SIO 224

Fluid Mechanics

1. Introduction

Earlier on in the class, we derived the equations of mass and momentum conservation. Conservation of mass is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

or, equivalently

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{2}$$

Note that if $\nabla \cdot \mathbf{v} = 0$, it immediately follows that $D\rho/Dt = 0$ so that the density of a particle does not change with time. This states that the medium is *incompressible* and is a commonly used approximation in fluid mechanics.

The conservation of linear momentum is

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{b} \tag{3}$$

Where $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{b}$ is the body force density (gravity from here on out) . Previously we considered the equilibrium state of the mantle when $\mathbf{v}$ is zero and the initial stress state is one of hydrostatic pressure:

$$\mathbf{T}_0 = -p_0 \mathbf{I}$$

and where the body force is $\mathbf{g}$. This leads to an equation for p_0:

$$\frac{\partial p_0}{\partial r} = -\rho_0 g_0 \tag{4}$$

where ρ_0 is the equilibrium density field and g_0 is given by

$$g_0(r) = \frac{1}{r^2} \int_0^r 4\pi G \rho_0 x^2 \, dx \tag{5}$$

2. Newtonian Fluids

To use the conservation of linear momentum in flow problems in the Earth's mantle, we need a constitutive relationship. In previous lectures, we considered relationships between stress and strain rate for various deformation mechanisms, some of which were nonlinear. Here we will confine attention to linear beavior between stress and strain rate – this type of fluid is called a Newtonian fluid. We first write the stress tensor as the sum of an isotropic part (pressure) and a deviatoric part:

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\sigma} \tag{6}$$

In general, a Newtonian fluid has a constitutive relationship that looks like

$$\boldsymbol{\tau} = \kappa (\nabla \cdot \mathbf{v}) \mathbf{I} + 2\eta \mathbf{D} \tag{7}$$

where κ is the bulk viscosity and η is the shear velocity and $\mathbf{D}$ is the strain rate tensor ($\dot{\boldsymbol{\epsilon}}$). It is common to ignore the bulk viscosity (and the first term would be zero for an incompressible fluid anyway) and we can write the stress strain relationship as

$$\mathbf{T} = -p\mathbf{I} + 2\eta\mathbf{D} \tag{8}$$

3. Navier Stokes equation

Substituting equation 8 into equation 3 and identifying the body force density as $\mathbf{g}$ gives

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla\cdot(-p\mathbf{I} + 2\eta\mathbf{D}) + \rho\mathbf{g} = -\nabla p + 2\nabla\cdot(\eta\mathbf{D}) + \rho\mathbf{g} \tag{9}$$

Note that if the effects of the flow on density can be neglected (as in incompressible flow) then $\mathbf{g} = g_0\hat{z}$. To make further progress we shall assume that η is constant resulting in

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \eta\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{10}$$

Now divide though by ρ and introduce the *kinematic viscosity*: $\nu = \eta/\rho$ which has units of a diffusivity (m^2/s)

$$\frac{D\mathbf{v}}{Dt} == -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{g} \tag{11}$$

which is the most common form of the Navier-Stokes equation.

4. Scaling and dynamic similarity

Inspection of equation 11 suggests that velocity solutions are functions of viscosity, density, geometry, etc. It turns out that combinations of quantities control the nature of flow. A particularly important combination is the *Reynolds number*. To see how this arises, we nondimensionalize equation 11 using a characteristic velocity, V, a characteristic length scale, L (which leads to a characteristic time, $T = L/V$) and a characteristic pressure. Pressure is a force per unit area. We use as a characteristic mass $M = \rho L^3$ so pressure has dimensions

$$\rho L^3\frac{L}{T^2}\frac{1}{L^2} = \rho\frac{L^2}{T^2} = \rho V^2$$

Also, we note that the operator ∇ has units of inverse length so we multiply by L to get a dimensionless operator. Finally, in applications to the shallow regions of the earth, both ρ and g_0 are considered constant so that the hydrostatic pressure is just given by $p_0 = \rho g_0 z$. We now denote non-dimensional quantities as

$$\mathbf{v}^* = \frac{\mathbf{v}}{V} \quad \mathbf{x}^* = \frac{\mathbf{x}}{L} \quad t^* = \frac{t}{T} \quad \nabla^* = L\nabla \quad p^* = \frac{p - p_0}{\rho V^2} \tag{12}$$

Note that p^* is chosen to be the pressure difference between the actual pressure and the hydrostatic pressure where $\nabla p_0 = \rho g_0\hat{z}$. In incompressible flow, this will cancel the last term in equation 10 so p^* represents the pressure generated by flow, e.g. the dynamic pressure. The result of doing this is

$$\frac{D\mathbf{v}^*}{Dt^*} = -\nabla^* p^* + \frac{\nu}{VL}\nabla^{*2}\mathbf{v}* \tag{13}$$

The dimensionless quantity VL/ν is called the Reynolds number denoted by Re so we have

$$\frac{D\mathbf{v}^*}{Dt^*} = -\nabla^* p^* + \frac{1}{Re}\nabla^{*2}\mathbf{v}* \tag{14}$$

Note that Re is the single number that governs the nature of flow in the system. The idea of dynamic similarity is that flows with the same Re will be similar (if they have the same boundary conditions). This means that is is possible to design lab experiements that mimic a desired scenario with the correct combination of parameters making up Re. The Reynolds number is just one of many dimensionless numbers that are used to characterize flow.

Physically, Re represents the balance between inertial and viscous forces. Viscous flows are characterized by $Re < 1$, laminar flows are characterized by $Re < 2000$ and higher Re systems are characterized by turbulent flows. Given the nature of flow in the mantle where velocites are small and viscosities are high, the Re is very small and inertial forces can be neglected. Physically this means that if the physical forcing of the flow is stopped, the flow will stop. Flow in a system where inertial forces can be neglected is called Stokes flow.

There are many other non-dimensional numbers, depending on which forces are included as well as coupling to other governing equations which can produce an equivalent controlling parameter. You may have heard of some of these before, such as the Mach number (Ma), the Ekman number (Ek), the Rossby number (Ro), the Rayleigh number (Ra). Many of the dimensionless groups don't usually play a role in mantle dynamics, like the Froude number, the Weber number, or the Strouhal number, but these play an important role in other fluid dynamic processes.

5. Some simple flows

When looking at Stokes flow, the Navier-Stokes equation reduces to the Stokes equation:

$$0 = -\nabla^* p^* + \frac{1}{Re} \nabla^{*2} \mathbf{v}* \tag{15}$$

where pressure variations within the fluid balance viscous forces. We shall consider some special cases of Navier-Stokes equation in which very simplified situations can be solved analytically.

Poiseuille Flow or Pipe Flow

Pipe flow is defined to be unidirectional, i.e. there is only a single non-zero component of velocity and that component is both independent of distance in the flow direction and has the same direction everywhere. The geometry is that of a long cylindrical pipe with length l and radius a so the appropriate coordinate system is cylindrical polar (r, θ, z). The pressures at each end of the pipe are P_1 and P_0 so the pressure gradient, dP/dz, is constant everywhere in the pipe.

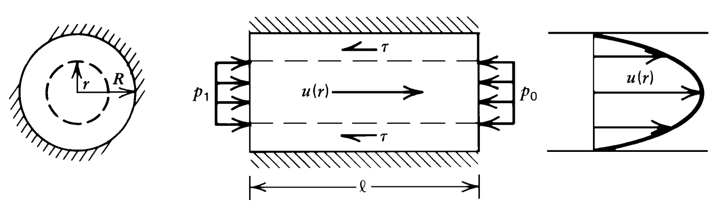


Figure 6.6 Poiseuille flow through a circular pipe.

The unidirectional nature of the problem means $v_r = 0$ and $v_\theta = 0$, thus the continuity equation is reduced to $\partial v_z/\partial z = 0$. This means that because of the incompressibility constraint, at any value of z the velocity must both be a constant value as well as have an identical velocity profile. Furthermore, any change in the flow will occur *everywhere* in the pipe *instantaneously*. However, even in the more general case of the Navier-Stokes equation that has an inertial term, $\rho\,(\partial v_z/\partial t + v_z \partial v_z/\partial z)$, one can see that for steady flow $(\partial v_z/\partial t = 0)$ the geometry of the problem and the incompressibility of the fluid specify that the inertial term is exactly zero. So Poiseuille Flow is not limited to the Stokes regime, but also occurs at higher Re and we'll see that this is important.

