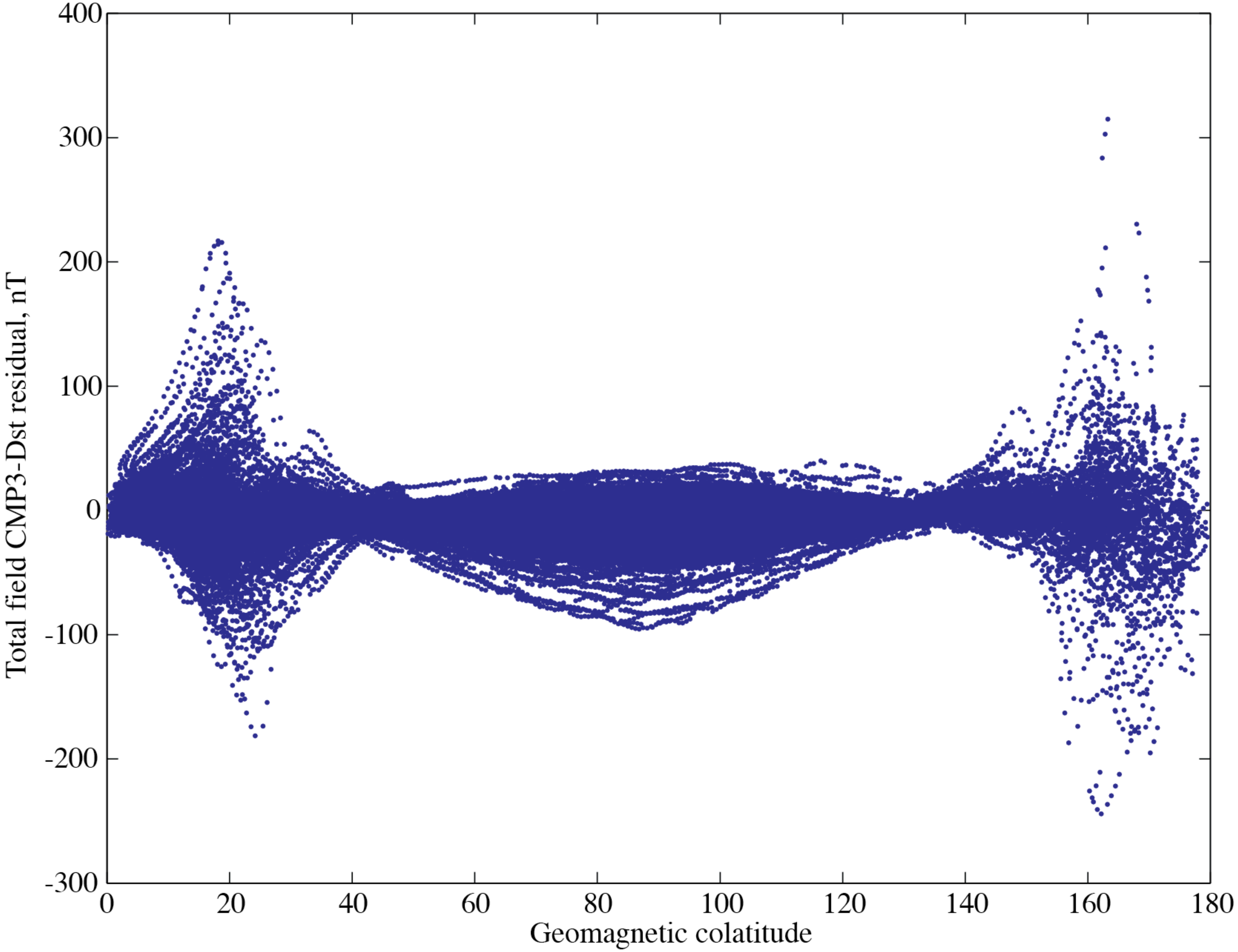


The Comprehensive Field Model allows one to estimate

- The main (core) field, and its secular variation.
- The crustal field due to remanent and induced magnetization.
- Ionospheric currents (daily variation)
- Field aligned and meridional currents, and seasonal variations.
- Equatorial electrojet.
- Coupling and induction of the above.

as a function of time and space. If they are then subtracted from the satellite data, we should just be left with the $P_1^0$ ring current signal.

Should… In practice it doesn't work near the poles.



