CHAPTER 4

KINEMATICS II: DEFORMATION

> We beat it out flat; we beat it back square; we battered it into every form known to geometry – but we could *not* make a hole in it. Then George went at it, and knocked it into a shape, so strange, so weird, so unearthly in its wild hideousness, that he got frightened and threw away the mast. Then we all three sat round it on the grass and looked at it.
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> JEROME K. JEROME (1889) *Three Men in a Boat (to say nothing of the Dog)*

4.1 Introduction

In the previous chapter we ended with the special case of rigid-body motion: this any motion for which distances between particles (points moving with the material) do not change with time. We now turn to the more interesting case in which these distances do change. The simplest example of this is a change in volume, in which all the distances change by an amount proportional to their size. A more important, and complicated, class of motions are ones which result in different distances changing in ways that depend, not just on the distance between the particles, but also on the direction of the vector between them. We refer to both of these as "deformation", though in common usage this word denotes only the second category.

This will be our first application of tensors, and long experience shows that this makes the subject difficult to learn. A student first working with this subject often tries to think in terms of vectors, perhaps because these can be easily visualized, and to focus on one-dimensional cases. Either approach is misleading; there is no substitute for using the mathematics of tensors in two and three dimensions.

4.2 Deformation

What matters in deformation is not total motion, but the relative motion of nearby particles, so we consider motions of the continuum relative to some particular (though arbitrary) particle, called the **reference particle**, which we label $\mathbf{x}_R$. The change in relative position between this particle and some other particle (labeled $\mathbf{x}$) is given by:

$$\mathbf{u}(\mathbf{x},t) = [\mathbf{r}^L(\mathbf{x},t) - \mathbf{r}^L(\mathbf{x}_R,t)] - [\mathbf{r}^L(\mathbf{x},0) - \mathbf{r}^L(\mathbf{x}_R,0)]$$

where, since we are looking at particles, we use the Lagrangean dscription of their motion. If $\mathbf{u}(\mathbf{x},t) = 0$ for all $\mathbf{x}$ and all t, we are back to a pure translation, since any line between two particles has an unchanging length and direction.

We therefore want to develop descriptions for more general forms of $\mathbf{u}$. We focus on three quantities which have very similar mathematical structure, and provide reasonable approximations to many actual behaviors: these are small deformation, rates of deformation, and homogeneous strain.

4.3 Small Deformation

To start, we define $\boldsymbol{\xi} = \mathbf{x} - \mathbf{x}_R$: the vector from the reference particle to another, arbitrary, one whose relative displacement we will be concerned with: effectively, we move the origin to $\mathbf{x}_R$. We do this because our first assumption is that we consider deformation in a small region only: so small that $\boldsymbol{\xi}$ is infinitesimal, which we indicate by writing it as $d\boldsymbol{\xi}$. We look at $\mathbf{u}$ and related quantities for a fixed (but nonzero) value of t – that is, we consider the material in two configurations: one at $t = 0$ and one at some other time. We do not, for now, consider how the material gets from one configuration to another; in the terms usually employed, we are concerned with deformation but not motion. In this case, $\mathbf{x}$ describes the locations in the initial state, and $\mathbf{r}$ in the final (second) state – the material and spatial descriptions thus become the undeformed and deformed states respectively, though it is arbitrary which state we call “undeformed”. By definition, when $d\boldsymbol{\xi} = 0$ (that is $\mathbf{x} = \mathbf{x}_R$), the vector $\mathbf{u}$ is always zero: the reference particle is always itself.

Our second assumption is that the deformation is smooth enough that we can write $\mathbf{u}$ as a Taylor series in $d\boldsymbol{\xi}$. Remembering that $d\boldsymbol{\xi}$ is a vector between particles (that is, it depends on particle labels $\mathbf{x}$), this Taylor series is written, pretending for the moment that the **x** axes define Cartesian coordinates:

$$u_i(d\boldsymbol{\xi}) = \frac{\partial u_i}{\partial x_j} d\xi_j + \text{higher-order terms} \tag{4.1}$$

where we are using, once again, the summation convention for repeated indices.

Our third assumption is that all we need from the Taylor series is the first term; this assumption amounts to requiring that the gradients in equation (4.1) be much less than one. And our fourth assumption is that the motions are small enough that the axes for **r** and **x** will locally coincide: this is a separate requirement from the one for small gradients. Remember that for $t = 0$, the reference state, these axes do coincide. This last assumption means that we may take **u** to be a function of **r** rather than of **x**.

The last two assumptions, combined, allow us to write the displacement **u** as

$$u_i = \frac{\partial u_i}{\partial r_j} d\xi_j \tag{4.2}$$

$$= \tfrac{1}{2}\left[\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right] d\xi_j + \tfrac{1}{2}\left[\frac{\partial u_i}{\partial r_j} - \frac{\partial u_j}{\partial r_i}\right] d\xi_j$$

$$\stackrel{\text{def}}{=} \varepsilon_{ij} d\xi_j + R_{ij} d\xi_j$$

The first expression gives the relative displacement in terms of the **displacement gradient**, which in coordinate-free terms is the tensor formed by the dyad $\nabla\mathbf{u}$. On the next line, we add, subtract, and regroup terms to get two combinations of these gradients that have special properties. In coordinate-free form equation (4.2) becomes

$$\mathbf{u} = (\nabla\mathbf{u})d\boldsymbol{\xi} = (\tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)d\boldsymbol{\xi} + \tfrac{1}{2}(\nabla\mathbf{u} - \mathbf{u}\nabla)d\boldsymbol{\xi} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon} d\boldsymbol{\xi} + \mathbf{R} d\boldsymbol{\xi} \tag{4.3}$$

where we have defined new quantities $\boldsymbol{\varepsilon} = \tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)$ and $\mathbf{R} = \tfrac{1}{2}(\nabla\mathbf{u} - \mathbf{u}\nabla)$. From equation (4.2) it is clear that $\boldsymbol{\varepsilon}$ is a symmetric tensor and **R** an antisymmetric one. The rest of this chapter will be largely devoted to exploring the properties of relative displacements described by these two tensors.