This 1-D version of the momentum equation in cylindrical coordinates is then

$$-\frac{dP}{dz} + \eta\nabla^2 v_z = 0$$

or

$$-\frac{dP}{dz} + \eta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = 0 \tag{16}$$

Integrating the above equation twice

$$v_z = \frac{1}{4\eta}\frac{dP}{dz}r^2 + c_1 \ln r + c_2 \tag{17}$$

where c_1 and c_2 are constants of integration. We specify boundary conditions of no-slip at the side walls and regularity of the solution everywhere – including $r = 0$ – which requries that $c_1 = 0$. The no slip condition requires that $v_z = 0$ ar $r = a$ allowing us to evaluate c_2 giving

$$v_z(r) = \frac{1}{4\eta}\frac{dP}{dz}\left(r^2 - a^2\right) \tag{18}$$

This means the velocity profile of the flow has a parabolic shape with a maximum in the center ($r = 0$) and is zero at the pipe walls. Also note that the flow is independent of the fluid density. The velocity is maximum at the center where $r = 0$ and is just

$$v_{max} = -\frac{1}{4\eta}\frac{dP}{dz}a^2 \tag{19}$$

Pressure gradients are normally defined to be negative, such that water flows from high pressure to low pressure, so when $P_1 > P_0$, v_{max} is a positive quantity. It is also useful to calculate the total flow rate through the pipe, so we integrate the velocity over the cross-section of the pipe

$$Q = \int_0^a 2\pi r v_z(r)dr = -\frac{dP}{dz}\frac{\pi a^4}{8\eta} = \frac{\pi a^4(P_1 - P_0)}{8\eta l} \tag{20}$$

The volumetric flow rate (units of volume/time or m^3/s) shows that for a given pressure gradient and viscosity, the flow through the pipe is proportional to the radius of the pipe to the fourth power. This is what Poiseuille demonstrated experimentally. The mean velocity is simply the total flow normalized by the cross-sectional area of the pipe

$$\bar{v} = -\frac{dP}{dz}\frac{\pi a^4}{8\pi a^2\eta} = -\frac{dP}{dz}\frac{a^2}{8\eta} = \frac{1}{2}v_{max} \tag{21}$$

The mean velocity is the result of the net force exerted on the fluid by the pressure gradient acting to overcome the viscous drag from the pipe walls. The force (per unit length) from pressure is

$$F_P = \pi a^2\frac{(P_0 - P_1)}{l} = -\pi a^2\frac{dP}{dz} \tag{22}$$

This shows that the mean flow, $\bar{v}$, is related to the pressure force by $\bar{v} = F_P/(8\pi\eta)$ and so it is *linearly* inversely proportional to η. Similarly, for a Newtonian fluid, viscous drag is proportional to the shear (tangential) stress, σ_{zr}, which we can evaluate near the wall of the pipe, $r = a$

$$\sigma_{zr}|_{r=a} = \eta\frac{\partial v_z}{\partial r}|_{r=a} = \eta\left(\frac{-a}{2\eta}\frac{dP}{dz}\right) = -\frac{a}{2}\frac{dP}{dz} \tag{23}$$

We can determine a friction factor, f, which describes the effect of drag. We use the shear stress evaluated at the wall as a characteristic stress and normalize that value by a characteristic pressure ($\frac{1}{2}\,\rho_f\bar{v}^2$) in which we use the mean velocity:

$$f = \frac{\sigma_{zr}|_{r=a}}{\frac{1}{2}\rho_f \bar{v}^2} = -\frac{4a}{\rho_f \bar{v}^2}\frac{dP}{dz} \tag{24}$$

If we substitute for just one of the $\bar{v}$, then f looks like

$$f = -\frac{4a}{\rho_f \bar{v}}\frac{1}{\bar{v}}\frac{dP}{dz} = -\frac{4a}{\rho_f \bar{v}}\frac{8\eta}{a^2\,(-dP/dz)}\frac{dP}{dz} = \frac{32\eta}{\rho_f \bar{v} a} \tag{25}$$

If we choose a characteristic length scale as the diameter of the pipe, $D = 2a$, then we have

$$f = \frac{64\eta}{\rho_f \bar{v} D} = \frac{64}{Re} \tag{26}$$

This relationship holds until the transition into the turbulent flow regime at $Re \sim 2000 - 3000$.

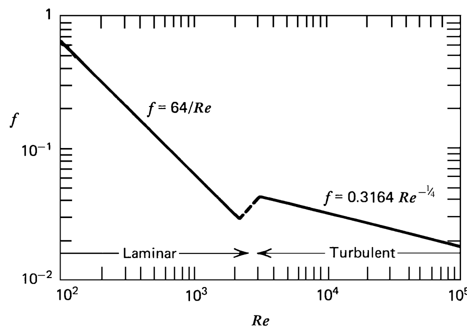


Figure 6.7 Dependence of the friction factor f on the Reynolds number Re for laminar flow, from Equation (6–41), and for turbulent flow, from Equation (6–42).

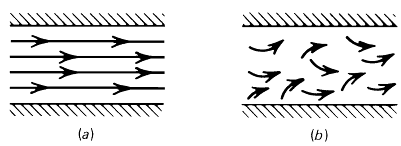


Figure 6.8 Illustration of the difference between (a) laminar and (b) turbulent flow. Laminar flow is steady, and the fluid flows parallel to the walls: lateral transport of momentum takes place on a molecular scale. Turbulent flow is unsteady and has many time-dependent eddies and swirls. These eddies are much more effective in the lateral transport of momentum than are molecular processes. Therefore, the friction factor (pressure drop) in turbulent flow is larger at a prescribed Reynolds number (flow velocity).

Channel Flow

Another unidirectional flow is the flow between two rigid plates driven by a pressure gradient. This is actually just Poiseuille flow in Cartesian geometry (with $\hat{z}$ the same direction as in cylindrical polar geometry) so the pressure gradient and resultant flow are both only in the x direction ($v_y = v_z = 0$) and the velocity profile varies with z. The geometry has the x-axis along the mid-plane of the channel, and since the channel has height h, the channel walls are at $\pm h/2$. As the flow is deemed incompressible,the continuity equation gives $\partial v_x/\partial x = 0$ and the Navier-Stokes equation is

$$-\frac{dP}{dx} + \eta\nabla^2 v_x = -\frac{dP}{dx} + \eta\frac{\partial^2 v_x}{\partial z^2} = 0 \tag{27}$$

Again, since dP/dx is constant, the integration is straightforward and integrating twice gives

$$v_x = \frac{1}{2\eta}\frac{dP}{dx}z^2 + c_1 z + c_2 \tag{28}$$

where the c's are constants of integration. The boundary conditions are from the mirror symmetry along the mid-plane ($v_x(z) = v_x(-z)$) and no-slip at the walls ($v_x(z = \pm h/2) = 0$). The mirror symmetry forces c_1 to be zero and c_2 is determined by the no slip condition. We end up with

$$v_x = \frac{1}{2\eta}\frac{dP}{dx}\left[z^2 - (h/2)^2\right] \tag{29}$$

The velocity profile is again parabolic in shape and constant everywhere. All the same insights from Poiseuille flow in a pipe are applicable here.

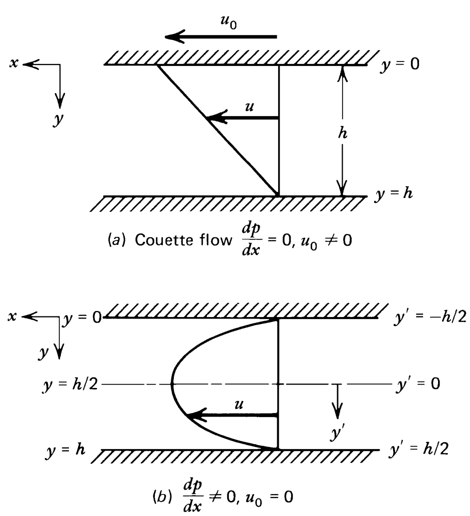


Figure 6.2 One-dimensional channel flows of a constant viscosity fluid.

Couette flow

Couette flow is similar to channel flow and has the same geometry but with an important modification. Instead of the pressure gradient driving the flow, it is driven by the motion of one of the boundaries and that motion is parallel to the direction of the channel (dP/dx is in fact absent from this problem). The assumption is that some external force is applied to move the wall and that applied force simply scales with the viscosity of the fluid. Depending on the reference frame you choose to do the problem in, the top or bottom plate can be moving at some velocity (V_0) or they can both move in opposite directions at ($V_0/2$). The most convenient choice for the coordinate system is to have a stationary plate at $z = 0$ and a moving plate at $z = h$ so again the channel has height h. The governing equations for a shear driven flow are even simpler than for channel flow since now $dP/dx = 0$. Again, from the assumption of incompressiblity, we have $\partial v_x/\partial x = 0$ and the Navier-Stokes equation becomes

$$\eta\nabla^2 v_x = \eta\frac{\partial^2 v_x}{\partial z^2} = 0 \tag{30}$$

Integrating twice gives the solution $v_x = c_1 z + c_2$. The boundary conditions are again no-slip velocity boundary conditions at the stationary and moving walls, so $v_x(z = 0) = 0$ and $v_x(z = h) = V_0$. c_2 is zero so the final solution is

$$v_x = V_0\frac{z}{h} \tag{31}$$

and the velocity profile is linear across the channel. The velocity profile in a shear driven flow is again identical for all values of x, varies linearly with distance from the moving wall, and is independent of both density and viscosity. Also note that the shear stress is also constant everywhere because

$$\sigma_{xz} = \eta \frac{\partial v_x}{\partial z} = \eta \frac{V_0}{h} \tag{32}$$

6. Classification of PDEs and types of Boundary Conditions

Any PDE can be classified using the method of characteristics which determines if the PDE is either hyperbolic, elliptic, or parabolic. Both Laplace's equation and Poisson's equation are classified as elliptical, and is a common class of equation one encounters in fluid dynamics. Other examples include the wave equation (hyperbolic) and the diffusion equation (parabolic). It is important to understand which class of equation you are attempting to solve, in particular if you are using numerical methods, because the stability or success of the numerical method applied to one class of equation may be a completely unstable or be an unsuccessful approach if applied to a different class of PDE.

The primary variable is the variable in the governing equation (either PDE or ODE) and every primary variable always has an associated secondary variable. The secondary variable is usually the derivative of the primary variable and always has a physical meaning that is often a quantity of interest. In fluid dynamics the primary variable is velocity and the secondary variable is stress. Another example is heat transfer in which the primary variable is temperature and the secondary variable is heat flux.

In order to obtain a solution to any PDE, boundary conditions must be specified. There are two types of boundary conditions that can be applied: those that specify the primary dependent variable on the boundary and those that specify a secondary variable on the boundary, and usually the derivative is taken normal to the boundary. The first type of boundary condition is called an *essential boundary condition* and when solving an elliptic class of equation it is known as a Dirichlet boundary condition. The second type of boundary condition is called a *natural boundary condition* and when solving an elliptic class of equation it is known as a Neumann boundary condition. It is quite ok, and even somewhat common, to have mixed types of boundary conditions along different parts of the boundary. For example, one portion of the boundary will specify a Dirichlet boundary condition and another portion will specify a Neumann boundary condition. However, it is impossible to specify both types of conditions at the same point of any portion of the boundary.