As before, we keep only the $P_1^0$ term but now the Gauss coefficients are functions of time $t$

$$\Phi_1^0(r,\theta,\phi) \;=\; a_o\left\{i_1^0(t)\left(\frac{a_o}{r}\right)^2 \;+\; e_1^0(t)\left(\frac{r}{a_o}\right)\right\}P_1^0(\cos\theta)$$

The field components are recovered in the usual way by differentiation

$$\mathbf{B}(r,\theta,\phi) = -\nabla\Phi_1^0(r,\theta,\phi) \qquad\qquad \nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)$$

yielding

$$B_r = -\frac{\partial\Phi}{\partial r} = \left[-e_1^0 + 2i_1^0\left(\frac{a}{r}\right)^3\right]\cos(\theta)$$

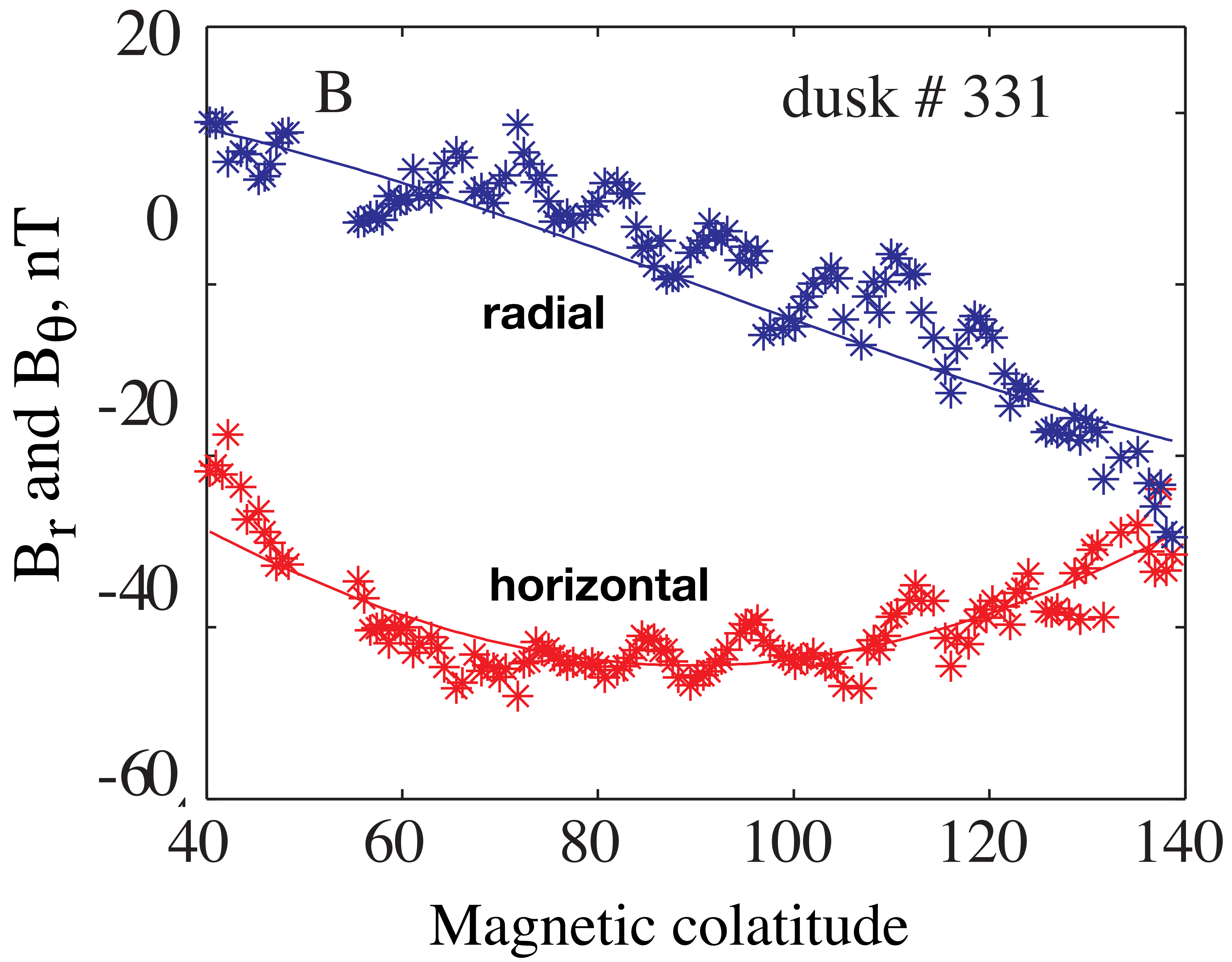
$$B_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \left[e_1^0 + i_1^0\left(\frac{a}{r}\right)^3\right]\sin(\theta)$$