4.3.1 Small Deformation in Two Dimensions

To begin this exploration of $\boldsymbol{\varepsilon}$ and **R**, we consider them in two dimensions – which makes it easy to draw pictures. The Cartesian components of $\boldsymbol{\varepsilon}$ are then:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} u_{1,1} & \tfrac{1}{2}(u_{1,2}+u_{2,1}) \\ \tfrac{1}{2}(u_{2,1}+u_{1,2}) & u_{2,2} \end{pmatrix}$$

where we have again used the convenient contractions (equation 2.24)

$$u_{i,j} \overset{\text{def}}{=} \frac{\partial u_i}{\partial r_j}$$

(Remember, again, that in general a spatial derivative might be $\partial/\partial\mathbf{x}$ or $\partial/\partial\mathbf{r}$; only for small deformations does this difference not matter).

Now suppose the only nonzero term is ε_{11}; then

$$u_1 = \varepsilon_{11} d\xi_1$$

describes the displacement field: there is only displacement in the u_1 direction. If $\varepsilon_{11} > 0$, this displacement increases as we move away from $d\xi_1 = 0$. This deformation is called a **uniaxial extension**. If $\varepsilon_{11} < 0$, we have displacement, also increasing away from $d\xi_1 = 0$, but towards the $d\xi_2$-axis: this is **uniaxial contraction**. The general term for these uniaxial motions is **uniaxial strain**. Of course, a nonzero ε_{22} gives the same kind of deformation field in the orthogonal direction.

This may all seem relatively obvious; experience shows that the next step is less so. This comes if we make ε_{12}, only, nonzero; then

$$u_1 = \varepsilon_{12} d\xi_2 \qquad u_2 = \varepsilon_{12} d\xi_1$$

so the amount of displacement in one direction depends on the location in the other coordinate.

This displacement is called a **pure shear strain**. In this deformation the $d\xi_1$ and $d\xi_2$ axes (or any lines parallel to them) are displaced so that the angle between them changes from the original $\pi/2$ to $\pi/2 - 2\varepsilon_{12}$. Another way of specifying a shear strain is then to use this angle change, which we write as $\gamma = 2\varepsilon_{12}$. The usual name for γ is the **engineering shear**, while ε_{12} is the **tensor shear**.

Figure 4.1 shows some of these simple two-dimensional deformations; for clarity we have to make them finite rather than infinitesimal, but since we have made them homogeneous they remain accurate. The tails of the arrows form a regular grid in the undeformed material; their heads show the positions of these particles after the deformation, so the arrows themselves show the displacement field $\mathbf{u}$. In addition to the two strain types

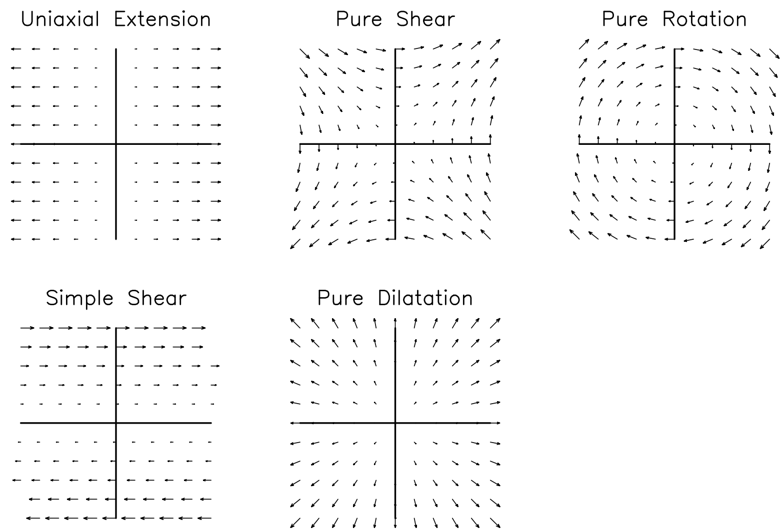

Figure 4.1: A variety of displacement fields, producing different kinds of strain. All these strains are finite, but since they are homogeneous (Section 4.5), they are appropriate for representing infinitesimal strains.

already described, we also show a **pure rotation**, in which $\boldsymbol{\varepsilon}$ is zero and $\mathbf{R}$ is not; a rigid-body motion. If we add this rotation to the pure shear, we get shear parallel to one axis, which is called **simple shear**: note that it includes both deformation and rotation. Simple shear has a special place in geodynamics, since it is the kind of deformation that takes place across diffuse plate boundaries when the motion is parallel to the boundary: crustal deformation in Southern California is one place where this is a good first approximation. This means that this deformation involves rotation, which has been measured by the deflection of paleomagnetic directions (e.g. Nicholson and Seeber (1989)).

Finally, Figure 4.1 shows the case in which $\varepsilon_{11} = \varepsilon_{22}$: this is often called **areal strain**. In the Earth, this kind of deformation is most typically found in volcanic areas. In this figure (and in fact in all these drawings) there is nothing special about the point in the center; if we took displacements relative to some location on the edge, we would the same pattern of relative displacements.

We just said that a nonzero $\mathbf{R}$ is a rotation; now we justify this. In two

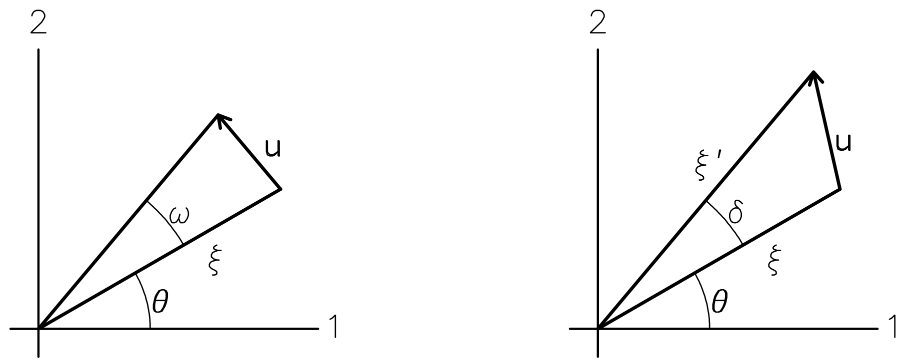

Figure 4.2: The left-hand diagram shows the motion of a vector under rotation; the right-hand one shows a vector that undergoes both rotation and change in length. See text for details.

dimensions **R** is

$$\mathbf{R} = \begin{pmatrix} 0 & \tfrac{1}{2}(u_{1,2} - u_{2,1}) \\ \tfrac{1}{2}(u_{2,1} - u_{1,2}) & 0 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

so there is only one component, ω, which represents the rotation angle. To demonstrate that, take a vector of length $d\xi$ which initially is at an angle θ to the 1-axis and is rotated by ω, as in the left panel of Figure 4.2. The end of the vector is displaced by

$$u_1 = d\xi[\cos(\theta + \omega) - \cos\theta]$$

$$u_2 = d\xi[\sin(\theta + \omega) - \sin\theta]$$

which can be written in matrix form as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\omega - 1 & -\sin\omega \\ \sin\omega & 1 - \cos\omega \end{pmatrix} \begin{pmatrix} d\xi\cos\theta \\ d\xi\sin\theta \end{pmatrix}$$

For $\omega \ll 1$ this becomes

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} d\xi_1 \\ d\xi_2 \end{pmatrix}$$

which shows that **R** describes just a rigid-body rotation – though the rotation has to be small for this description to work.