This is actually quite a powerful, and useful, thing to know, especially in situations like Couette flow and channel flow, which have the same geometry. It is actually possible to combine the simple solutions from both problems because 1) they are both linear ODEs we can use the principle of superposition and 2) the solutions were arrived upon by applying the *same type* of boundary condition. Both problems specified the velocity on the walls and therefore both applied Dirichlet boundary conditions. We can then write the solution of Couette flow that now includes a pressure gradient by simply transforming the channel flow solution to a coordinate system with the bottom wall at $z = 0$ (see equation 28)

$$v_x = V_0 \frac{z}{h} + \frac{1}{2\eta} \frac{dP}{dx} \left[z^2 - hz \right] \tag{33}$$

A simple model of asthenospheric counterflow is motivated by a shear flow driven by plate motions on the surface. The shear flow sets up a pressure gradient in the the opposite direction which drives an associated channel flow underneath the shear flow (a return flow). This is the same as the above problem, except the direction of the pressure gradient is reversed

$$v_x = V_0 \frac{z}{h} - \frac{1}{2\eta} \frac{dP}{dx} \left[z^2 - hz \right] \tag{34}$$

It is interesting to note that Turcotte and Schubert show that confining the return flow to the asthenosphere requires the sea floor to rise as you move away from a ridge to provide the correct pressures. Clearly, much of the return flow must be substantially deeper.

7. Viscous flow past a sphere

The most famous application of Stokes flow is that of viscous flow around a sphere. In a laboratory reference frame, the sphere sinks through a viscous fluid and this is actually the fluid dynamics inside a viscometer which is an instrument used to measure viscosity. Recently, such viscosity experiments have actually been done at high pressure in a multi-anvil device.

The solution to the problem of a sinking Stokes sphere is done numerous places (basically every book on fluid dynamics that exists). We begin with the dimensional form of the Stokes equation in spherical polar geometry, with the coordinate system that has $\theta = 180°$ in the flow direction, i.e. the fluid approaches the sphere from $z = \infty$ with velocity $-V_0$ in the z-direction. The problem is solved in the reference frame of the sphere (so flow is moving past the sphere) and the sphere has radius a. The problem has an azimuthal symmetry such that v_ϕ=0 and $\partial/\partial\phi$ = 0

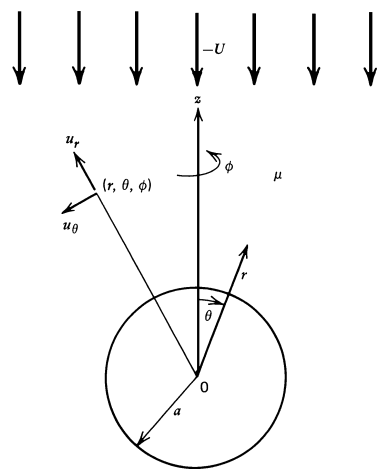


Figure 6.31 Steady flow of a viscous fluid past a sphere.

In spherical geometry, the governing equations are:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) = 0$$
$$-\frac{\partial P}{\partial r} + \eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) - \frac{2}{r^2}v_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)\right] = 0 \qquad (35)$$
$$-\frac{1}{r}\frac{\partial P}{\partial\theta} + \eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\theta}{\partial\theta}\right) + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{1}{r^2\sin^2\theta}v_\theta\right] = 0$$

We can solve these equations subject to these 4 boundary conditions: the no-slip velocity boundary conditions ($v_r = 0$ and $v_\theta = 0$ at $r = a$), and the far-field boundary conditions ($v_r \to -V_0\cos\theta$ as $r \to \infty$ and $v_\theta \to V_0\sin\theta$ as $r \to \infty$). This is one of those systems of PDE's that is obvious how to solve it when someone else has already found the solution. According to Turcotte and Schubert, the nature of the boundary conditions suggests that the solution is of the form

$$v_r = f(r)\cos\theta \quad \text{and} \quad v_\theta = g(r)\sin\theta \qquad (36)$$

Substituting these functions into the governing equations we obtain

$$
\begin{aligned}
&-\frac{1}{2r}\frac{d}{dr}\left(r^2 f\right) = g \\
&-\frac{\partial P}{\partial r} + \frac{\eta\cos\theta}{r^2}\left[\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 4(f+g)\right] = 0 \\
&-\frac{\partial P}{\partial \theta} + \frac{\eta\sin\theta}{r}\left[\frac{d}{dr}\left(r^2\frac{dg}{dr}\right) - 2(f+g)\right] = 0
\end{aligned} \tag{37}
$$

To elliminate the pressure, we apply the $\partial/\partial\theta$ and $\partial/\partial r$ derivatives to the second two of these equations respectively, and then subtract. This gives

$$
\begin{aligned}
&-\frac{1}{2r}\frac{d}{dr}\left(r^2 f\right) = g \\
&\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{4(f+g)}{r^2} + \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r^2\frac{dg}{dr}\right) - \frac{2(f+g)}{r}\right] = 0
\end{aligned} \tag{38}
$$

The solutions of functions f and g can be found by assuming simple powers of r

$$
f = cr^n \tag{39}
$$

where c is a constant. Substituting this into the continuity equation (the first of 38) gives

$$
g = -\frac{c(n+2)}{2}r^n \tag{40}
$$

Now the functions f and g can be substituted into the remaining momentum equation (the second of 38) and it produces a simple algebraic expression which has several roots for n

$$
n(n+3)(n-2)(n+1) = 0 \quad \text{which gives} \quad n = 0, -3, 2, -1 \tag{41}
$$

This gives the full description for the linear combinations of $f(r)$ and $g(r)$ using the values of n above

$$
\begin{aligned}
f =& c_1 + \frac{c_2}{r^3} + \frac{c_3}{r} + c_4 r^2 \\
g =& -c_1 + \frac{c_2}{2r^3} - \frac{c_3}{2r} - 2c_4 r^2
\end{aligned} \tag{42}
$$

These can be substituted into the expressions for velocity to give

$$
\begin{aligned}
v_r =& (c_1 + \frac{c_2}{r^3} + \frac{c_3}{r} + c_4 r^2)\cos\theta \\
v_\theta =& (-c_1 + \frac{c_2}{2r^3} - \frac{c_3}{2r} - 2c_4 r^2)\sin\theta
\end{aligned} \tag{43}
$$

We can apply the boundary conditions to solve for the constants. Applying the far field velocity boundary conditions gives

$$
c_1 = -V_0 \quad \text{and} \quad c_4 = 0 \tag{44}
$$

Applying the no-slip condition at $r = a$ gives

$$
c_2 = -\frac{a^3 V_0}{2} \quad \text{and} \quad c_3 = \frac{3aV_0}{2} \tag{45}
$$

This gives the final expressions for the velocity components

$$
\begin{aligned}
v_r =& -V_0\left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right)\cos\theta \\
v_\theta =& V_0\left(1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right)\sin\theta
\end{aligned} \tag{46}
$$

These can be substituted back into the original equation for momentum in the θ direction and integrating with respect to θ

$$P(\theta) = \frac{3\eta a V_0}{2r^2}\cos\theta \tag{47}$$

The solution for flow is now given as both components of velocity as well as pressure have been solved for. Stokes flow is a balance of viscous forces and pressure and the net effect of these forces describes the amount of drag the sphere has with respect to the surrounding flow. Since we know the solution to the flow, we can calculate these forces and determine the drag on the sphere. There are two contributions to the drag, one from pressure and one from viscous stresses.

$$D = D_P + D_v \tag{48}$$

In order to calculate the contribution from pressure, we need the component of the force in the direction that pressure is pushing on the sphere. This equates to the vertical component of the pressure in the negative z direction projected on the surface of the sphere at radius a, or just

$$P\cos\theta = \frac{3\eta V_0}{2a}\cos^2\theta \tag{49}$$

We need to integrate this pressure over the surface of the sphere, but since it only acts on the cross-sectional area of the sphere ($\pi a^2 \sin\theta$) we have

$$D_P = \int_0^\pi (P\cos\theta)\, 2\pi a^2 \sin\theta d\theta = 3\pi\eta a V_0 \int_0^\pi \sin\theta\cos^2\theta d\theta = 2\pi\eta a V_0 \tag{50}$$

The viscous contribution to the drag has two components, one from the normal stresses and one from the tangential stress, so we need to apply the constitutive relation ($\sigma = \eta\dot{\epsilon}$) using the strain rates in spherical polar coordinates

$$\begin{aligned} (\sigma_{rr})_{r=a} &= 2\eta\left(\frac{\partial v_r}{\partial r}\right)_{r=a} \\ (\sigma_{r\theta})_{r=a} &= \eta\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)_{r=a} \end{aligned} \tag{51}$$

These are easily found by substituting in the solutions for the velocity components

$$(\sigma_{rr})_{r=a} = 0 \quad \text{and} \quad (\sigma_{r\theta})_{r=a} = \frac{3\eta V_0 \sin\theta}{2a} \tag{52}$$

There are no normal stresses because the boundary conditions define the sphere to be rigid due to the boundary condition. It is a property of incompressible fluid that the deviatoric stress acting across a rigid boundary is wholly tangential. The tangential stress is in the θ direction all along the sphere, but we need the component in the negative z direction so use the $\sin\theta$ projection

$$\sigma_{r\theta}\sin\theta = \frac{3\eta V_0 \sin^2\theta}{2a} \tag{53}$$

Once again, integrate the product of this quantity with the cross-sectional area of the sphere

$$D_v = \int_0^\pi \left(\frac{3\eta V_0\sin^2\theta}{2a}\right) 2\pi a^2 \sin\theta d\theta = 3\pi\eta a V_0 \int_0^\pi \sin^3\theta d\theta = 4\pi\eta a V_0 \tag{54}$$

Notice that the contribution to drag from viscous stresses is exactly double the contribution from pressure forces. It is more common to report the drag coefficient, c_D, defined the total drag normalized by both a characteristic pressure ($\frac{1}{2}\rho_f V_0^2$) and cross-sectional area of the sphere (πa^2)

$$c_D \equiv \frac{D}{\frac{1}{2}\rho V_0^2 \pi a^2} = \frac{D_P + D_v}{\frac{1}{2}\rho V_0^2 \pi a^2} = \frac{6\pi\eta a V_0}{\frac{1}{2}\rho V_0^2 \pi a^2} = \tfrac{1}{2}\left(\rho V_0 a\right)/\eta = \frac{24}{Re} \tag{55}$$

Notice that the Reynolds number appears in the denominator. These sinking sphere experiments can be done at various Re and it is very striking that the predicted Stokes drag coefficient holds remarkably well up until $Re \sim 1$ when inertial effects begin to become important.