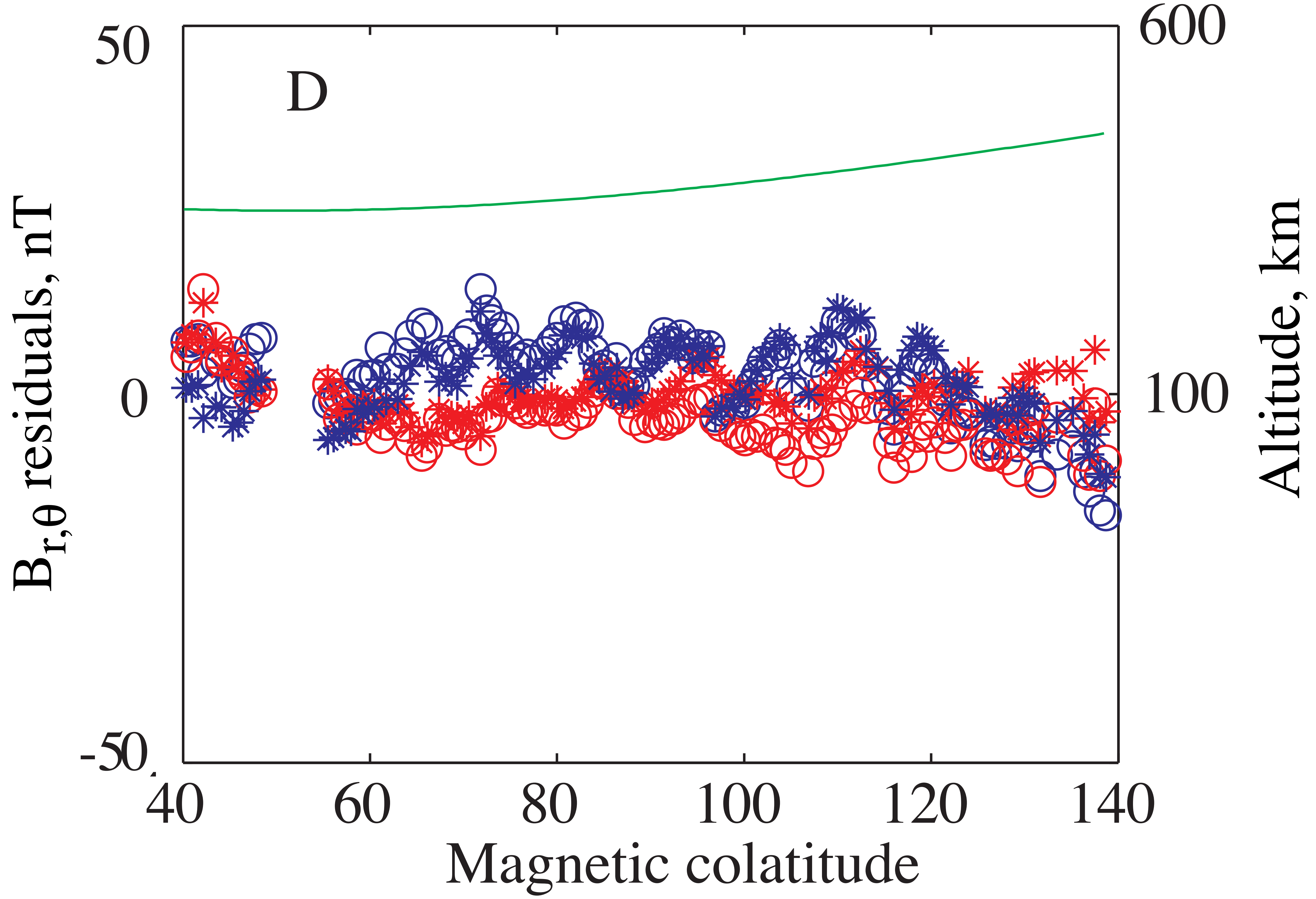
$$B_\phi = 0 \quad .$$

$$B_r = -\frac{\partial \Phi}{\partial r} = \left[-e_1^0 + 2i_1^0\left(\frac{a}{r}\right)^3\right]\cos(\theta)$$

