CHAPTER X

Second Half Review 2017

Here is a quick overview of what we covered in the second half of the class. Remember that the final covers the whole course but there will naturally be a bias towards the second half.

1. Gravity (chapter 4)

Gravity is based on the inverse square force law (Newton's Law). Dividing the inverse square law by one mass gives the gravitational field, which has units of acceleration. We can find the total acceleration due to gravity at any point for any body by dividing up the body into small elements of mass δm, computing δg, then performing the (vector) sum of all the contributions from all the elements. In the limit that the elements become infinitesimally small, g can be found by integrating over the mass distribution. However, it is easier to sum over the gravitational potential, V, a scaler defined as gravitational potential energy per unit mass, or the work done to take a test mass from infinity to the point of interest. As for all potential fields, the field can be obtained from $-\nabla V$, but be careful to note that r is positive up and z is positive down, while gravity is usually defined as positive down.

Gauss' Law says that the integral over a closed surface of the gravity field perpendicular to the surface is equal to $-4\pi Gm$ where m is the enclosed mass. We can use Gauss' Law to show that (a) a spherically symmetric mass looks like a point mass on the outside and (b) inside a spherically symmetric shell of mass gravity is everywhere zero.

With these results we can use the inverse square law to compute the acceleration due to gravity within a spherical planet. The acceleration due to gravity at radius $r < a$ inside the Earth is

$$g(r) = \frac{G}{r^2}\int_0^r 4\pi\rho(x)x^2\,dx \qquad (X.1)$$

Naturally, as we go away from the surface of a spherical body ($r > a$), the point mass assumption holds; the acceleration due to gravity outside the body is therefore

$$g(r) = \frac{GM}{r^2} \qquad (X.2)$$

We are actually interested at looking at gravity anomalies due to variations in density in the Earth, but to do this we have to have a reference state relative to which we measure anomalies. This reference state includes the effect of the Earth's equatorial bulge and the effects of centrifugal acceleration. It is easiest to do this using the gravitational potential (as it is a scalar). You should understand qualitatively how a spherical harmonic fit is done and why J_2 is the dominant term (though it is still small relative to 1: $J_2 = 1.08270 \times 10^{-3}$). Including the centrifugal acceleration gives

$$g(r,\lambda) = \frac{GM}{r^2}\left(1 - \frac{3J_2}{2}\left(\frac{a}{r}\right)^2(3\sin^2\lambda - 1) - \frac{\Omega^2 a^3}{GM}\left(\frac{r}{a}\right)^3\cos^2\lambda\right) \qquad (X.3)$$

This is the derivative of the "geopotential"

$$U(r,\lambda) = -\frac{GM}{r}\left(1 - \frac{J_2}{2}\left(\frac{a}{r}\right)^2(3\sin^2\lambda - 1) + \frac{1}{2}\frac{\Omega^2 a^3}{GM}\left(\frac{r}{a}\right)^3\cos^2\lambda\right) \qquad (X.4)$$

where the last term is due to the centrifugal acceleration. Note that the term $\Omega^2 a^3/GM$ is 3.46775×10^{-3} and so is small relative to 1. You had some homework problems to manipulate these equations recognizing which terms are small and using the binomial expansion.

We then had a diversion on moments of inertia. For spherical planets, the crucial equation is:

$$C = \frac{8\pi}{3} \int_0^a \rho(r) r^4 \, dr \qquad (X.5)$$

The datum that is used is usually not C but C/Ma^2 which is a dimensionless number (M is the mass of the planet). Remember that the mass of a spherically symmetric planet is just

$$M = 4\pi \int_0^a \rho(r) r^2 \, dr \qquad (X.6)$$

You should be able to show that $C/Ma^2 = 0.4$ for a homogeneous planet and that it is less than this for a planet which is denser towards the center.