The final thing that is useful to do is calculate the terminal velocity of the sphere. As the sphere can be rising or sinking, it has many applications in geological fluid dynamics such as settling of crystals in a magma, or rise of a plume head in the mantle. Archimedes principle describes the buoyancy force of an object as the density contrast with respect to a background fluid, in this case a rising sphere

$$F = (\rho_f - \rho_s) g \left(\frac{4}{3}\pi a^3\right) \tag{56}$$

and by setting this force equal to the drag force, the terminal upward velocity is obtained.

$$v_{term} = \frac{2a^2(\rho_f - \rho_s)g}{9\eta} \tag{57}$$

It is important to recognize that the velocity depends on the radius squared.

8. The Stream Function

The stream function, ψ, is both an illustrative and useful approach to apply to fluid dynamics as it can provide relatively quick solutions to 2-D incompressible flow problems. The major drawback of the stream function is that it is basically limited entirely to 2-D incompressible flow problems. The stream function is like a potential field in that only the difference in ψ between two points has any physical meaning (the absolute value of ψ is arbitrary). Lines of constant ψ are called stream lines and give an excellent visual representation of the flow, however, only in a 2-D geometry. In 2-D, the incompressibility constraint is

$$\nabla \cdot \mathbf{v} = 0 \quad \text{or} \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{58}$$

The definition of the stream function is

$$v_x = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = \frac{\partial \psi}{\partial x} \tag{59}$$

Obviously, the stream function satisfies the continuity equation since

$$\frac{-\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0 \tag{60}$$

The stream function can also be substituted into the Stokes equation

$$\begin{aligned} 0 =& \frac{dP}{dx} + \eta\left(\frac{\partial^3 \psi}{\partial^2 x \partial y} + \frac{\partial^3 \psi}{\partial y^3}\right) \\ 0 =& -\frac{dP}{dy} + \eta\left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial y^2 \partial x}\right) \end{aligned} \tag{61}$$

Now eliminate the pressure term using the same technique that was applied earlier when solving for the flow around a Stokes sphere, i.e. take partial derivatives w.r.t. the other dimension

$$\begin{aligned} 0 =& \frac{\partial}{\partial y}\left[\frac{dP}{dx} + \eta\left(\frac{\partial^3 \psi}{\partial^2 x \partial y} + \frac{\partial^3 \psi}{\partial y^3}\right)\right] \\ 0 =& \frac{\partial}{\partial x}\left[-\frac{dP}{dy} + \eta\left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial y^2 \partial x}\right)\right] \end{aligned} \tag{62}$$

then subtract the resulting equations:

$$0 = \frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} \tag{63}$$

Rearranging the derivatives we now have

$$0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi \tag{64}$$

This equation can be recognized as the Laplacian operator (∇^2) being applied twice to ψ

$$0 = \left(\nabla^2\right)\left(\nabla^2\right)\psi \tag{65}$$

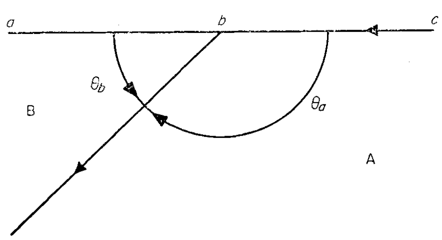
This is known as the Biharmonic operator ($\left(\nabla^2\right)^2 = \nabla^4$) which we can use to write

$$0 = \nabla^4 \psi \tag{66}$$

There are well-known solutions to this equation and it is also valid for non-Cartesian geometries.

9. Corner Flow

The situation of a subduction zone is in some ways analogous to one variation of the classic corner flow problem in fluid dynamics. In this version, two rigid plates (infinite in extent) converge at a point where the advancing plate (plate A) dips at an angle below the back-arc plate (plate B). We will use the point of convergence as the origin of a 2-D cylindrical coordinate system with plates on the surface (the line at $\theta = 0$). The angle that plate A makes between itself on the surface and the dipping portion is defined as θ_a and the "'dip angle" between plates A and B is defined as θ_b (and assumed to be acute). Plate B is assumed to remain stationary while plate A is moving on the surface at velocity $v_r = -V_0$ (towards the origin) and along the dip angle at $v_r = V_0$ (away from the origin). For both plates, the velocities in the θ direction are assumed to be zero ($v_\theta = 0$). Notice that there are no body forces in this problem, and that the Stokes flow is driven entirely by the velocity boundary conditions (which themselves are driven by some applied force but since it is not a body force it is irrelevant).



The governing equations for Stokes flow are simply $\nabla \cdot \boldsymbol{\tau} = 0$ and $\nabla \cdot \mathbf{v} = 0$. Expanding the momentum equation out into the components of total stress

$$\begin{aligned} \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} &= 0 \end{aligned} \tag{67}$$

We can use the constitutive relationship between total stress and strain rate for an incompressible fluid, $\tau = -P\mathbf{I} + 2\eta\mathbf{D}$

$$
\begin{aligned}
\tau_{rr} =& - P + \sigma_{rr} = -P + 2\eta\dot{\epsilon}_{rr} \\
\tau_{\theta\theta} =& - P + \sigma_{\theta\theta} = -P + 2\eta\dot{\epsilon}_{\theta\theta} \\
\tau_{r\theta}(=&\tau_{\theta r}) = \sigma_{r\theta} = 2\eta\dot{\epsilon}_{r\theta}
\end{aligned} \tag{68}
$$

Now rewrite the total stress with the strain rate in terms of the velocity gradients

$$
\begin{aligned}
\tau_{rr} =& - P + 2\eta\frac{\partial v_r}{\partial r} \\
\tau_{\theta\theta} =& - P + 2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \\
\tau_{r\theta} =& \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)
\end{aligned} \tag{69}
$$

Notice that if we add the normal components of stress together we get

$$
\tau_{rr} + \tau_{\theta\theta} = -2P + 2\eta\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \tag{70}
$$

The 2^{nd} term on the RHS vanishes since $\nabla \cdot \mathbf{v} = 0$, and, because the fluid is isotropic, the expressions for the normal stresses become

$$
\tau_{rr} = -P \quad \text{and} \quad \tau_{\theta\theta} = -P \tag{71}
$$

Using these allows us to express the momentum equation entirely in terms of P and $\tau_{r\theta}$

$$
\begin{aligned}
-\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{r\theta} = 0 \\
-\frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{\partial}{\partial r}\tau_{r\theta} = 0
\end{aligned} \tag{72}
$$

and this can be rewritten in terms of velocity gradients as

$$
\begin{aligned}
-\frac{\partial P}{\partial r} + \eta\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) = 0 \\
-\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) = 0
\end{aligned} \tag{73}
$$

The equations 69 and 71 put severe constraints on any allowed velocity function. To proceed, we use the stream function where, in cylindrical coordinates

$$
v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = -\frac{\partial \psi}{\partial r} \tag{74}
$$

The geometry of our problem suggests that the solution for velocity may be independent of r. To test this, we write the stream function as

$$
\psi = R(r)T(\theta) \tag{75}
$$

and calculate $\partial v_r/\partial r$ which, by the first of equation 69, must be zero (the second of equation 69 is then automatically satisfied). This is only achieved if $R(r) \propto r$ which results in both v_r and v_θ being independent of r as expected. In 2-D cylindrical coordinates, the Laplacian is

$$
\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \tag{76}
$$

Using this result (twice), the substitution for ψ into the biharmonic equation with $R \propto r$ results in

$$\frac{d^4T}{d\theta^4} + 2\frac{d^2T}{d\theta^2} + T = 0 \tag{77}$$

Our problem has now been reduced to a 4^{th} order ODE which has a general solution of the form

$$T(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta \tag{78}$$

We can solve this as there are 4 boundary conditions but these are given as velocities so we need v_r and v_θ

$$\begin{aligned} v_r =& \frac{\partial T(\theta)}{\partial\theta} = A\cos\theta - B\sin\theta + C(\sin\theta + \theta\cos\theta) + D(\cos\theta - \theta\sin\theta) \\ v_\theta =& -T(\theta) \end{aligned} \tag{79}$$

where we have absorbed the scaling in R into the constants. At this point its a good idea to break the problem into two portions and solve for the stream function in each domain. The obvious choice for the two domains is the "back-arc region" formed by the (acute) dip angle between the subducting plate and overriding plate and the "fore-arc region" underneath the subducting plate. The flows are identical along the boundary of the subducting plate, and this line is known as the separatrix. The boundary conditions are then

$$\begin{aligned} v_r(\theta = 0) =& -V_0 \quad \text{in the fore-arc region} \\ v_r(\theta = 0) =& 0 \quad \text{in the back-arc region} \\ v_\theta(\theta = 0) =& 0 \quad \text{in both regions} \\ v_\theta(\theta = \theta_b) =& 0 \quad \text{along the separatrix} \\ v_r(\theta = \theta_b) =& V_0 \quad \text{along the separatrix} \end{aligned} \tag{80}$$

Each region has 4 boundary conditions to solve for the 4 unknowns constants, and after a lot of algebra one arrives at the solution

$$\begin{aligned} \psi_a =& -rV_0\frac{[(\theta_a - \theta)\,\sin\theta - \theta\sin\,(\theta_a - \theta)]}{\theta_a + \sin\theta_a} \equiv -rV_0 f_a(\theta) \quad \text{in the fore-arc region} \\ \psi_b =& rV_0\frac{[(\theta_b - \theta)\,\sin\theta_b\sin\theta - \theta_b\theta\sin\,(\theta_b - \theta)]}{\theta_b^2 - \sin^2\theta_b} \equiv rV_0 f_b(\theta) \quad \text{in the back-arc region} \end{aligned} \tag{81}$$

The velocities in each region are readily obtained through differentiation of ψ: $v_r = -V_0 f_a'(\theta)$ and $v_\theta = V_0 f_a(\theta)$ in the fore arc and $v_r = V_0 f_b'(\theta)$ and $v_\theta = -V_0 f_b(\theta)$ in the back arc.