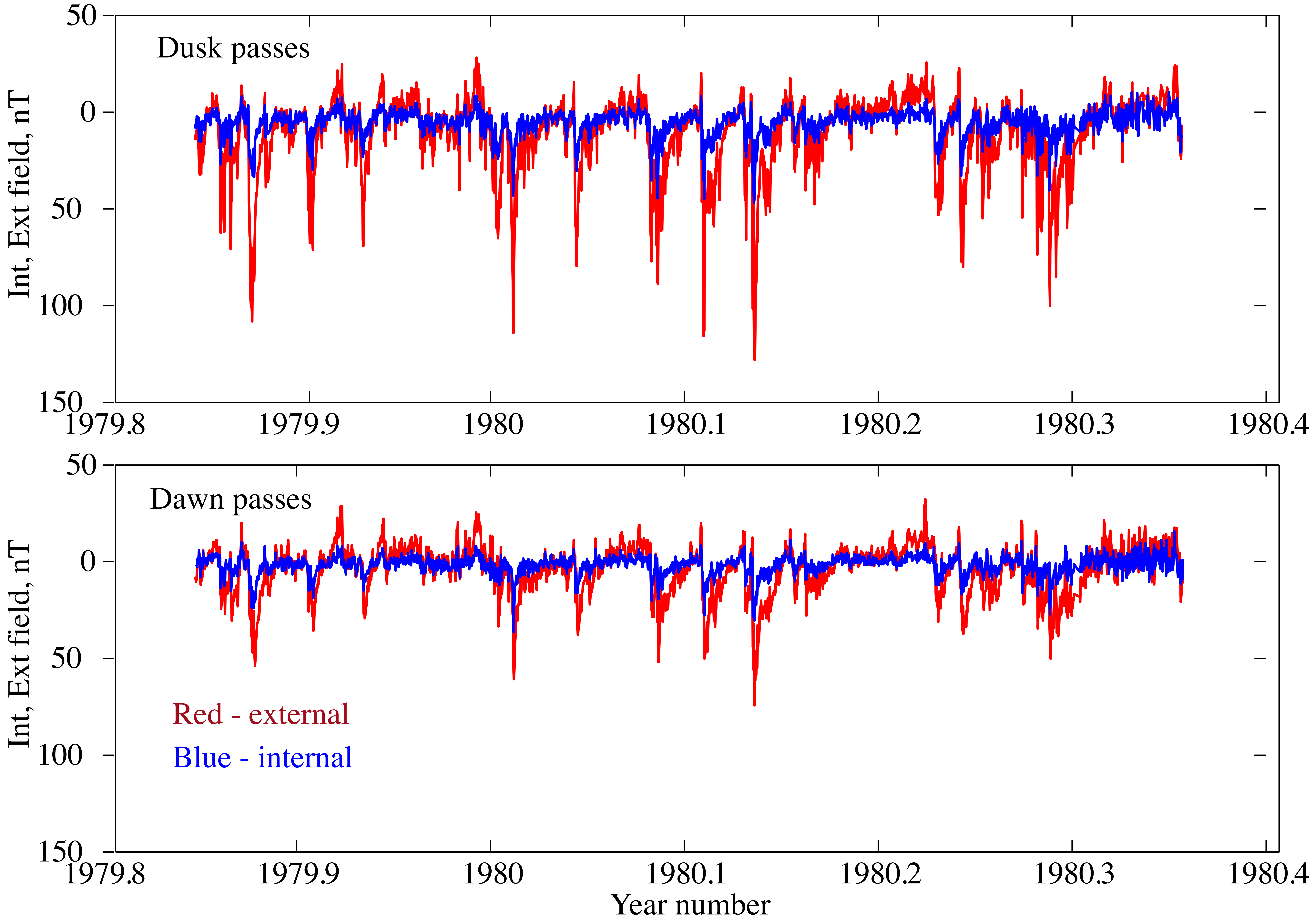
$$B_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \left[e_1^0 + i_1^0\left(\frac{a}{r}\right)^3\right]\sin(\theta)$$

$$\longrightarrow \begin{bmatrix} -\cos(\theta) & 2(a/r)^3\cos(\theta) \\ \sin(\theta) & (a/r)^3\sin(\theta) \end{bmatrix} \begin{bmatrix} e_1^0 \\ i_1^0 \end{bmatrix} = \begin{bmatrix} B_r \\ B_\theta \end{bmatrix}$$