4.3.2 Transforming Small Strains

It is instructive to work though how the Cartesian components of $\boldsymbol{\varepsilon}$ and $\mathbf{R}$ change in two dimensions as we change the direction of the coordinate axes; though the algebra is slightly tedious, we end up with the result which justifies calling $\boldsymbol{\varepsilon}$ a tensor. Take, as before, a vector $d\boldsymbol{\xi}$ at an angle θ to the 1-axis; as a result of the deformation it becomes a new vector $d\boldsymbol{\xi}'$. (In figure 4.2 we show these, and label them, as finite vectors). We define the **stretch** to be the ratio of lengths of these vectors, and the **extension** to be this minus one:

$$e \stackrel{\text{def}}{=} \frac{|d\boldsymbol{\xi}'|}{|d\boldsymbol{\xi}|} - 1$$

Since $d\boldsymbol{\xi}' = d\boldsymbol{\xi} + \mathbf{u}$, we can write a quantity related to e

$$\begin{aligned} &\frac{(d\xi_1 + u_1)^2 + (d\xi_1 + u_2)^2 - (d\xi_1^2 + d\xi_2^2)}{d\xi_1^2 + d\xi_2^2} \\ &= \frac{((1+e)|d\boldsymbol{\xi}|)^2 - |d\boldsymbol{\xi}|^2}{|d\boldsymbol{\xi}|^2} = (1+e)^2 - 1 \approx 2e \end{aligned} \tag{4.4}$$

where the last approximation is valid for e small. We next write the displacements u_i in terms of the displacement gradients; for example, for u_1,

$$u_1 = \frac{\partial u_1}{\partial r_1} d\xi_1 + \frac{\partial u_1}{\partial r_2} d\xi_2$$

which means that

$$(u_1 + d\xi_1)^2 - d\xi_1^2 = 2\frac{\partial u_1}{\partial r_1} d\xi_1^2 + 2\frac{\partial u_1}{\partial r_2} d\xi_1 d\xi_2 \tag{4.5}$$

plus higher-order terms involving products of displacement gradients. Substituting expressions like equation (4.5) into 4.4 we find that

$$2e = 2\frac{\partial u_1}{\partial r_1}\left(\frac{d\xi_1}{|d\boldsymbol{\xi}|}\right)^2 + 2\frac{\partial u_2}{\partial r_2}\left(\frac{d\xi_2}{|d\boldsymbol{\xi}|}\right)^2 + 2\left(\frac{\partial u_1}{\partial r_2} + \frac{\partial u_2}{\partial r_1}\right)\frac{d\xi_1 d\xi_2}{|d\boldsymbol{\xi}|^2}$$

which implies

$$e(\theta) = \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta + 2\varepsilon_{12}\sin\theta\cos\theta \tag{4.6}$$

since, for example, $d\xi_1 = |d\boldsymbol{\xi}|\cos\theta$. Since e is just a uniaxial strain, equation (4.6) is also an expression for $\varepsilon_{11}(\theta)$, the $_{11}$ component of strain in a coordinate system rotated counterclockwise by θ from the original. For example, equation (4.6) implies that for $\theta = 90°$ we will get $\varepsilon_{11}(90°) = \varepsilon_{22}$, which is obviously the case. For pure shear, with $\varepsilon_{11} = \varepsilon_{22} = 0$ and $\varepsilon_{12} \neq 0$, $\varepsilon_{11}(\theta)$ has a four-lobed pattern, with two of the lobes being negative. This behavior is apparent if we look at the pure shear shown in Figure 4.1; at 45° to the axes there is just uniaxial extension and contraction.

This shows how the extension, and the extensional components, transform for a rotation of the axes. To derive how the shear strain will change for a rotation, we look first at the change in orientation of a single vector, which is described by the angle given, for small displacements, by (Figure 4.2):

$$\delta = \frac{u_1(-\sin\theta) + u_2\cos\theta}{|d\boldsymbol{\xi}|}$$

that is, the projection of $\mathbf{u}$ onto a direction perpendicular to $d\boldsymbol{\xi}$, divided by $|d\boldsymbol{\xi}|$. Again using the expression for u_1 and u_2 in terms of displacement gradients, and that $d\xi_1 = |d\boldsymbol{\xi}|\cos\theta$, we get that, for small displacements,

$$\begin{aligned}\delta &= \frac{\partial u_1}{\partial r_1}(-\sin\theta\cos\theta) + \frac{\partial u_1}{\partial r_2}(-\sin 2\theta) + \frac{\partial u_2}{\partial r_2}(\sin\theta\cos\theta) + \frac{\partial u_2}{\partial r_1}cos^2\theta \\ &= (\varepsilon_{22} - \varepsilon_{11})(\sin\theta\cos\theta) + (\varepsilon_{12} + \omega)(-\sin^2\theta) + (\varepsilon_{12} - \omega)(\cos^2\theta) \\ &= (\varepsilon_{22} - \varepsilon_{11})\sin\theta\cos\theta + \varepsilon_{12}(\cos^2\theta - \sin^2\theta) - \omega\end{aligned}$$

so the change in direction depends on $\boldsymbol{\varepsilon}$ and $\mathbf{R}$; reasonably, there is no dependence on θ for the part involving $\mathbf{R}$, as expected for a rigid-body rotation. If we add 90° to θ, we get the change in orientation for an orthogonal line, which is:

$$-(\varepsilon_{22} - \varepsilon_{11})\sin\theta\cos\theta - \varepsilon_{12}(\cos^2\theta - \sin^2\theta) - \omega$$

Subtracting these two expressions to get the change in angle between the (originally orthogonal) lines, and dividing by two to get the tensor shear, we get

$$\varepsilon_{12}(\theta) = (\varepsilon_{22} - \varepsilon_{11})\sin\theta\cos\theta + \varepsilon_{12}(\cos^2\theta - \sin^2\theta) \tag{4.7}$$

which shows, among other things, that $\varepsilon_{11} - \varepsilon_{22}$ is just as much a shear as ε_{12} is.