What we are really interested in is the differential pressure between the top and the bottom of the slab which can cause the slab to "float". The situation is illustrated in the following figure

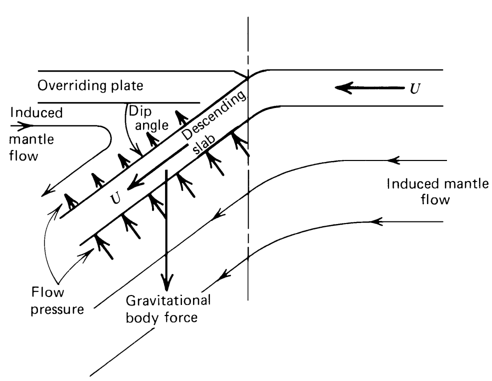


Figure 6.17 Forces acting on a descending lithosphere.

In order to obtain the pressure, we need to go back to the momentum equation and use the fact that pressure is related through equation 72 to the shear (tangential) stress which is given by

$$\begin{aligned} \tau_{r\theta} =& \eta \frac{V_0}{r} \left[-f_a''(\theta) - f_a(\theta) \right] \quad \text{in the fore-arc region} \\ \tau_{r\theta} =& \eta \frac{V_0}{r} \left[f_b''(\theta) + f_b(\theta) \right] \quad \text{in the back-arc region} \end{aligned} \tag{82}$$

Integrating the second of equation 72 allows us to obtain the pressure solution

$$\begin{aligned} P_a(r,\theta) =& \frac{2V_0\eta}{r} \frac{[\sin\theta - \sin(\theta - \theta_a)]}{\theta_a + \sin\theta_a} \quad \text{in the fore-arc region} \\ P_b(r,\theta) =& -\frac{2V_0\eta}{r} \frac{[\theta_b \sin(\theta_b - \theta) - \sin\theta_b \sin\theta]}{\theta_b^2 - \sin^2\theta_b} \quad \text{in the back-arc region} \end{aligned} \tag{83}$$

Inspection of these solutions reveals that P_a is always a positive quantity and P_b is always a negative quantity. A positive pressure below the subducting plate implies compression or upward force on the surface. A negative pressure in the mantle wedge indicates that there is a suction between the subducting plate and overriding plate. This corner flow suction acts as a hydrodynamic lift that is proportional to the pressure difference above and below the slab. The lift is found by integrating $P(r,\theta)$ along the dip angle, θ_a, over a length l. The torque exerted by lift is balanced by gravity through the weight of the slab with thickness h and density $\Delta\rho$.

$$\begin{aligned} T_{flow} =& \int_0^l [P_a(r,\theta_a) - P_b(r,\theta_b)]\, r\, dr = 2V_0\eta l \left[\frac{\sin\theta_b}{(\pi - \theta_b) + \sin\theta_b} + \frac{\sin^2\theta_b}{\theta_b^2 - \sin^2\theta_b} \right] \\ T_{gravity} =& \frac{1}{2}\, \Delta\rho g h l^2 \cos\theta_b \end{aligned} \tag{84}$$

Both torques can be normalized by a characteristic torque, $2V_0\eta l$, which then allows one to find the critical dip angle, θ_c that determines when the torque derived from gravity is balanced by the lift generated by circulation in the mantle wedge. For any angle smaller than θ_c, the torque exerted on the slab by mantle flow will exceed the weight of the slab, and assuming the velocities remain constant, a positive feedback will occur such that θ decreases to zero. This critical angle was determined by *Stevenson and Turner, (1977),* to be 63° for which they found the net torque was about 2 times the characteristic torque. Assuming a 100 km thick slab that is 600 km in length subducts at 6 cm/yr and has $\Delta\rho$ =80 kg/m^3 gives $2 \sim \Delta\rho g h l/(4\eta V_0)$ which can be used to estimate the upper mantle viscosity: $\eta = \pi 10^{21}$ Pa s.

Clearly, $\theta_c = 63°$ is too large as many slabs are observed to have dip angles shallower than this estimate, so obviously there must be many other important factors. One of the more important factors is the non-Newtonian rheology of the mantle wedge as studied by *Tovish et al. (1978)* who found this reduced $\theta_c = 54°$ for a power law fluid with n=3. There are also reasons for θ_c to be larger, as slabs with finite lateral extent allow for a 3-D component of the mantle flow (i.e the toroidal flow) around slab edges which reduces the pressure differential *Dvorkin et al. (1993).*

10. Postglacial Rebound

We can get important information about the fluid behavior of the mantle by looking at its response to loading and unloading. Of course, mountains are an example of a load, but mountain building takes so long that there really is no dynamic response. On the other hand, loading of the Earth by ice sheets followed by unloading during rapid ice sheet melting results in a dynamic response with a time-scale of thousands of years, one that is sensitive to the viscous properties of the (upper) mantle. This process is shown in the following figure

For example, during the last ice-age, Scandinavia was covered with a thick ice sheet which depressed the surface causing mantle material to flow. When the ice sheet melted about 10,000 years ago, the surface rebounded. The rate of rebound has been found by dating elevated beaches.

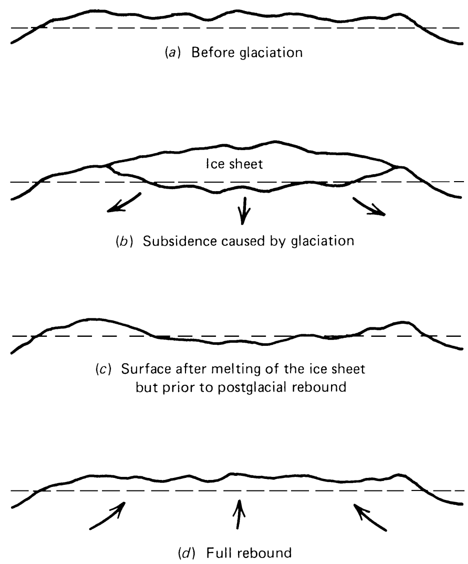


Figure 6.14 Subsidence due to glaciation and the subsequent postglacial rebound.

To get an idea of how rebound data can constrain the viscosity of the mantle, we consider a much simplified model: a semi-infinite viscous fluid half-space subject to an initial periodic surface displacement:

$$w = w_m \cos\left(\frac{2\pi x}{\lambda}\right) \tag{85}$$

where λ is the wavelength and $w_m \ll \lambda$. The displacement of the surface leads to a horizontal pressure gradient due to the hydrostatic load. When the surface is displaced upward (negative), the pressure is positive and fluid is driven away from this region. When the surface is displaced down (positive), the pressure is negative. This corresponds to the case whan a load has been removed.

We can solve this problem by considering the biharmonic equation. Since the initial displacement has a consinusoidal dependence, it is reasonable to anticipate that the ψ must behave similarly but perhaps with some phase lag, i.e., it is not clear if $\psi \propto \cos(2\pi x/\lambda)$ or $\sin(2\pi x/\lambda)$ or some combination. It turns out the the behavior is sinusoidal so we shall assume this to be the case. We can now apply the method of separation of variables to the biharmonic equation using a trial form of

$$\psi = \sin\left(\frac{2\pi x}{\lambda}\right) M(z) \tag{86}$$

Substituting into the biharmonic equation results in

$$\frac{d^4 M}{dz^4} - 2k^2 \frac{d^2 M}{dz^2} + k^4 M = 0 \tag{87}$$

where $k = 2\pi/\lambda$. Solutions of a constant coefficient ODE are of the form $M \propto \exp(mz)$ and substitution gives

$$m^4 - 2k^2 m^2 + k^4 = (m^2 - k^2)^2 = 0 \tag{88}$$

so

$$m = \pm k \tag{89}$$

These two solutions are incomplete and the two additional solutions needed are of the form $z \exp(\pm mz)$ (see also equation 78) so the general solution for ψ is

$$\psi = \sin(kx)\left[A\exp(-kz) + Bz\exp(-kz) + C\exp(kz) + Dz\exp(kz)\right] \tag{90}$$

If we require that the solution remains finite as $z \to \infty$ then $C = D = 0$ so ψ becomes

$$\psi = \sin(kx)e^{-kz}\left[A + Bz\right] \tag{91}$$

The velocities can be determined from equation 59:

$$\begin{aligned} v_x =& \sin(kx)e^{-kz}\left[k(A+Bz) - B\right] \\ v_z =& k\cos(kx)e^{-kz}\left[A + Bz\right] \end{aligned} \tag{92}$$

To mimic the fact that the part of the mantle that behaves like a fluid is overlain by a rigid lithosphere, it is appropriate to force v_x to be zero at the surface (actually, we should apply the no slip bounday condition at the perturbed boundary but we are assuming that the vertical displacement of the surface is small so, to first order, we can apply this at $z = 0$). This means that $B = kA$. So now we have

$$\begin{aligned} \psi =& A\sin(kx)e^{-kz}\left[1 + kz\right] \\ v_x =& A\sin(kx)e^{-kz}k^2 z \\ v_z =& Ak\cos(kx)e^{-kz}\left[1 + kz\right] \end{aligned} \tag{93}$$