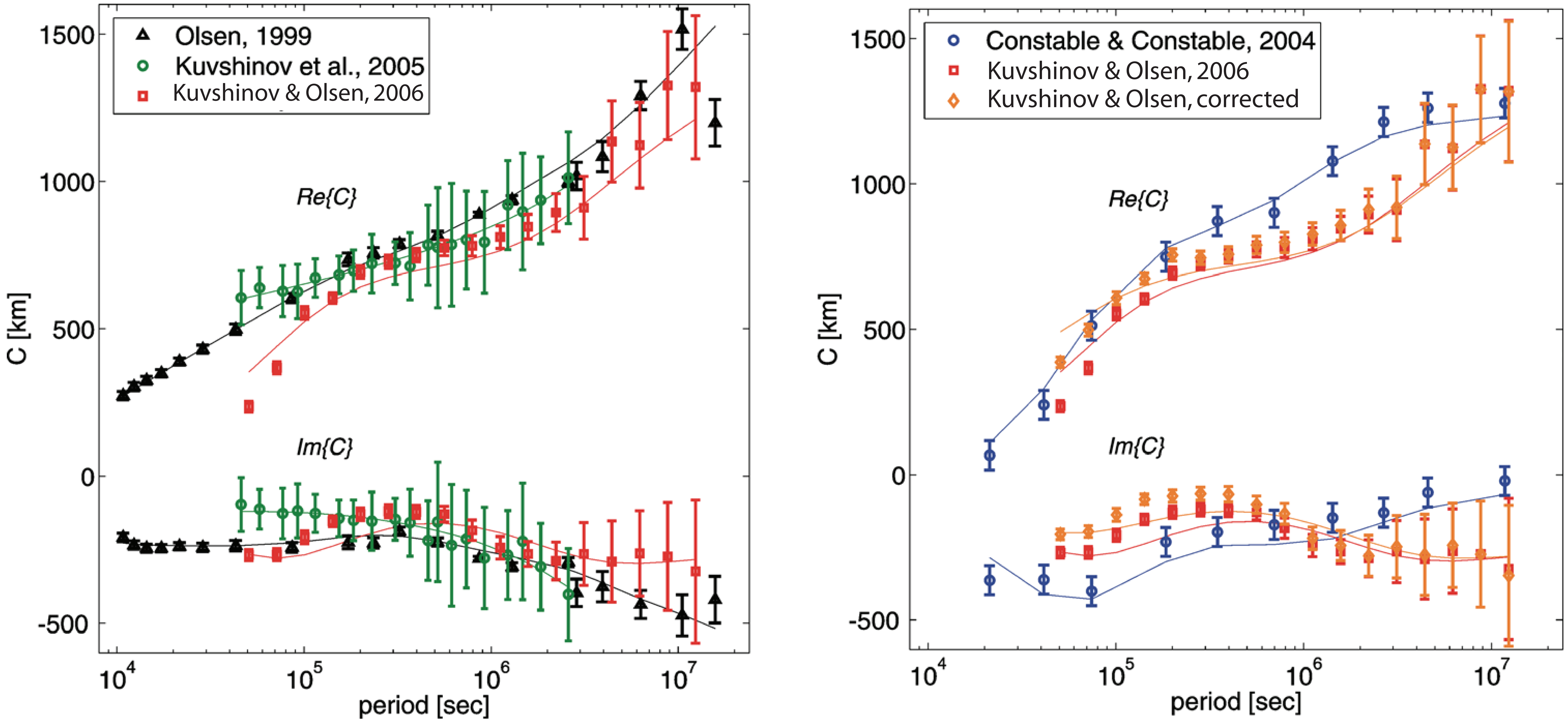


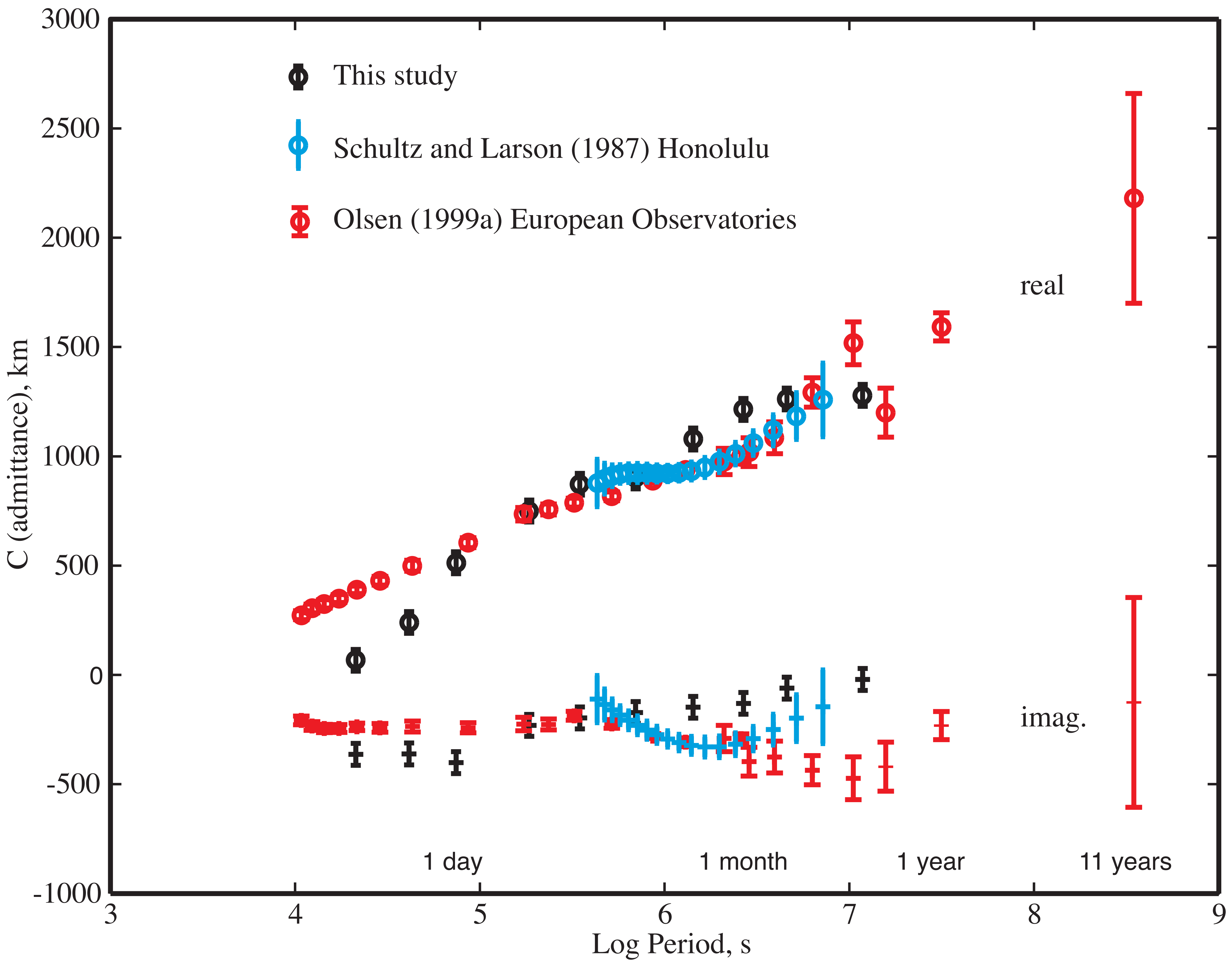
Putting together all the estimates from every pass give a time series that we can Fourier transform