The full transformation matrix for the Cartesian components of strain in two dimensions (known as the Mohr transformation) is therefore:

$$\begin{pmatrix} \varepsilon'_{11} \\ \varepsilon'_{12} \\ \varepsilon'_{22} \end{pmatrix} = \begin{pmatrix} \cos^2\theta & 2\sin\theta\cos\theta & \sin^2\theta \\ -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{22} \end{pmatrix} \tag{4.8}$$

or

$$\begin{pmatrix} \varepsilon'_{11} \\ \varepsilon'_{12} \\ \varepsilon'_{22} \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin 2\theta & \sin^2\theta \\ -\tfrac{1}{2}\sin 2\theta & \cos 2\theta & \tfrac{1}{2}\sin 2\theta \\ \sin^2\theta & -\sin 2\theta & \cos^2\theta \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{22} \end{pmatrix}$$

Because the Cartesian components of $\boldsymbol{\varepsilon}$ transform in this way, we know that $\boldsymbol{\varepsilon}$ is indeed a tensor, since obedience to such a transformation rule defines the tensor character of an entity.

This transformation in turn can be used to find additional ways of parameterizing the strain; we describe these for the two-dimensional case, but some results carry over to three dimensions.

If we consider $\Delta_A = \varepsilon_{11}+\varepsilon_{22}$, we can see from equation (4.8) that this will not vary with θ at all – that is, this quantity is invariant, or more properly **an invariant of $\boldsymbol{\varepsilon}$**. (There are other invariants, but they involve powers of the components). Δ_A is the areal strain that we described earlier; this name is given because for small strains it is the ratio of areas in the deformed and undeformed states, minus one. In three dimensions, the equivalent ($\Delta_V = \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$) is called the **dilatation**, and is related to the change in volume in the same way.

If (still in three dimensions) we subtract the dilatation from the strain tensor to form $\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon} - \tfrac{1}{3}\Delta_A \mathbf{I}$, we have the **deviatoric strain $\boldsymbol{\varepsilon}^D$**, which can also be written as $\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon} - \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{I}$, where $\mathbf{I}$ is the identity tensor, and $\mathrm{Tr}(\boldsymbol{\varepsilon})$ is the trace of the strain tensor: for Cartesian components the sum of the diagonal terms.

From equation (4.7), we can see that at 45° to our original (and arbitrary) choice of axes the shear strain would be $\tfrac{1}{2}(\varepsilon_{22}-\varepsilon_{11})$: as noted above, this is thus just as much a shear as ε_{12}. We thus can express the strain as:

$$[\tfrac{1}{2}\Delta_A, \tfrac{1}{2}(\varepsilon_{22}-\varepsilon_{11}), \varepsilon_{12}] \stackrel{\text{def}}{=} [\tfrac{1}{2}\Delta_A, \tfrac{1}{2}\gamma_1, \tfrac{1}{2}\gamma_2)$$

where γ_1 and γ_2 are the engineering shear strains: γ_2 corresponds to the angle change between two orthogonal lines aligned with the coordinate axes, while γ_1 corresponds to the angle change between two orthogonal lines at 45° to those axes. This areal/shear parameterization can be useful because materials usually respond differently to shear than to change in area or volume: for example, in rocks shear leads to failure, but dilatation does not, at least in compression.

Yet another representation comes when we realize that there is an orientation of coordinate axes that will make $\varepsilon_{12} = 0$. Let the angle of the axes (relative to the original set) be $\theta = \theta_p$; then equation (4.7) shows that to make the shear zero we have to have

$$\tfrac{1}{2}(\varepsilon_{22}-\varepsilon_{11})\sin 2\theta_p+\varepsilon_{12}\cos 2\theta_p=0$$

which means that the angle is given by

$$\theta_p=\tfrac{1}{2}\arctan\left[\frac{2\varepsilon_{12}}{\varepsilon_{11}-\varepsilon_{22}}\right] \tag{4.9}$$

For axes that make this angle to the original axes, only ε_{11} and ε_{22} are nonzero, so yet another way to express strain is as

$$[\theta_p,\varepsilon_{11}(\theta_p),\varepsilon_{22}(\theta_p)]$$

These are termed the **principal axis strains**: the principal axes are those for which $\varepsilon_{12}=0$, which is to say, the axes for which $\boldsymbol{\varepsilon}$ is diagonal (in Cartesian coordinates). While this set of numbers does not directly transform to other coordinates, it is very often useful to plot at strains in this way, with arrows (pointing in or out) along these directions.

4.3.3 The Mohr's Circle Representation

Yet another way to look at two-dimensional strains is the **Mohr's circle** construction. This is a geometrical way of expressing the transformation 4.8 and of showing a particular state of strain. From equation (4.9) we can see that

$$\frac{\sin 2\theta_p}{\cos 2\theta_p}=\frac{\varepsilon_{12}}{\tfrac{1}{2}(\varepsilon_{11}-\varepsilon_{22})}$$

This expression in turn suggests reparameterizing these two shear strain components as

$$\varepsilon_{12}=\varrho\sin 2\theta_p \qquad \tfrac{1}{2}(\varepsilon_{11}-\varepsilon_{22})=\varrho\cos 2\theta_p$$

where $\varrho^2=\varepsilon_{12}^2+(\varepsilon_{11}-\varepsilon_{22})^2/4$. Substituting these expressions into equation (4.7) we get

$$\varepsilon_{12}(\theta)=-\varrho\cos 2\theta_p\sin 2\theta+\varrho\sin 2\theta_p\cos 2\theta=\varrho\sin 2(\theta_p-\theta) \tag{4.10}$$

Similarly, we can write equation (4.6) as

$$\varepsilon_{11}(\theta)=\tfrac{1}{2}\varepsilon_{11}(1+\cos 2\theta)+\tfrac{1}{2}\varepsilon_{22}(1-\cos 2\theta)+\varepsilon_{12}\sin 2\theta$$