In order to evaluate the final constant A, we need to relate the hydostatic pressure head to the normal stress at the top boundary. Thus

$$\sigma_{zz}(z=0) = -\rho g w = P - 2\eta\frac{\partial v_z}{\partial z} \tag{94}$$

Consider the horizontal force balance:

$$0 = -\frac{\partial P}{\partial x} + \eta\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}\right] \tag{95}$$

Using equation 93 for v_x and integrating the result wrt x gives the pressure on $z = 0$:

$$P = 2\eta A k^2 \cos(kx) \tag{96}$$

Equation 93 for v_z can be differentiated wrt z and then evaluated which gives $\partial v_z/\partial z = 0$ at $z = 0$ Thus equation 94 simplifies to

$$w(z=0) = -\frac{2\eta A}{\rho g}k^2\cos(kx) \tag{96}$$

Note that the vertical component of velocity at the surface is just $\partial w/\partial t$ (we should evaluate at $z = w$ but we can linearize and identify this with v_z evaluated at $z = 0$). Using equation 93 gives

$$v_z(z=0) \simeq \frac{\partial w}{\partial t}(z=0) = Ak\cos(kx) \tag{97}$$

Combining this with equation 96 gives at $z = 0$

$$\frac{\partial w}{\partial t} = -w\frac{\lambda\rho g}{4\pi\eta} \tag{98}$$

This can be integrated with the initial condition that $w = w_m$ at $t = 0$ to give

$$w = w_m e^{-t/\tau} \tag{99}$$

where

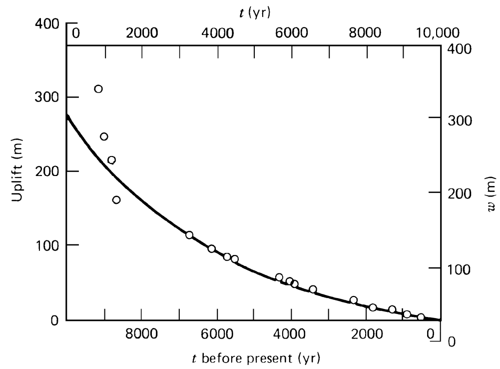


Figure 6.16 Uplift of the mouth of the Angerman River, Sweden, as a function of time before the present compared with the exponential relaxation model, Equation (6–104), for $w_{m_0} = 300$ m less 30 m of uplift yet to occur, $\tau_r = 4400$ years, and an initiation of the uplift 10,000 years ago.

$$\tau = \frac{4\pi\eta}{\rho g \lambda} \tag{100}$$

Once the relaxation time τ can be estimated from observations, we can find out what the viscosity is. Applying this to elevated beach terraces in Scandinavia gives the result shown in the figure above. This gives an initiation of uplift 10,000 years ago and a relaxation time of 4400 years. With an initial depression of 300m, there are less than 30 m of uplift yet to occur. A reasonable value for the wavelength of deglaciation in Fennoscandia is 3000 km. Using equation 100, these data give $\eta = 1.1 \times 10^{21}$ Pa s. The value of 10^{21} Pa s is sometimes called the "Haskell" value after an early paper on this subject (Haskell, 1935). This value is an average of a range of depths. Modern studes of PGR take account of the history and geometry of the ice load (as well as we know it) , they may also need to take account of the sea-level change associated with melting of ice sheets globally. This requires numerical modelling of the entire system.

11. Rayleigh Taylor Instability

Some of the most interesting natural phenomena of fluid dynamics are waves and instabilities. Among these is the classic instability at an interface of two fluids (ρ_1 and ρ_2) which are unstably stratified (i.e. the fluid with ρ_2 is on the bottom and $\rho_2 < \rho_1$). This is the set-up used by Nobel prize winning physicist Lord Rayleigh who first described the physics of the instability. He was also one of the key people (along with Reynolds and Stokes) in establishing the idea of hydrodynamic similarity for flows. A famous example of a RTI is the development of salt diapirs illustrated in the following figure:

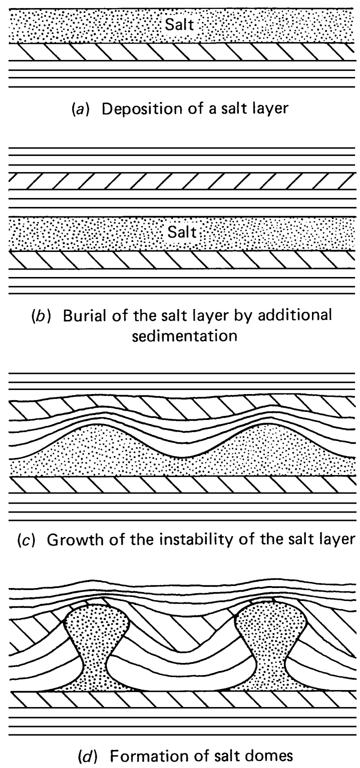


Figure 6.19 Diapiric formation of salt domes due to the gravitational instability of a light salt layer buried beneath heavier sedimentary rocks.

Consider the figure on the following page. The unperturbed density interface, w, is at $y = 0$ and there are walls at $y = \pm b$ which provide no-slip boundary conditions (y is positive downwards). For now we will assume that the two fluids have the same viscosity. Since this is Stokes flow problem in 2-D, the stream function can be used to solve for the velocity.

The general solution for the stream function in the top and bottom fluids is best expressed in terms of hyperbolic trig functions

$$\psi_1 = \sin(kx)\,(A_1 \cosh(ky) + B_1 \sinh(ky) + C_1\, y \cosh(ky) + D_1\, y \sinh(ky)) \tag{101}$$

where $k = 2\pi/\lambda$. The boundary conditions are

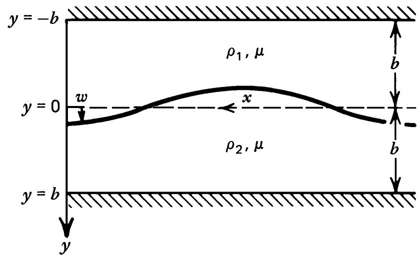


Figure 6.21 The Rayleigh–Taylor instability of a dense fluid overlying a lighter fluid.

$$
\begin{aligned}
u_{x1}(y=-b) &= 0 \\
u_{y1}(y=-b) &= 0 \\
u_{x2}(y=b) &= 0 \\
] \quad u_{y2}(y=b) &= 0 \\
u_{x1}(y=0) &= u_{x2}(y=0) \\
u_{y1}(y=0) &= u_{y2}(y=0) \\
\sigma_{xy1}(y=0) &= \sigma_{xy2}(y=0)
\end{aligned} \tag{102}
$$

and the shear stress is the usual $\sigma_{xy} = \eta\,(\partial u_x/\partial y + \partial u_y/\partial x)$. This means that to solve for the constants, the appropriate derivatives of the stream functions need to be taken to obtain velocities. However, for right now notice that there are only 7 boundary conditions and 8 constants, so the best we can do at this point is get stream functions for both layers into expressions with a single unknown constant (after lots of tedious algebra... - see Turcotte and Schubert for the full details).

$$
\begin{aligned}
\psi_1 = A_1 \sin(kx)\,\{&\cosh(ky) + \\
&\left[\frac{y}{kb^2}\,tanh(kb)\sinh(ky) + \left(\frac{y}{b}\cosh(ky) - \frac{1}{kb}\sinh(ky)\right)\left(\frac{1}{kb} + \frac{1}{\sinh(kb)\cosh(kb)}\right)\right] \\
&\times \left[\frac{1}{\sinh(kb)\cosh(kb)} - \frac{1}{k^2b^2}\,tanh(kb)\right]^{-1}\Bigg\}
\end{aligned} \tag{103}
$$

The expression for ψ_2 is obtained by replacing y with $-y$ and A_1 with A_2.

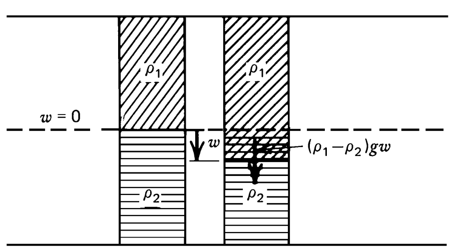


Figure 6.22 The buoyancy force associated with the displacement of the interface.