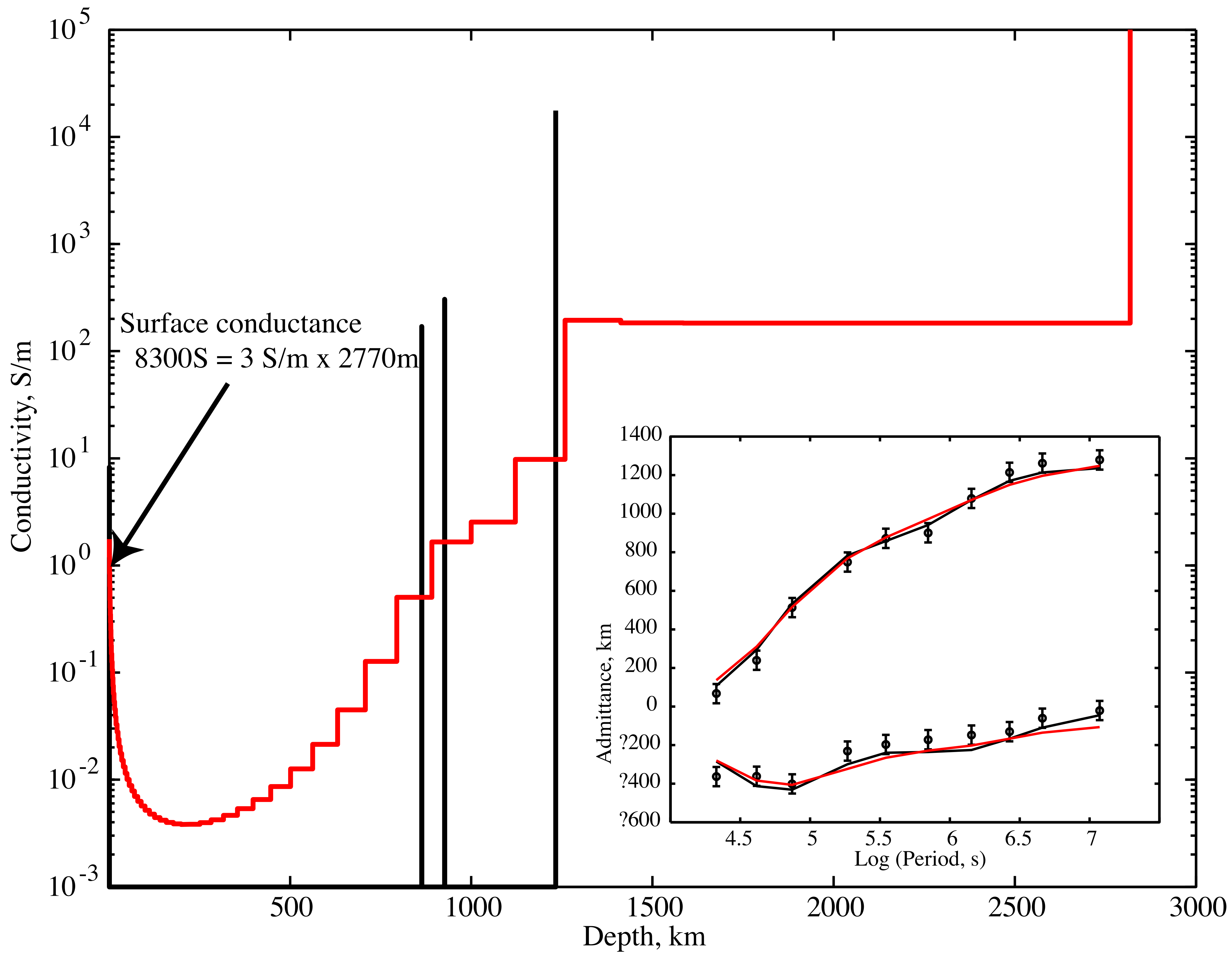


Examples of satellite responses:



Satellite response ("this study") compared with observatory GDS response, and model. Satellite response samples the ocean at short period.



# SIOG 231
# GEOMAGNETISM AND ELECTROMAGNETISM

Lecture 9b

Least Squares Fitting

2/1/2022

Some data $\mathbf{d} = (d_1, d_2, d_3, ..., d_M)^T$ with errors: $\sigma = (\sigma_1, \sigma_2, ..., \sigma_M)^T$

We have mathematics (could be a computer program) $f$ which maps a model onto predicted data.

$$\hat{\mathbf{d}} = f(\mathbf{x}, \mathbf{m})$$

This is called the "forward functional" because it maps **m**, which may be infinite dimensional, onto a single data point at a time. In practice **m** is of size *N* (which could still be large). The **x** tells the functional about the data.

$$\mathbf{m} = (m_1, m_2, ..., m_N)^T \qquad \mathbf{x} = (x_1, x_2, x_3, ..., x_M)$$

We might like a method which finds an **m** that exactly fits the data, but data have errors and **m** never captures all the complexities of the real world. So we need a measure of how well our model fits our data. For various reasons the sum-squared misfit is a good measure:

$$\chi^2 = \sum_{i=1}^{M} \frac{1}{\sigma_i^2} \left[d_i - f(x_i, \mathbf{m})\right]^2$$

$$\chi^2 = \sum_{i=1}^{M} \frac{1}{\sigma_i^2} \left[ d_i - f(x_i, \mathbf{m}) \right]^2$$

Our misfit measure can be written in matrix notation

$$\chi^2 = ||\mathbf{W}(\mathbf{d} - \hat{\mathbf{d}})||^2 = ||\mathbf{W}\mathbf{d} - \mathbf{W}f(\mathbf{m})||^2$$

$$\mathbf{W} = \mathrm{diag}(1/\sigma_1, 1/\sigma_2, ....1/\sigma_M) \quad .$$

Least squares minimizes this misfit with respect to all model parameters simultaneously. If the errors are zero-mean, independent, and normally distributed, $\chi^2$ is Chi-squared distributed with *M-N* degrees of freedom, and least squares gives a maximum likelihood and unbiased estimate of **m**.

In practice the errors are rarely so well behaved, but least squares is fairly tolerant.

Linear problems: Now the forward functional can be written as a matrix **F** of coefficients. The misfit here is given by the vector of residuals, **r**.

$$\hat{\mathbf{d}} = \mathbf{F}\mathbf{m} \qquad \mathbf{d} = \mathbf{F}\mathbf{m} + \mathbf{r} \qquad \mathbf{r} = \mathbf{d} - \mathbf{F}\mathbf{m}$$