We then discussed the "geoid" which is the equipotential surface which most closely coincides with sea level. The "reference geoid" is defined by

$$r_0 = a \left[1 + \frac{(2f - f^2)}{(1-f)^2} \sin^2\lambda \right]^{-\frac{1}{2}} \simeq a(1 - f \sin^2\lambda) \qquad (X.7)$$

where f is the flattening $(a-c)/a$ and has the value 3.35282×10^{-3} (again small relative to 1). "Geoid anomalies" are the deviation of the actual geoid away from the reference geoid and are measured in meters. They are given by

$$\Delta N = -\frac{U_R - U_0}{g_R}$$

where U_R is the potential on the reference geoid, U_0 is the observed potential, and g_R is the acceleration due to gravity on the reference geoid. g_R is given by the international gravity formula. The long wavelength part of the geoid is actually related to large-scale convection in the Earth's mantle.

We then talked about gravity anomalies. First of all, you should remember that all we measure is the perturbation to the vertical component of gravity (so don't forget that cosine term!). We looked at the anomaly due to a buried sphere. By integrating an infinite a slab in cylindrical coordinates we modeled the gravitational effect of slowly varying topography. This is called the *Bouguer gravity formula*. If topography has a height h and a density ρ_c (independent of depth) its contribution to g will be given by

$$\Delta g = 2\pi G \rho_c h \qquad (X.8)$$

When we are on topography of height h, we also have to correct for the fact that gravity is lower than on the reference ellipsoid. This effect is given by

$$\Delta g_h = \frac{2hg_R}{r_0} \qquad (X.9)$$

The "free air" anomaly is given by

$$\Delta g_{fa} = g_{obs} - g_R + \Delta g_h \qquad (X.10)$$

The Bouguer gravity anomaly also corrects for the gravitational attraction of topography:

$$\Delta g_B = \Delta g_{fa} - 2\pi G \rho_c h \qquad (X.11)$$

You should know how these anomalies behave when you have compensated and uncompensated topography.

The final section of the notes looked at geoid anomalies associated with isostatically compensated structure. The important equation is

$$\Delta N = -\frac{2\pi G}{g_R} \int_0^h z \Delta\rho(z) dz \qquad (X.12)$$

You should also know what Pratt and Airy compensation are and be able to use the above equation to compute the geoid anomaly for each type of compensation.

2. Geomagnetism (chapter 5)

We began with a description of the main field – which is dominantly dipolar. You should know the geomagnetic coordinate system, including inclination, declination, and colatitude. The topic of generation of magnetic field by dynamo action is a natural for an essay question. and we would expect you to be able to do some dimensional analysis to estimate time-scales of diffusion and convection of the magnetic field (e.g., under what conditions is frozen flux valid). You should understand why we think Earth's magnetic field originates with a dynamo in the outer core, and qualitatively how dynamo action works. You should know the diffusion equation for magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nu_m \nabla^2 \mathbf{B}$$

where $\nu = 1/\mu_o \sigma$.

You should qualitatively understand the spherical harmonic expansion solution to the magnetic field potential, and that the dipole term is given by

$$V_{GAD} = a g_1^0 \left(\frac{a}{r}\right)^2 P_1^0(\cos\theta) = a g_1^0 \left(\frac{a}{r}\right)^2 \cos\theta = \frac{\mu_0 M \cos\theta}{4\pi r^2} \qquad (X.13)$$

where M is the dipole moment and GAD stands for "geocentric axial dipole hypothesis". You should know what GAD is and why it is important (particularly in paleomag). You should also be able to derive this equation by considering a dipole to be made of two equal and opposite magnetic poles.

Be able to derive:

$$\tan I = 2 \cot \theta_m$$

and the equivalent equation for intensity. You should know about the different kinds of remanent magnetization and the mechanisms by which rocks get magnetized. You should also be able to calculate paleomagnetic pole positions from measurements of declination and inclination of a remanent magnetization (or, equivalently, predict the declination and inclination at a site given the paleomagnetic pole). This requires the use of the cosine rule for spherical triangles:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

where b and c are two sides of a spherical triangle and A is the included angle. a is the third side of the triangle.