We need to use some physics to add a constraint in order to solve for the final constant. If the interface, w, is perturbed, there will be a buoyancy force associated with the amount of fluid that is displaced from the reference position, and this buoyancy force will generate a dynamic pressure (in the case where the two fluids are stably stratified, the buoyancy force is acts a restoring force). This additional constraint is described as

$$(\rho_1 - \rho_2)gw = (P_2 - P_1)\Big|_{y=0} \tag{104}$$

In order to solve for pressure at the interface, we need to go back to when the momentum equation was written in terms of the stream function but before the pressure had been eliminated.

$$\begin{aligned} 0 =& \frac{dP}{dx} + \eta\left(\frac{\partial^3\psi}{\partial^2 x\partial y} + \frac{\partial^3\psi}{\partial y^3}\right) \\ 0 =& -\frac{dP}{dy} + \eta\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial y^2\partial x}\right) \end{aligned} \tag{105}$$

and after substituting in ψ_1, P_1 can then evaluated at $y = 0$, giving

$$\begin{aligned} P_1(x)\Big|_{y=0} =& \frac{2\eta A_1 k}{b}\cos(kx)\left(\frac{1}{kb} + \frac{1}{\sinh(kb)\cosh(kb)}\right) \\ &\times\left[\frac{1}{\sinh(kb)\cosh(kb)} - \frac{1}{k^2b^2}tanh(kb)\right]^{-1} \end{aligned} \tag{106}$$

At this point, one should recognize that most of that complicated expression is only a function of kb and these are not variables in the system, so $P_1(x)$ at $y = 0$ can be rewritten more simply as

$$P_1(x)\Big|_{y=0} = \frac{2\eta A_1}{b}\cos(kx)f(kb) \tag{107}$$

where

$$f(kb) = k\left(\frac{1}{kb} + \frac{1}{\sinh(kb)\cosh(kb)}\right)\times\left[\frac{1}{\sinh(kb)\cosh(kb)} - \frac{1}{k^2b^2}tanh(kb)\right]^{-1} \tag{108}$$

An equivalent expression for $P_2(x)$ can be found using ψ_2 and it turns out that

$$P_1(x)\Big|_{y=0} = -P_2(x)\Big|_{y=0} \quad \text{so then} \quad (P_2 - P_1)\Big|_{y=0} = -2P_1(x)\Big|_{y=0} \tag{109}$$

This means that

$$(\rho_1 - \rho_2)gw = -2P_1(x)\Big|_{y=0} \tag{110}$$

Thus, if the fluids are unstably stratified ($\rho_2 < \rho_1$), a downward displacement of the interface ($w > 0$) establishes a pressure gradient across the interface that promotes further deflection, and thus the instability will grow. If the two fluids are stably stratified, the pressure gradient would be in the opposite direction and resist the motion of the interface.

$$(\rho_1 - \rho_2)gw = -\frac{4\eta A_1}{b}\cos(kx)f \tag{111}$$

Solving for A_1 gives

$$A_1 = -\frac{(\rho_1 - \rho_2)gwb}{4\eta}\left[f cos(kx)\right]^{-1} \tag{112}$$

The same procedure can be done using $P_2(x)$ evaluated at $y = 0$ to solve for A_2 and then both of the last remaining constants have been determined.

When the interface is deflected, the time rate of change of the deflection, dw/dt, must be equal to the vertical component of velocity at the fluid interface (otherwise a void would open but that can't happen because this is an incompressible fluid). It doesn't matter if it's fluid 1 or fluid 2:

$$\frac{\partial w}{\partial t} = u_y(y=0) \tag{113}$$

(valid if the deflection of the interface is small) and then using the definition of the stream function

$$\frac{\partial w}{\partial t} = \left.\frac{\partial \psi}{\partial x}\right|_{y=0} \tag{114}$$

Now that all the constants are known, we can take $(\partial\psi/\partial x)$ and set it equal to the time rate of change of the deflection to calculate the rate of growth of this instability.

$$\frac{\partial w}{\partial t} = kA_1 \cos(kx) \tag{115}$$

and substituting in for A_1 gives

$$\frac{\partial w}{\partial t} = -w\frac{(\rho_1-\rho_2)gb}{4f\eta} \tag{116}$$

This is straightforward to integrate and solve for w

$$w = w_0 e^{t/\tau_a} \quad \text{where} \quad \tau_a = \frac{4f\eta}{(\rho_1-\rho_2)gb} \tag{117}$$

This can be evaluated at the limits of very long wavelengths and very short wavelengths

$$\begin{aligned}\tau_a &= \frac{24\eta}{(\rho_1-\rho_2)gb}\left(\frac{1}{kb}\right)^2 \quad \text{for} \quad \lambda >> b\\ \tau_a &= \frac{4\eta}{(\rho_1-\rho_2)gb}kb \quad \text{for} \quad \lambda << b\end{aligned} \tag{118}$$

The expression can also be differentiated to find the wavelength that has the maximum growth rate (corresponding to a minimum for τ_a)

$$\tau_a = \frac{13.04\eta}{(\rho_1-\rho_2)gb} \tag{119}$$

which corresponds to $\lambda_{min} = 2.568\,b$. Often times in nature, there are many perturbations simultaneously in the system so the question becomes which wavelength is dominant and will grow the fastest (a perfectly flat spectrum with all wavelengths represented is called "white noise"). Another approach to find the most unstable wavelength is to study the growth rate of individual wavelengths in isolation of all other wavelengths as this ensures the measured growth rate is associated with a single wavelength rather than any possible combination due to non-linear effects that may be present.

This simple scaling relationship for very long and very short wavelengths can be applied to a few natural examples. The first example is that of a salt dome which is very common in the southeastern United States and formed due to the instability of a salt layer underneath an more recently deposited sediments.. These salt domes have a characteristic spacing and originate from a salt layer about 3-5 km deep. Assuming the fastest growing instability was responsible for creating the salt domes, a density difference of 300 kg/m^3 and an viscosity of the overlying sediments as 10^{20} - 10^{21} Pa s, this gives

$$\tau_a = \frac{13.04\eta}{(\rho_1-\rho_2)gb} = 10,080 - 100,000\,\text{years} \tag{120}$$

One can also estimate the rise time for a granitic diapir from $\sim$20 km, $\Delta\rho = 20$kg/m^3, and η between 10^{22} - 10^{23} Pa s, which gives $\tau_a \sim 1-10$ million years.

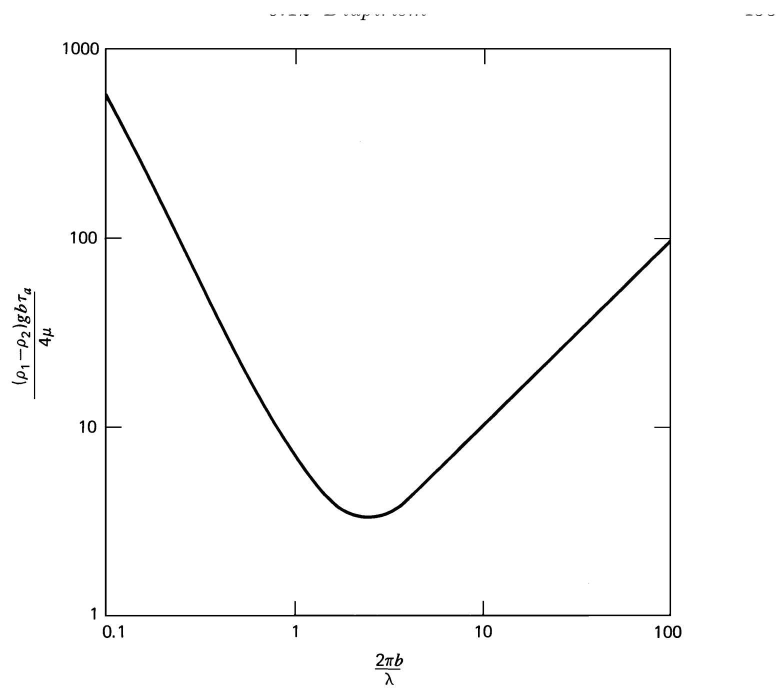


Figure 6.23 Dimensionless growth time of a disturbance as a function of dimensionless wave number for the Rayleigh–Taylor instability.

12. The Propagator Matrix Method

Imagine you have a layered stack of fluids and each fluid has a different density and viscosity and we want to know the flow throughout the entire system. The brute force method is to solve for the flow independently in each layer and enforce that the stresses and velocities are continuous across each interface (in case you're wondering, the jumps in viscosity will cause strain rates to be discontinuous). This results in lots of matching the two flows at an interface to solve for the constants which gets tedious very quickly. There is however, a more elegant method to arrive at the solution which involves a formalism that does all this automatically. This is known as the Propagator Matrix Method. In theory it is possible to use this method to arrive at the analytic solution for simple models (i.e. 2 layers), but this method is readily adapted into an algorithm so in practice the analytic solution is simply evaluated by a computer (i.e. semi-analytic).

Consider flow in a 2-D box of length L that is periodic in L. It is possible then to Fourier analyze the stresses, velocities, and density contrasts. For now we will consider the case when the flow is driven only by buoyancy forces, but later we will recognize that the method is applicable to more general situations including Couette flow, etc. The buoyancy forces and deformations of the boundaries can also be described as an appropriate Fourier series (e.g. $\sim \cos kx$) and similarly for the horizontal velocities (e.g. $\sim \sin kx$). Then for a wavenumber $k_n = 2\pi n/L$ and remember that τ is the total stress, we have

$$
\begin{aligned}
u_z(x,z) &= \sum_{n=1}^{\infty} u_z^n(z)\cos k_n x \\
u_x(x,z) &= \sum_{n=1}^{\infty} u_x^n(z)\sin k_n x \\
\tau_{zz}(x,z) &= \sum_{n=1}^{\infty} \tau_{zz}^n(z)\cos k_n x \\
\tau_{xx}(x,z) &= \sum_{n=1}^{\infty} \tau_{xx}^n(z)\cos k_n x \\
\tau_{xz}(x,z) &= \sum_{n=1}^{\infty} \tau_{xz}^n(z)\sin k_n x \\
P(x,z) &= \sum_{n=1}^{\infty} P^n(z)\cos k_n x \\
\Delta\rho(x,z) &= \sum_{n=1}^{\infty} \Delta\rho^n(z)\cos k_n x
\end{aligned} \tag{121}
$$

where P is the dynamic (or non-hydrostatic) pressure. Now use this set of equations to solve for the flow within one homogeneous layer by substituting the Fourier expressions into the governing equations, for example, the continuity equation ($\partial u_x/\partial x + \partial u_z/\partial z = 0$) becomes

$$
\sum_{n=1}^{\infty}\left[+k_n u_x^n(z) + \frac{d}{dz}u_z^n(z)\right]\cos k_n x = 0 \tag{122}
$$

where $+k_n$ is the Fourier derivative operator. In order to perform the same substitution into the momentum equation, $\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}_b = 0$, the constitutive equation needs to be transformed into an equivalent Fourier representation

$$
\begin{aligned}
\tau_{zz}(x,z) &= \sum_{n=1}^{\infty}\left(2\eta\frac{d}{dz}u_z^n(z) - P^n(z)\right)\cos k_n x \\
\tau_{xx}(x,z) &= \sum_{n=1}^{\infty}\left(2\eta k_n u_x^n(z) - P^n(z)\right)\cos k_n x \\
\tau_{xz}(x,z) &= \sum_{n=1}^{\infty}\eta\left(\frac{d}{dz}u_x^n(z) - k_n u_z^n(z)\right)\sin k_n x
\end{aligned} \tag{123}
$$