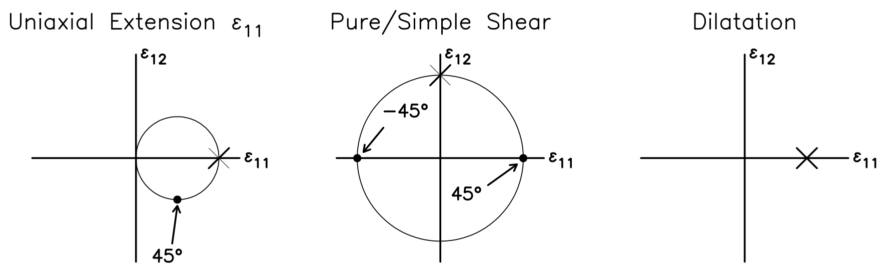

Figure 4.3: Mohr's circle representation of three states of strain.

$$= \tfrac{1}{2}(\varepsilon_{11} + \varepsilon_{22}) + \tfrac{1}{2}(\varepsilon_{11} - \varepsilon_{22})\cos 2\theta + \varepsilon_{12}\sin 2\theta$$

$$= \tfrac{1}{2}(\varepsilon_{11} + \varepsilon_{22}) + \varrho\cos 2(\theta_p - \theta) \tag{4.11}$$

If we plot ε_{11} and ε_{12} along two axes as a function of θ, using equations (4.10) and (4.11), the curve defined by these equations is a circle of radius ϱ, centered at $\tfrac{1}{2}(\varepsilon_{11} + \varepsilon_{22})$. This is called the **Mohr's circle for two dimensions** (there is a more elaborate version for three dimensions), and it is a useful way to display strain because it makes clear the possible range of both extension and shear. For example, it at once shows that for $\theta = -\theta_p$ the shear is zero, with the two extensional strains being $\varepsilon_{11}(\theta_p)$ and $\varepsilon_{22}(\theta_p)$, given by where the circle cuts the ε_{11} axis. (These points are 180° apart in terms of angle measured around the Mohr circle, and so 90° apart in terms of orientations of the coordinate axes, because of the factor of two in equations (4.10) and (4.11). Figure 4.3 shows three of the strain states from Figure 4.1 drawn in this way; in each diagram, the cross corresponds to the strain state for unrotated axes, and the dots to rotations of axes by the amount they are labeled by. Thus, at 45° to a uniaxial extension the shear is maximal; at ±45° to the coordinates for pure shear, we have uniaxial extension and contraction. And since there is no shear for dilatation, there is no Mohr's circle: just a point away from the origin.

Note that the Mohr's circle construction, and indeed all the parameterizations we have considered, hold, not just for $\boldsymbol{\varepsilon}$ in two dimensions, but for *any* two-dimensional symmetric tensor. We will see later how it can be applied to stress – which is, indeed, where it is more usually met with.

4.3.4 Small Deformation: Three-Dimensional Results

We now return to our three-dimensional results 4.2 and 4.4, and look first at the tensor **R**. The Cartesian components of this can be written as a matrix

$$\mathbf{R} = \begin{pmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{pmatrix}$$

where the indexing of the "vector" $\boldsymbol{\nu}$ (which we have not yet demonstrated to be a vector) is done so that we can write

$$R_{ik} = \epsilon_{ikm}\nu_m$$

where ϵ is the permutation symbol. But since we have (equation 2.10)

$$\epsilon_{ikp}\epsilon_{ikm} = \epsilon_{pik}\epsilon_{mik} = 2\delta_{pm}$$

we can write

$$\epsilon_{ikp}R_{ik} = \epsilon_{ikp}\epsilon_{ikm}\nu_m = 2\delta_{pm}\nu_m = 2\nu_p$$

giving, finally, $\nu_m = \frac{1}{2}\epsilon_{ijm}R_{ij}$ for $\boldsymbol{\nu}$ given **R**. But, substituting into this the definition of R_{ij} gives

$$\nu_m = \tfrac{1}{2}\epsilon_{mij}u_{j,i}$$

which is, in coordinate-free form, $\boldsymbol{\nu} = \frac{1}{2}\nabla \times \mathbf{u}$; $\boldsymbol{\nu}$ is the (polar) vector that is the curl of displacement. This corresponds to a small rigid-body rotation about the $\boldsymbol{\nu}$ axis, by an amount $|\boldsymbol{\nu}|$. **R** is thus a (small) rotation tensor, while $\boldsymbol{\varepsilon}$ describes the deformation. Note that if $\boldsymbol{\varepsilon}$ is zero everywhere the material deforms as a rigid body, in which case the only possible solution for **R** is that it is everywhere constant: different rotations in different places are not possible without some nonzero strain.

The other result is for $\boldsymbol{\varepsilon}$, and is that the expression in terms of principal strains, which we demonstrated for two dimensions, extends to three. The result comes from equation (2.21), the result that a symmetric matrix, can be decomposed into the product of an orthogonal matrix U and a diagonal matrix D:

$$\boldsymbol{\varepsilon} = UDU^T$$

which means that there is some set of axes in which the components of $\boldsymbol{\varepsilon}$ are

$$\begin{pmatrix} \varepsilon_I & 0 & 0 \\ 0 & \varepsilon_{II} & 0 \\ 0 & 0 & \varepsilon_{III} \end{pmatrix}$$

The axes in this coordinate system are the **principal axes** of the strain tensor; in this particular coordinate system there are no shears, only extensions or contractions. We already saw this representation (derived more laboriously) for the two-dimensional case.

4.3.5 Strain Compatibility

As we will see in Chapter 10, some methods of solving problems in elasticity compute only the strain. But if we do this, we need to make sure that the strains we find are in fact as implied by equation (10.1); that is, the strains have to be derivable from a vector displacement field **u**. As in our discussion of line integrals of vector fields (Section 2.7.1), the strain field must when integrated along two different paths, give the same relative displacements between two points. In Sections 10.4, we will allow some differences along different paths to represent discontinuities in the material – but for now we assume that there are none. The **equations of compatibility** express the conditions that the strain field must obey to be integrable in a path-independent way. Though these equations are most usually used for elastic strains, they in fact apply to any small deformation.

In Section 2.7.1, we showed that an integrable vector field is the gradient of a scalar one, and that whether or not it is such a gradient is determined by seeing if certain derivatives of the vector field, namely the curl, are zero. There we used already-established results from vector calculus, but we could have gotten the same results by just assuming a scalar field and looking at combinations of second derivatives of it. In the same way, we derive the equations of compatibility by taking different spatial second derivatives of strain and seeing that the resulting third derivatives of **u** are the same, giving combinations that we can say ought to be zero.

