SIO 224

Buoyancy driven flow in spherical geometry

1. Introduction

These notes derive the equations used for buoyancy-driven flow in the mantle. The idea is that we take a velocity field from seismic tomography and convert it to a density field. This density field will give rise to buoyancy forces that will drive flow. The buoyancy forces will be balanced by the viscous forces associated with flow and, if the form and size of viscosity is known, the flow field can be determined within the mantle along with the perturbations in its boundaries (dynamic topography). From these, we can calculate the geoid or free air gravity field. The viscosity can then be adjusted until the geoid and dynamic topography are fit. For analytic convenience, it is usually assumed that viscosity is a function only of depth and is Newtonian (linear). It is then convenient to work in expansions of the variables in spherical harmonics. Deriving the equations is tedious and the literature is full of derivations with many typos (I don't guarantee that the following derivation is typo-free). The following notes are general and, in particular, the appendix gives a general discussion of manipulating vector pde's in spherical geometry. It is included because we use some of the results derived there but I don't expect you to read it. The derivation is long and intricate so I wish you luck! Note the μ is viscosity rather than η – I just didn't have the strength to change all the equations.

2. Basic equations

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{b}; \quad \mathbf{b} \equiv \mathbf{g} = -\nabla \phi \tag{1}$$

$$\nabla^2 \phi = 4\pi G \rho \tag{2}$$

Note:
1) Inertial terms are negligible (infinite Prandtl number)
2) Assume we have a Newtonian fluid (also incompressible)
3) Reference state is hydrostatic: $\mathbf{T}_0 = -p_0 \mathbf{I}$ so $\nabla p_0 = -\rho_0 \nabla \phi_0$.
For Newtonian fluid:

$$\mathbf{T} = 2\mu \dot{\boldsymbol{\epsilon}} - p\mathbf{I} \tag{3}$$

and $p = p_0 + \delta p$; $\rho = \rho_0 + \delta\rho$; $\phi = \phi_0 + \delta\phi$; $\mathbf{g}_0 = -\nabla \phi_0$. Note

$$\mathbf{g}_0 = -\hat{\mathbf{r}} g(r) \qquad \text{and} \qquad g(r) = \frac{4\pi G}{r^2} \int_0^r \rho_0 x^2 \, dx \tag{4}$$

The non-hydrostatic stress is:

$$\mathbf{T}' = 2\mu \dot{\boldsymbol{\epsilon}} - \delta p \mathbf{I} \tag{5}$$

Equations 1 and 2 become:

$$0 = \nabla \cdot (2\mu\dot{\boldsymbol{\epsilon}}) - \nabla \delta p + \delta\rho \mathbf{g}_0 - \rho_0 \nabla \delta\phi \tag{6}$$

$$\nabla^2 \delta\phi = 4\pi G \delta\rho \tag{7}$$

Expand $\delta\phi$ and $\delta\rho$ in spherical harmonics:

$$\delta\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \delta\Phi_l^m(r) Y_l^m(\theta, \phi) \tag{8}$$

$$\delta\rho = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \delta\rho_l^m(r) Y_l^m(\theta, \phi) \tag{9}$$

Expand $\mathbf{v}$ in vector spherical harmonics:

$$\mathbf{v} = \hat{\mathbf{r}}\, U + \nabla_1\, V - \hat{\mathbf{r}} \times (\nabla_1\, W) \tag{10}$$

with $\nabla_1 = \hat{\boldsymbol{\theta}}\partial/\partial\theta + \hat{\boldsymbol{\phi}}\,\mathrm{cosec}\,\theta\partial/\partial\phi$

$$U = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_l^m(r) Y_l^m(\theta, \phi) \quad \text{etc.} \tag{11}$$

Note that (ignoring l, m indices on U and V)

$$\nabla \cdot \mathbf{v} = \left(\frac{dU}{dr} + \frac{2U}{r} - \frac{l(l+1)V}{r} \right) Y_l^m$$

This is zero for an incompressible fluid so

$$\frac{dU}{dr} = -\frac{1}{r}(2U - l(l+1)V) = -F$$

The traction vector $\mathbf{t} = \mathbf{T}' \cdot \hat{\mathbf{r}}$ is also expanded in vector spherical harmonics:

$$\mathbf{t} = \hat{\mathbf{r}}\, R + \nabla_1\, S - \hat{\mathbf{r}} \times (\nabla_1\, T) \tag{12}$$

with

$$R = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_l^m(r) Y_l^m(\theta, \phi) \quad \text{etc.} \tag{13}$$

From the constitutive relation ship, we find

$$R = 2\mu \frac{dU}{dr} - \delta p; \quad S = \mu X; \quad T = \mu Z \tag{14}$$

$$\text{with} \quad X = \frac{dV}{dr} + \frac{U - V}{r} \quad \text{and} \quad Z = \frac{dW}{dr} - \frac{W}{r}$$