These series can be simplified by taking advantage of the orthoganality of the trigonometric basis functions. Multiplying through by $\cos(k_m x)$ and integrating with respect to x from 0 to L results in a complete decoupling of the equations for each term of the Fourier series. A new set of equations for each m can be written using notation in which the $_n$ subscripts are dropped and the dependence on z and m are now implicit. The constitutive relationship now becomes

$$
\begin{aligned}
\tau_{zz} &= 2\eta D u_z - P \\
\tau_{xx} &= 2\eta k u_x - P \\
\tau_{xz}(x,z) &= \eta\left(D u_x - k u_z\right)
\end{aligned} \tag{124}
$$

where $D = d/dz$. The equations for mass and momentum become

$$
\begin{aligned}
0 =& Du_z + ku_x \\
0 =& -k\tau_{xx} + D\tau_{xz} \\
0 =& D\tau_{zz} + k\tau_{xz} - \Delta\rho g
\end{aligned}
\tag{125}
$$

The continuity equation can be substituted into the equation for vertical stress giving it now as

$$\tau_{zz} = -2\eta k u_x - P \tag{126}$$

and this allows P to be eliminated from the problem by subtracting the normal stresses from each other $\tau_{zz} - \tau_{xx}$

$$\tau_{xx} = \tau_{zz} + 4\eta k u_x \tag{127}$$

This expression can be substituted into the horizontal momentum equation giving

$$D\tau_{xz} = k\tau_{zz} + 4\eta k^2 u_x \tag{128}$$

We finally arrive at a system of 4 coupled equations which can be written as

$$D \begin{bmatrix} u_z \\ u_x \\ \tau_{zz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} 0 & -k & 0 & 0 \\ k & 0 & 0 & 1/\eta \\ 0 & 0 & 0 & -k \\ 0 & 4\eta k^2 & k & 0 \end{bmatrix} \begin{bmatrix} u_z \\ u_x \\ \tau_{zz} \\ \tau_{xz} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Delta\rho g \\ 0 \end{bmatrix} \tag{129}$$

This reduces to a coupled set of linear O.D.E.s. At this point it's also a good idea to rescale the problem so that all the values of the dependent variables are of the same magnitude, so a reference viscosity η_0 is introduced and now x can be defined

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} u_z \\ u_x \\ \frac{\tau_{zz}}{2\eta_0 k} \\ \frac{\tau_{xz}}{2\eta_0 k} \end{bmatrix} \tag{130}$$

So finally we have $D\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$ a

$$\mathbf{A} = \begin{bmatrix} 0 & -k & 0 & 0 \\ k & 0 & 0 & 2k/\eta^* \\ 0 & 0 & 0 & -k \\ 0 & 2\eta^* k & k & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \Delta\rho/2\eta_0 k \\ 0 \end{bmatrix} \tag{131}$$

where $\eta^* = \eta/\eta_0$. This system should look familiar to you from the notes on computing geoid kernels. We write the solution in terms of propagator matrices:

$$\mathbf{x} = \mathbf{P}(z, z_0)\mathbf{x}(z_0) + \int_{z_0}^{z} \mathbf{P}(z, \xi)\mathbf{b}(\xi) d\xi \tag{132}$$

, where

$$\mathbf{P}(z, z_0) = \exp \mathbf{A}(z - z_0) \tag{133}$$

where the matrix exponential can be evaluated as a series – or analytically if $\mathbf{A}$ is independent of z in the range z_0 to z:

$$\mathbf{P}(z, z_0) = \begin{bmatrix} C & 0 & S/\eta^* & 0 \\ 0 & C & 0 & S/\eta^* \\ S\eta^* & 0 & C & C \\ 0 & S\eta^* & 0 & C \end{bmatrix} + k(z - z_0) \begin{bmatrix} -S & -C & -C\eta^* & -S\eta^* \\ C & S & S\eta^* & C\eta^* \\ -C/\eta^* & -S/\eta^* & -S & -C \\ S/\eta^* & C/\eta^* & C & S \end{bmatrix} \tag{134}$$

where $S = \sinh[k(z - z_0)]$ and $C = \cosh[k(z - z_0)]$. If we discretise the system into several layers then

$$\mathbf{x} = \mathbf{P}(z, z_0)\mathbf{x}(z_0) + \sum_{i=1}^{n} \mathbf{P}(z, \xi_i)\mathbf{b}(\xi_i)\Delta\xi_i \tag{135}$$

where the buoyancy has been discretized over n horizontal layers with ξ_i as the z coordinate in the center of the i'th layer and $\Delta\xi_i$ is the i'th layer thickness. Also, the boundary conditions need to be specified and enter the solution through the $\mathbf{x}(z_0)$ term. If the bottom of the box is at $z = 0$ and the box has height H, then, for example, free slip velocity boundary conditions correspond to

$$\begin{aligned} x(0) &= [0,\ u_x(0),\ \tau_{zz}(0),\ 0]^T \quad \text{at the bottom} \\ x(H) &= [0,\ u_x(H),\ \tau_{zz}(H),\ 0]^T \quad \text{at the top} \end{aligned} \tag{136}$$

where the vertical velocities and shear stresses are specified to be zero and the non-zero terms (horizontal velocities and vertical stresses) are to be determined as part of the solution. Alternatively, the horizontal surface velocity could be specified, $x(H) = [0,\ u_x(k)\big|_{(z=H)},\ \tau_{zz}(H),\ \tau_{xz}(H)]^T$, but in this case the imposed surface velocity must be represented as a Fourier expansion.

13. Rayleigh Taylor Instability - Redux

The Propagator Matrix Method can be used to solve for flow in the Rayleigh Taylor problem with an arbitrary number of layers, up until the flow an interface has been deformed so much that there are no longer coherent layers. Sometimes the question of interest is how large the deflection will be on boundaries several layers away from the unstable layers deflection.

Also, this framework can be used to quickly assess the growth rate of instabilities in horizontally layered fluids. Let's revisit the simple case of two fluid layers of different densities with an interface at $z = l$. We want to consider the motion of the interface as a function of the wavelength of the perturbation, the density contrast, and the viscosity contrast. This will provide a growth rate or relaxation time, depending on whether the fluids are unstably or stably stratified, respectively. Because within each fluid layer both the density and viscosity are uniform, the solution of the flow just above the interface $\mathbf{x}(l^+)$ is propagated to the surface $\mathbf{x}(H)$ through the propagator matrix (and vice versa). Similarly, the flow just below the interface $\mathbf{x}(l^-)$ is propagated to the bottom $\mathbf{x}(0)$, giving

$$\begin{aligned} \mathbf{x}(l^-) &= \mathbf{P}(l, 0)\mathbf{x}(0) \\ \mathbf{x}(l^+) &= \mathbf{P}(l, H)\mathbf{x}(H) \end{aligned} \tag{137}$$

In the case when the fluid layers extend to $\pm\infty$, the solution to the propagator matrix equation provides the intrinsic relaxation timescale, τ_r, of the interface. Both the velocity and stress must be continuous across the layer interface. Deflection of the interface generates a buoyancy force proportional to $\Delta\rho = \rho_1 - \rho_2$ that can be described as an equivalent pressure perturbation

$$\left[\mathbf{x}(l^+) - \mathbf{x}(l^-)\right] = \Delta P(k) = \frac{\Delta\rho g \tau_r u_z}{2\eta_0 k} \tag{138}$$

The pressure perturbation enters through the vertical stress at the interface and is also normalized to be of the same magnitude. For a sinusoidal perturbation these flows are given as

$$\mathbf{x}(l^-) = \begin{bmatrix} u_z^- \\ u_x^- \\ \tau_{zz}^-/2\eta_0 k \\ \tau_{xz}^-/2\eta_0 k \end{bmatrix} \quad \text{and} \quad \mathbf{x}(l^+) = \begin{bmatrix} u_z^+ \\ u_x^+ \\ \tau_{zz}^+/2\eta_0 k + \Delta\rho g \tau_r u_z/2\eta_0 k \\ \tau_{xz}^+/2\eta_0 k \end{bmatrix} \tag{139}$$

The $\Delta\rho$ only needs to be accounted for on one side of the interface as vertical stress on the other side of the interface will naturally adjust. Also, if the same pressure perturbation were applied to the other side it would either be double counting (if in the same direction) or canceled out due to a mirror symmetry (if in the opposite direction). The boundary conditions in this case would be $u_z(-\infty) = 0$ and $u_z(+\infty) = 0$ assuming the fluid is at rest at $\pm\infty$. Substituting these into the propagator matrix equations gives

$$\begin{aligned} u_z(-\infty) = 0 &= -\cosh(kz)u_z + \left(\frac{\tau_{zz}}{2\eta_0 k} + \frac{\Delta\rho g \tau_r u_z}{2\eta_0 k}\right)\frac{1}{\eta_1^*}\sinh(kz) + (\text{terms of order } kz) \\ u_z(+\infty) = 0 &= \cosh(kz)u_z + \frac{\tau_{zz}}{2\eta_0 k}\frac{1}{\eta_2^*}\sinh(kz) + (\text{terms of order } kz) \end{aligned} \tag{140}$$

Terms that are of order kz must vanish as $z \to \infty$. Subtracting these equations and evaluating with the appropriate limit as $z \to \infty$ gives

$$\tau_r = \frac{2\eta_0 k}{\Delta\rho g}(\eta_1^* + \eta_2^*) \tag{141}$$

which agrees with the result obtained previously when the fluids' viscosity is the same ($\eta_1^* = \eta_2^*$).