For example, one second derivative of strain is

$$\frac{2\partial^2 \varepsilon_{12}}{\partial r_1 \partial r_2} = \frac{\partial^3 u_1}{\partial r_2^2 \partial r_1} + \frac{\partial^3 u_2}{\partial r_1^2 \partial r_2}$$

and another is

$$\frac{\partial^2 \varepsilon_{11}}{\partial r_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial r_1^2} = \frac{\partial^3 u_1}{\partial r_2^2 \partial r_1} + \frac{\partial^3 u_2}{\partial r_1^2 \partial r_2}$$

which means that

$$\frac{\partial^2 \varepsilon_{11}}{\partial r_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial r_1^2} - \frac{2\partial^2 \varepsilon_{12}}{\partial r_1 \partial r_2} = 0$$

In two dimensions this is the only equation for strain compatibility; in three, if we start with ε_{13} or ε_{23}, we get two more equations of the same type.

Similarly we can find, in three dimensions,

$$2\left[\frac{\partial^2\varepsilon_{12}}{\partial r_1\partial r_3}+\frac{\partial^2\varepsilon_{31}}{\partial r_2\partial r_1}\right]=2\left[\frac{\partial^2\varepsilon_{11}}{\partial r_2\partial r_3}+\frac{\partial^2\varepsilon_{23}}{\partial r_1^2}\right]$$

and permutation of indices yields two more equations, of this type. We end up with a total of six equations in the six unknown strain components, which in index form are all some variation of

$$\varepsilon_{ij,mn}+\varepsilon_{mn,ij}-\varepsilon_{im,jn}-\varepsilon_{jn,im}=0$$

4.3.5.1 Integrating Strain to Displacement

Given compatible strains, integrating them to get displacement does require some care, as we can best illustrate with a simple example, working in two dimensions and assuming that the strains are constant (a type of strain we discuss further in Section 4.5). Since $\varepsilon_{11}=\partial_1 u_1$, it is tempting, but wrong, to integrate it to $u_1=\varepsilon_{11}r_1$; we also need to consider the other derivative $\partial_2 u_1$, which is part of ε_{12}.

In fact, if ε_{11}, ε_{22}, and ε_{12} are constant, we can write the integrals of the first two as

$$u_1=\varepsilon_{11}r_1+f_1(r_2)+c_1 \qquad u_2=\varepsilon_{22}r_2+f_2(r_1)+c_2 \tag{4.12}$$

where the c's are constants of integration, and the f's are arbitrary functions. We determine the f's from the third strain component, which we also hold constant at ε_{12}; but then, by the definition of shear strain, the displacements in equation (4.12) must satisfy

$$\varepsilon_{12}=\frac{1}{2}\left(\frac{\partial f_1}{\partial r_2}+\frac{\partial f_2}{\partial r_1}\right)$$

For the derivatives of the f's to be constant, they must be linear functions of their arguments:

$$f_1(r_2)=d_1r_2 \qquad f_2(r_1)=d_2r_1$$

with $\frac{1}{2}(d_1+d_2)=\varepsilon_{12}$, so that the final expressions for the displacement are

$$u_1=\varepsilon_{11}r_1+d_1r_2+c_1 \qquad u_2=\varepsilon_{22}r_2+(2\varepsilon_{12}-d_1)r_1+c_2$$

So we end up with three arbitrary constants, c_1, c_2, and d_1. These describe a rigid-body motion, with c_1 and c_2 giving a translation and d_1 a rotation: just the part of the motion that the strains cannot determine. This generalizes to three dimensions; if we have only strains and want displacements, we must introduce some constraints on translation and rotation to get a unique solution. Of course, in general strains may be inhomogeneous, and it then becomes important both to ensure compatibility and to introduce the constraints appropriately; see Prescott (1981) and Segall and Matthews (1988) for geodetic applications and Haines and Holt (1993) for geologic ones.

4.4 Rates of Motion

We get results very similar to those for small strains if we consider the rate of motion of a continuum – which, for geophysics, can often be the more relevant description. For example, if we are trying to relate current seismicity to deformation, we want the instantaneous rate of deformation: the few decades (or at most few centuries) of geophysical measurement are, on geological time, instantaneous.

Formally, though nonrigorously, we can see that over an infinitesimal time dt the displacement must be small. Then locally we may, as we did before, take only the first term of a Taylor series expansion of the relative velocity (instead of the displacement). We can ignore differences between axes, since over an infinitesimal time they will not change. Then we have

$$dv_i=\frac{\partial v_i}{\partial r_k}dr_k=\frac{\partial v_i}{\partial x_k}d\xi_k$$

where $\mathbf{dv}$ is the relative velocity between a reference point (or particle) and one $d\boldsymbol{\xi}$ away. These partial derivatives define the **velocity gradient tensor** $\mathbf{L}=v_{i,k}=\nabla\mathbf{v}$; unlike the case for the small-strain tensor (Section 4.3). these gradients need not themselves be infinitesimal. We can make exactly the same decomposition as we did for small strain, though we give the tensors different symbols:

$$\mathbf{L}=\mathbf{D}+\mathbf{W}$$

where $\mathbf{W} = \frac{1}{2}(\nabla\mathbf{v} - \mathbf{v}\nabla)$ is the **spin tensor** and $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{v} + \mathbf{v}\nabla)$ is the **rate-of-deformation tensor** – names we shall now proceed to justify.

By exactly the same procedure as we applied to **R**, we can see that the components of **W** can be written as a vector, given by $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v}$; this is half the **vorticity vector** $\nabla \times \mathbf{v}$ as defined in fluid mechanics (confusingly, **W** is sometimes called the vorticity tensor). This quantity, again, need not be infinitesimal, unlike the small-rotation tensor **R**.

To show the meaning of **W**, consider a small circular area (disk) of diameter δ and area S, centered on the reference particle. The mean velocity around the edge of this disk, which is a line we denote by Γ, is

$$\bar{v} = \frac{1}{2\pi\delta}\int_\Gamma \mathbf{v}\cdot d\mathbf{l} = \frac{1}{2\pi\delta}\int_S \nabla \times \mathbf{v} d\mathbf{A}$$

by Stokes' theorem. For $\nabla \times \mathbf{v}$ constant across the surface S, the mean velocity around the edge is

$$\bar{v} = \frac{1}{2\pi d}\pi\delta^2\nabla \times \mathbf{v} = \delta \cdot \tfrac{1}{2}(\nabla \times \mathbf{v})$$

which is just what we would expect if the disk rotated as a rigid body with angular velocity $\boldsymbol{\omega} = \frac{1}{2}(\nabla \times \mathbf{v})$. **W** thus describes the local angular velocity, or spin, of the material, including both rigid-body rotation and any spin caused by local deformation.