You should understand the difference between induced and remanent magnetization, and and the various processes by which rocks acquire a remanent magnetization. You should understand the concepts of secular variation, magnetic reversals, and how the reversal tim scale relates to oceanic magnetic anomalies.

You should qualitatively understand the concept of geomagnetic induction and the result that for a uniform conductivity σ and a single frequency ω the magnetic and electric fields fall off exponentially with a characteristic distance given by the skin depth z_s:

$$z_s = \sqrt{\frac{2}{\sigma \mu_o \omega}} \quad \approx \quad 500\text{m}\sqrt{1/(\sigma f)} \qquad (X.14)$$

3. Internal constitution (chapter 6)

We first showed how to compute pressure within the earth and I would expect you to be able to derive the equation:

$$\frac{dp}{dr} = -\rho(r)g(r) \qquad (X.15)$$

and to be able to compute pressure in some simple planetary bodies. We then discussed how temperature would vary in a vigorously convecting region. In such a region, there is no time for diffusion to be effective (see the argument based on Peclet number) so the body of the adiabatically convecting region is isentropic, i.e., $ds/dr = 0$ where entropy s is a function of temperature and pressure. You should then be able to derive the equation governing temperature in such a region:

$$\left(\frac{dT}{dr}\right)_{ad} = -\frac{\alpha g T}{C_p} = -\frac{gT\gamma}{\phi} \qquad (X.16)$$

or, alternatively

$$\left(\frac{\partial T}{\partial \rho}\right)_s = \gamma \frac{T}{\rho}$$

where ϕ is the seismic parameter (see section 2.4) and

$$\gamma = \frac{\alpha \phi}{C_p}$$

is Gruneisen's ration which is a dimensionless number typically between 1 and 2. These equations can be used to extrapolate temperature across the lower mantle or outer core given a boundary condition on temperature somewhere. (You should also know what the Adams-Williamson equation is).

The way we get boundary conditions on temperature is to use experimental data on the behavior of materials at high temperature and pressures. We first discussed likely compositions for the mantle, then we discussed the high pressure experimental techniques. We followed this with a discussion of phase diagrams and identification of the phase transformations that correspond to the various seismic discontinuities in the mantle. This allows us to construct a temperature profile for the mantle. We also find that there is no strong evidence for compositional stratification of the mantle. We believe the core is dominantly iron and the inner core is solidified outer core material. This allows us to guess a temperature at the inner core boundary and extrapolate temperature across the outer core to the under side of the core-mantle boundary. Comparing with our profile for the mantle shows that there must be a strong boundary layer with a steep temperature gradient at the base of the mantle. This region coincides with the seismic D" region. All of this stuff is a natural for an essay question.

4. Summary

The equations on this review sheet and the midterm review sheet and the math cheat sheet given out earlier are strong contenders for your cheat sheet. I recommend practicing for the exam by redoing the problem sheets. Remember to look for key words in questions such as "homogeneous", "linear", "exponential", etc. Remember to draw a diagram, define the variables, and then tell us how you are going to solve a problem. Do this at the beginning of a question could well earn you most of the marks before you get lost in algebra!

The exam will consist of a choice of 6 out of 8 short questions and 2 out of 4 long questions (one of which will be an essay question). The short questions should take you about 15 minutes and the long question should take about 30 minutes.

Remember to read the questions carefully and to answer all parts of all questions (some questions will have parts a,b,c, etc.). Indicate on the top of the test which questions you want to be graded – and don't forget to put your name on the test! Remember to bring a calculator to the exam – the final part of a question may require plugging in some numbers to get a real answer – and I will expect you to get the units of your answer correct! (No units, unless the answer is dimensionless, is an incorrect unit.)

Remember, your cheat sheet can have no worked problems or derivations, no pictures, and no essays (or even complete sentences). It should just have equations on it with maybe a couple of words by each one.