Let $\mathbf{d} = \nabla \cdot \mathbf{T}'$ and, from the appendix, we find that

$$
\left.
\begin{aligned}
d_r &= \left\{ \frac{dR}{dr} + \frac{1}{r}\left[-6F\mu - l(l+1)S\right] \right\} Y_l^m \\
d_\theta &= \left\{ \frac{dS}{dr} + \frac{1}{r}\left[3\mu F + R + 3S - \frac{\mu V}{r}(l+2)(l-1)\right] \right\} \frac{\partial Y_l^m}{\partial \theta} \\
&\quad + \left\{ \frac{dT}{dr} + \frac{1}{r}\left[3T - \frac{\mu W}{r}(l+2)(l-1)\right] \right\} im \operatorname{cosec}\theta Y_l^m \\
d_\phi &= \left\{ \frac{dS}{dr} + \frac{1}{r}\left[3\mu F + R + 3S - \frac{\mu V}{r}(l+2)(l-1)\right] \right\} im \operatorname{cosec}\theta Y_l^m \\
&\quad - \left\{ \frac{dT}{dr} + \frac{1}{r}\left[3T - \frac{\mu W}{r}(l+2)(l-1)\right] \right\} \frac{\partial Y_l^m}{\partial \theta}
\end{aligned}
\right\} \tag{15}
$$

and equation 6 now reads

$$
\left.
\begin{aligned}
d_r - \delta\rho g - \rho_0 \frac{d\delta\Phi}{dr} Y_l^m = 0 \\
d_\theta - \frac{\rho_0}{r}\delta\Phi \frac{\partial Y_l^m}{\partial \theta} = 0 \\
d_\phi - \frac{\rho_0}{r}\delta\Phi im \operatorname{cosec} Y_l^m = 0
\end{aligned}
\right\} \tag{16}
$$

and equation (7) becomes

$$
\frac{1}{r^2}\left(\frac{d}{dr} r^2 \frac{d\delta\Phi}{dr}\right) - l(l+1)\frac{\delta\Phi}{r^2} = 4\pi G\delta\rho \tag{17}
$$

(This is because $\nabla^2 = 1/r^2[\partial/\partial r(r^2\partial/\partial r) + \nabla_1^2]$ and $\nabla_1^2 Y_l^m = -l(l+1)Y_l^m$.) Collecting terms gives

$$
\frac{dR}{dr} + \frac{1}{r}\left[-6F\mu - l(l+1)S\right] - \delta\rho g - \rho_0 \frac{d\delta\Phi}{dr} = 0 \tag{18}
$$

$$
\frac{dS}{dr} + \frac{1}{r}\left[3\mu F + R + 3S - \frac{\mu V}{r}(l+2)(l-1)\right] - \rho_0\delta\Phi = 0 \tag{19}
$$

$$
\frac{dT}{dr} + \frac{1}{r}\left[3T - \frac{\mu W}{r}(l+2)(l-1)\right] = 0 \tag{20}
$$

and

$$
\frac{dU}{dr} = -F = -\frac{1}{r}(2U - l(l+1)V) \tag{21}
$$

$$
\frac{dV}{dr} = \frac{S}{\mu} - \frac{U-V}{r} \tag{22}
$$

$$
\frac{dW}{dr} = \frac{T}{\mu} + \frac{W}{r} \tag{23}
$$

Toroidal case (equations 20 and 23). Let $y_1 = W$ and $y_2 = rT$

$$\frac{d}{dr}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & 1/\mu \\ \mu(l+2)(l-1) & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{24}$$

This form is convenient for numerical solution.
Poloidal case (equations 17,18,19,21,22). Let $y_1 = U$,$y_2 = V$,$y_3 = rR$,$y_4 = rS$, $y_5 = r\delta\Phi$, $y_6 = r^2 d\delta\Phi/dr$

$$\frac{d\mathbf{y}}{dr} = \frac{1}{r}\left(\mathbf{A}\,\mathbf{y} + \mathbf{f}\right) \tag{25}$$

The 6×6 coefficient matrix $\mathbf{A}$ is given by

$$\begin{bmatrix} -2 & l(l+1) & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & \frac{1}{\mu} & 0 & 0 \\ 12\mu & -6l(l+1)\mu & 1 & l(l+1) & 0 & -\rho_0 \\ -6\mu & 2[2l(l+1)-1]\mu & -1 & -2 & \rho_0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & l(l+1) & 0 \end{bmatrix} \tag{26}$$

$$\mathbf{f} = \begin{bmatrix} 0 \\ 0 \\ \delta\rho g r^2 \\ 0 \\ 0 \\ 4\pi G r^3 \delta\rho \end{bmatrix}$$

Use dimensionless variable $\nu = ln(r/a)$ so $r = ae^{\nu}$ and so

$$\frac{d\mathbf{y}}{d\nu} = r\frac{d\mathbf{y}}{dr} \tag{27}$$

(Can also divide $y_3 \to y_6$ by a reference viscosity multiply y_5 and y_6 by a reference density to make everything dimernsionless). Propagator matrices are typically used to solve these systems.