To better see the meaning of **D**, consider an infinitesimal line $d\boldsymbol{\xi}$, which in spatial coordinates is $d\mathbf{r}(t)$. The length of this is the dot product $d\mathbf{r}\cdot d\mathbf{r}$, and the rate of change of length is thus

$$\frac{d}{dt}(d\mathbf{r}\cdot d\mathbf{r}) = 2\cdot d\mathbf{r}\cdot\frac{d}{dt}(d\mathbf{r}) \tag{4.13}$$

But we can write, at any time, $d\mathbf{r}$ as

$$d\mathbf{r} = (\mathbf{r}\nabla)\cdot d\boldsymbol{\xi} \quad \text{i.e.} \quad dr_i = \frac{\partial r_i}{\partial x_j}d\xi_j$$

which implies that

$$\frac{d}{dt}(d\mathbf{r}) = \frac{d}{dt}(\mathbf{r}\nabla)\cdot d\boldsymbol{\xi} + (\mathbf{r}\nabla)\cdot\frac{d\boldsymbol{\xi}}{dt} = \frac{d}{dt}(\mathbf{r}\nabla)\cdot d\boldsymbol{\xi}$$

because $d\boldsymbol{\xi}$ is constant (it always includes the same particles). But, we can write this as

$$\frac{d}{dt}(d\mathbf{r}) = \frac{d\mathbf{r}}{dt}\nabla\cdot d\boldsymbol{\xi} = \mathbf{v}\nabla\cdot d\boldsymbol{\xi}$$

where $\mathbf{v}\nabla$ is, in Cartesian components, $\partial v_i/\partial x_j$. This is a gradient with respect to material coordinates (particles), not spatial coordinates. However, it is the case that

$$dv_i = \frac{\partial v_i}{\partial x_j} d\xi_j \qquad \text{and} \qquad dv_i = \frac{\partial v_i}{\partial r_j} dr_j$$

which means that we can write

$$\frac{d}{dt}(d\mathbf{r}) = \mathbf{L} d\mathbf{r}$$

Finally, substituting this into equation (4.13), we get

$$\frac{d}{dt}(d\mathbf{r} \cdot d\mathbf{r}) = 2 \cdot d\mathbf{r} \cdot \mathbf{L} \cdot d\mathbf{r} = 2d\mathbf{r} \cdot \mathbf{D} \cdot d\mathbf{r}$$

where the antisymmetric part of $\mathbf{L}$ has canceled in the dot product. Thus $\mathbf{D}$ is associated with rate of change of length; if $\mathbf{D} = 0$ then there is no change of length in any direction, so that where this is true the motion is that of a rigid body. $\mathbf{D}$, being symmetric, has exactly the same properties as $\boldsymbol{\varepsilon}$ does: we can define rates of extension, rates of shear, and principal axes just as we did above.

It is important to realize that, as in the small-strain case, $\mathbf{D}$ and $\mathbf{W}$ are not completely independent. Consider two different motions that have the same $\mathbf{D}$. If we subtract these, the motion that is the difference between them has to have $\mathbf{D} = 0$ everywhere – but this means that this difference describes a rigid-body motion, for which $\mathbf{W}$ is constant. Knowing $\mathbf{D}$ everywhere thus means that we know $\mathbf{W}$ to within a constant amount: knowledge of the rate of deformation determines, up to a constant, the local spin, a point first made in a geophysical context by Haines (1982) and since developed considerably (e.g., Holt and Haines (1995)).

4.5 Homogeneous Strain

Our final category of deformation is **finite homogeneous strain**, which is actually what we have already been using in the figures. Fortunately it possesses almost the same mathematical structure as small strains and rates of deformations. Many actual patterns of large strains are not homogeneous; but homogeneous finite strain is often a good approximation for such cases if applied over a small enough region. We will see an example in

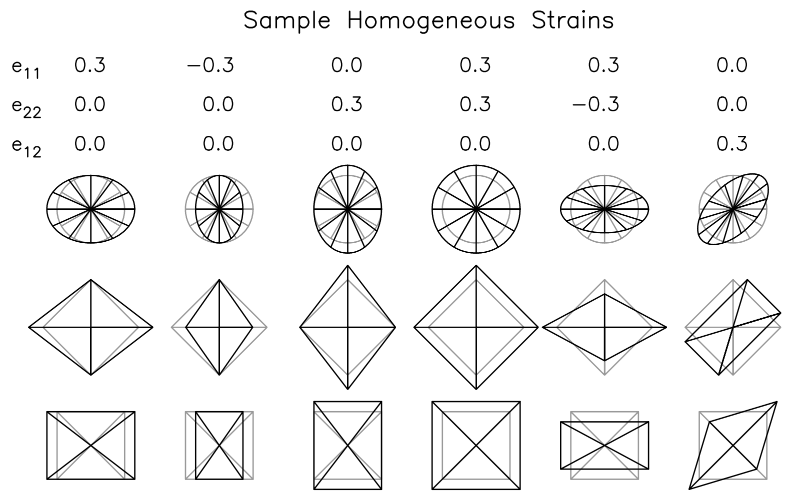

Figure 4.4: Samples of homogeneous strain. In each column the tensor components are given in the first three rows; the next three show the deformation of a circle with radii added, and a diamond shape (a square rotated by 45 degrees) and a square with added. In each picture the gray is the original shape, the black the strained one. Note in the last two columns how equal and opposite extensions are (after rotation) equivalent to shear.

the next section; other applications are found in the analysis of deformed structures (such as fossils) in deformed rocks; even though the deformation of these can be very complicated, on a small scale it can be spatially uniform enough for homogeneous strain to be an adequate approximation.

Homogeneous strain can be defined geometrically as any deformation in which all straight lines in the undeformed material remain straight in the deformed one. The equation for a straight line is $c_i x_i = d$ where the c_i's and d are arbitrary constants and we have used the summation convention; we can write this in vector form as $\mathbf{c}\cdot\mathbf{x} = d$, though we need to remember that $\mathbf{c}$ is not a geometrical vector, but just a triplet of numbers. Then we can write the geometrical constraint as

$$\mathbf{c}\cdot\mathbf{x} = d \qquad \text{and} \qquad \mathbf{c}'\cdot\mathbf{r} = d'$$

where $\mathbf{x}$ is, as usual, the coordinates in the undeformed configuration, and $\mathbf{r}$ the spatial coordinates (in either one). Algebraically this means that the relation between these two sets of coordinates must be

$$r_i = A_{ij} x_j \tag{4.14}$$

where A is an arbitrary 3×3 real-valued matrix.