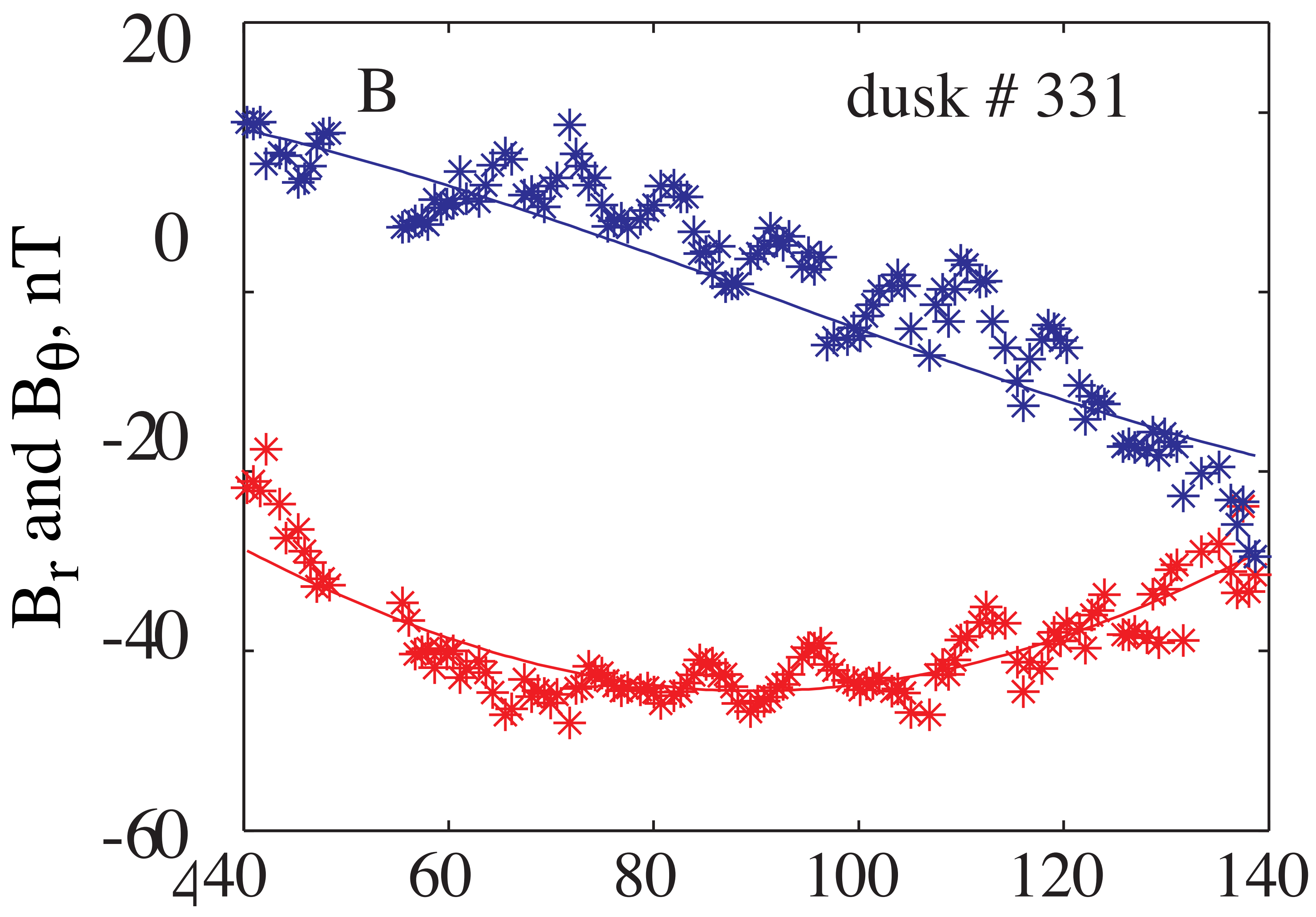
It can (and will) be shown that the model that minimizes **r** in a least squares sense is given by

$$\mathbf{m}_* = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{d}$$

We have assumed all the errors are the same. Note also this only works if $M > N$.

Here is an example of a linear problem you have seen:

$$\begin{bmatrix} B_r \\ \\ B_\theta \end{bmatrix} = \begin{bmatrix} -\cos(\theta) & 2(a/r)^3\cos(\theta) \\ \\ \sin(\theta) & (a/r)^3\sin(\theta) \end{bmatrix} \begin{bmatrix} e_1^0 \\ \\ i_1^0 \end{bmatrix}$$



In practice we have more than one pair of data points, in this case 123 pairs, each with a particular value of colatitude and radial distance.

$$\begin{bmatrix} B_r(\theta_1, r_1) \\ B_\theta(\theta_1, r_1) \\ B_r(\theta_2, r_2) \\ B_\theta(\theta_2, r_2) \\ ... \\ B_r(\theta_{123}, r_{123}) \\ B_\theta(\theta_{123}, r_{123}) \end{bmatrix} = \begin{bmatrix} -\cos(\theta_1) & 2(a/r_1)^3 \cos(\theta_1) \\ \sin(\theta_1) & (a/r_1)^3 \sin(\theta_1) \\ -\cos(\theta_2) & 2(a/r_2)^3 \cos(\theta_2) \\ \sin(\theta_2) & (a/r_2)^3 \sin(\theta_2) \\ ... & ... \\ -\cos(\theta_{123}) & 2(a/r_{123})^3 \cos(\theta_{123}) \\ \sin(\theta_{123}) & (a/r_{123})^3 \sin(\theta_{123}) \end{bmatrix} \begin{bmatrix} e_1^0 \\ i_1^0 \end{bmatrix}$$