3. Propagators

Consider the equation:

$$\frac{d\mathbf{y}}{d\nu} = \mathbf{A}\cdot\mathbf{y} + \mathbf{f} \tag{28}$$

We write the solution to the homogeneous eqution ($\mathbf{f} = 0$)

$$\mathbf{y}(\nu) = \mathbf{P}(\nu, \nu_0)\cdot\mathbf{y}(\nu_0) \tag{29}$$

where $\mathbf{P}$ is known as a "propagator matrix". For the simple matrices of the type given here, $\mathbf{P}$ can be calculated analytyically, or alternatively, it can be computed using Runge-Kutta or some other such numerical method for integrating systems of 1st order ODEs. Note

$$\mathbf{P}(\nu_0,\nu_0)=\mathbf{I};\quad \mathbf{P}(\nu_0,\nu)\cdot\mathbf{P}(\nu,\nu_0)=\mathbf{I};\quad \mathbf{P}^{-1}(\nu_0,\nu)=\mathbf{P}(\nu,\nu_0) \tag{30}$$

Also

$$\frac{d\mathbf{P}}{d\nu}=\mathbf{A}\cdot\mathbf{P} \tag{31}$$

From

$$\frac{d}{d\nu}[\mathbf{P}^{-1}\cdot\mathbf{P}]=\frac{d\mathbf{P}^{-1}}{d\nu}\cdot\mathbf{P}+\mathbf{P}^{-1}\cdot\mathbf{A}\cdot\mathbf{P}=0 \tag{32}$$

we get

$$\frac{d\mathbf{P}^{-1}}{d\nu}=-\mathbf{P}^{-1}\cdot\mathbf{A} \tag{33}$$

but

$$\mathbf{P}^{-1}\cdot\frac{d\mathbf{y}}{d\nu}=\mathbf{P}^{-1}\cdot\mathbf{A}\cdot\mathbf{y}+\mathbf{P}^{-1}\cdot\mathbf{f}=-\frac{d\mathbf{P}^{-1}}{d\nu}\cdot\mathbf{y}+\mathbf{P}^{-1}\cdot\mathbf{f} \tag{34}$$

Therefore

$$\frac{d}{d\nu}\left[\mathbf{P}^{-1}\cdot\mathbf{y}\right]=\mathbf{P}^{-1}\cdot\mathbf{f} \tag{35}$$

Integrating gives

$$\left[\mathbf{P}^{-1}\cdot\mathbf{y}\right]_{\nu_0}^{\nu}=\mathbf{P}^{-1}(\nu,\nu_0)\cdot\mathbf{y}(\nu)-\mathbf{P}^{-1}(\nu_0,\nu_0)\cdot\mathbf{y}(\nu_0)=\int_{\nu_0}^{\nu}\mathbf{P}^{-1}(\zeta,\nu_0)\cdot\mathbf{f}(\zeta)\,d\zeta \tag{36}$$

so

$$\mathbf{P}^{-1}(\nu,\nu_0)\cdot\mathbf{y}(\nu)=\mathbf{y}(\nu_0)+\int_{\nu_0}^{\nu}\mathbf{P}(\nu_0,\zeta)\cdot\mathbf{f}(\zeta)\,d\zeta \tag{37}$$

or

$$\mathbf{y}(\nu)=\mathbf{P}(\nu,\nu_0)\cdot\mathbf{y}(\nu_0)+\int_{\nu_0}^{\nu}\mathbf{P}(\nu,\zeta)\cdot\mathbf{f}(\zeta)\,d\zeta \tag{38}$$

From this equation, we can compute $\mathbf{y}$ at any depth once $\mathbf{P}$ is known. We are not quite done since we must still specify the boundary conditions to be satisfied.