We may now usefully apply another standard result from linear algebra: the **polar decomposition theorem**, which states that any real matrix can be decomposed into a product of two others:

$$A = RB_1 = B_2 R$$

where the matrix R is orthogonal and B_1 and B_2 are symmetric, though not the same. In three dimensions R corresponds to a rotation; B_1 and B_2, being symmetric, can be decomposed into another orthogonal matrix U and a diagonal matrix D:

$$A = RU_1 D_1 U_1^T = U_2 D_2 U_2^T R$$

From equation (4.14) for finite homogeneous strain, this means that any such strain can be produced in two ways:

A The RB_1 decomposition corresponds to first deforming the material in a way given by B_1, which can be expressed as different extensions along the directions of the principal axes, followed by a rotation given by R.

B The B_2R decomposition corresponds to first rotating the material in a way given by R and then deforming it a way given by B_2 – again, this is expressible in terms of (different) extensions along (different) principal axes.

Figure 4.4 shows some of the same homogeneous strains in terms of shape or area changes of simple figures: circles, squares, and diamonds.

4.6 Inhomogeneous Finite Strain in Cartography: Tissot's Indicatrix

An interesting application of finite homogeneous strain in two dimensions is the use of strain ellipses to show the deformations inherent in different map projections. Why we say "strain" for map projections is simply that any projection of the Earth's spherical surface onto a plane has to involve

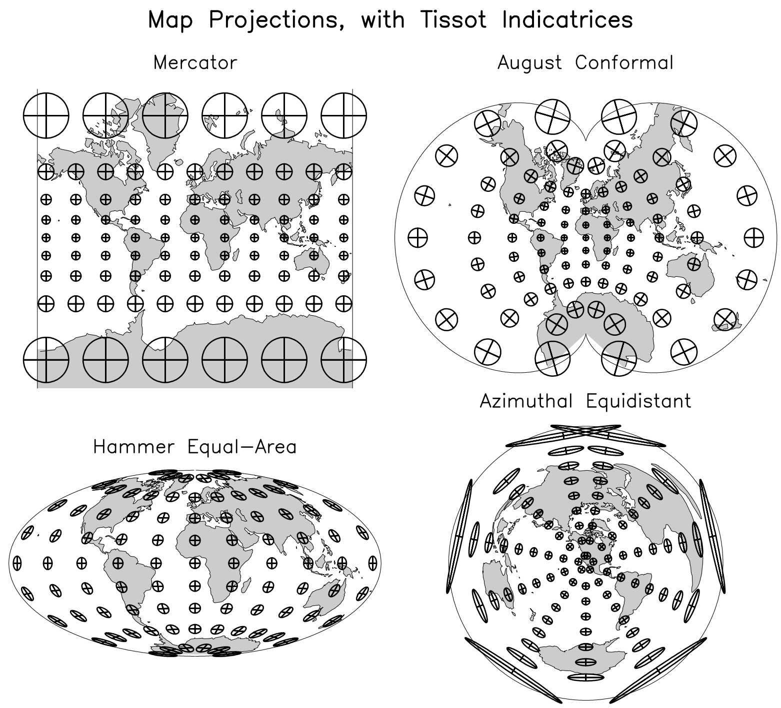

Figure 4.5: Map projections with Tissot indicatrices. The indicatrices were computed by projecting 5-km circles around particular latitude-longitude points onto the map plane, and then magnifying their scale by 100 to make them visible. The "August" projection maps the sphere into a two-cusped epicycloid. The land-sea distribution is shown in the normal aspect, except for the azimuthal equidistant projection, for which the map is centered on SIO.

deformation. The use of strain in cartography seems to have been independently invented by Tissot some time after all the mathematics for strain results had been worked out by Cauchy; it is still rare for the cartographic literature to reference continuum mechanics rather than Tissot's work.

For our purposes a map projection can be considered a function that maps[1] the surface of a sphere (or an ellipsoid if we want to be more accurate) onto a plane. We can put this into vector form by considering a three-dimensional unit vector **e**, which must end on the surface of the sphere, and

[1] Unfortunately there is no easy way to avoid the dual use of "map" in both its specifically cartographic and more general mathematical sense.

saying that a projection maps this to a two-dimensional vector $\mathbf{p}$: $\mathbf{p} = \mathbf{P}(\mathbf{e})$. However, it is probably more useful to view a projection as a mapping from one pair of coordinates (the latitude and longitude ϕ and λ) to another pair, the Cartesian coordinates of $\mathbf{p}$: $\mathbf{p} = \mathbf{P}(\phi, \lambda)$. Note that while $\mathbf{p}$ is a vector in two dimensions, the two spherical coordinates are not.

Mapping a spherical surface to a plane involves distortion; to make this concrete, imagine that we have a globe made from a sheet of rubber, and flatten this sheet into a plane. This will distort the sheet (indeed tear it), and these distortions will be both large and nonuniform, making this a case of finite inhomogeneous strain. However, locally we can regard the distortion as nearly constant, and treat it using the results of the previous section for finite homogeneous strain: locally, each distortion can be described as a combination of rotation, and stretching along two orthogonal axes. This means that a circle traced around some point on the sphere will be mapped into an ellipse (perhaps rotated) in the plane that the sphere is mapped into. The cartographic term for this strain ellipse is **Tissot's indicatrix**.

Figure 4.5 shows four examples of map projections, with the indicatrices shown on a grid of latitude-longitude points. The top two projections are **conformal**, meaning that they produce, locally, no change of shape, or equivalently, no changes in any angles. The indicatrices therefore retain their original circular shape, but their size changes because of the scale variations that must occur in any conformal projection from the sphere to the plane. The upper left map is the familiar Mercator projection, showing the equally familiar exaggeration of scale close to the poles. Note that for this map there is no rotation anywhere: this, and conformality, are what make the Mercator projection useful for many purposes. The upper right panel shows a (deservedly) less familiar conformal projection, to illustrate that conformality can coexist with relative rotations. The lower left map is an **equal-area** projection, so the indicatrices all have the same area, but are increasingly distorted (and rotated) away from the center. Finally, the lower right map is an azimuthal equidistant projection, which has the specialized property that lines from the center show true great-circle distance and direction from that point; as the varying shapes and sizes of the indicatrices show, this projection is neither conformal nor equal-area.