4. Boundary Conditions

We have continuity of $\mathbf{t}$ and $\mathbf{v}$ (though a barrier to flow such as at a chemical boundary would have $v_r = 0$. The CMB is probably a stress-free boundary. The surface could be modelled as such or a velocity boundary condition representing plate motion could be imposed. Assuming stress free boundaries $t_\theta = t_\phi = 0$ and t_r must be zero on the deformed boundary. We use a linearization to get behavior on the deformed boundary:

$$\mathbf{y}_{ref} = \mathbf{y}_{def} - \frac{d\mathbf{y}}{dr}\delta r \tag{39}$$

At the surface ($r = a$)

$$t_r(a) = -\rho_a g(a)\delta r(a) \tag{40}$$

where ρ_a is the density at the surface. At the CMB ($r = c$)

$$t_r(c) = -\Delta\rho g(c)\delta r(c) - \rho_c \delta\phi(c) \tag{41}$$

where $\Delta\rho$ is the density jump at the CMB. The second term in the above equation arises because of a perturbation in the gravitational potential. We therefore have the following values for $\mathbf{y}$:

$$\mathbf{y} = [v_r, v_\theta, rt_r, rt_\theta, r\delta\Phi, r^2 d\delta\Phi/dr] \tag{42}$$

$$\mathbf{y}(a) = [0, v_\theta(a), -\rho_a g(a) a\delta r(a), 0, a\delta\Phi(a), a^2 d\delta\Phi(a)/dr] \tag{43}$$

$$\mathbf{y}(c) = [0, v_\theta(c), -\Delta\rho g(c) c\delta r(c) - c\rho_c\delta\Phi(c), 0, c\delta\Phi(c), c^2 d\delta\Phi(c)/dr] \tag{44}$$

In general, we don't solve for a specific form of $\delta\rho$ in the $\mathbf{f}$ vector. We solve for a sheet mass anomaly at various depths – this allows us to construct a "kernel" then we convolve the kernel with a particular $\delta\rho$ to get the final answer. That is, we represent $\delta\rho$ as

$$\delta\rho(r) = \sum_i \delta(r - h_i)\sigma_i \tag{45}$$

and we solve equation 38 for a bunch of $\mathbf{f}_i$ where each one is of the form

$$\mathbf{f}_i = [0, 0, h_i^2 g, 0, 0, 4\pi G h_i^3]\delta(r - h_i)\sigma_i \tag{46}$$

Note that equation 38 for one of the $\mathbf{f}_i$ can be written

$$\mathbf{y}(\nu) = \mathbf{P}(\nu, \nu_0)\mathbf{y}(\nu_0) + \mathbf{P}(\nu, h_i) \cdot \mathbf{f}_i \tag{47}$$

and in particular

$$\mathbf{y}(a) = \mathbf{P}(a, c)\mathbf{y}(c) + \mathbf{P}(a, h_i) \cdot \mathbf{f}_i \tag{48}$$

The idea is to solve this equation for the unknown elements of $\mathbf{y}(a)$ and $\mathbf{y}(c)$ then we can solve the general equation (47) for the velocity (flow field) and perturbation to the gravitational potential (from which we can compute the geoid). Equation (48) gives six constraints but from 43 and 44 it seems we have eight unknowns. For a sheet mass anomaly, Poisson's equation reduces to Laplaces equation (except at the anomaly itself) and we can solve analytically for the perturbation in the gravitational potential. The general solution to Laplaces equation (for a spherical harmonic component of degree l) is of the form

$$\delta\Phi = \sum_l \left[A_l r^l + B_l \frac{1}{r^{l+1}}\right] Y_l^m \tag{49}$$

To satisfy the potential at infinity and the origin, it is clear the first term controls the potential below the sheet mass while the second term gives the potential above the sheet mass. In terms of the potential at the sheet mass, we have

$$\begin{aligned}\delta\Phi_l^m(r) =&\delta\Phi_l^m(h_i)\left(\frac{h_i}{r}\right)^{l+1} \quad \text{for} \quad r > h_i \\ =&\delta\Phi_l^m(h_i)\left(\frac{r}{h_i}\right)^{l} \quad \text{for} \quad r < h_i\end{aligned} \tag{50}$$

It is also easy to show that there is a jump in the potential gradient at the depth of the sheet mass. Note that Poisson's equation is $\nabla \cdot \nabla \delta\Phi = 4\pi G \delta\rho$ so using Gauss' theorem gives

$$\int \hat{\mathbf{n}} \cdot \nabla \delta\Phi dS = \int 4\pi G \delta\rho \, dV$$

so the jump in gradient is given by

$$\left[\frac{d\delta\Phi}{dr}\right]_-^+ = 4\pi G\sigma \tag{51}$$

Differentiating equation 49 and evaluating the difference in gradient at the sheet mass using 50 and 51 gives

$$\frac{4\pi G\sigma_l^m h_i}{2l+1} = \delta\Phi_l^m(h_i) \tag{52}$$

which gives the potential at the sheet mass in terms of the sheet mass load (σ).

In reality, a sheet mass load will give rise to topgraphy on the surface and CMB which also appear as sheet mass loads in the potential. Thus (using equation 50)

$$\delta\Phi(a) = \frac{4\pi G}{2l+1}\left[a(\frac{a}{a})^l\rho_a\delta r(a) + c(\frac{c}{a})^{l+1}\Delta\rho\delta r(c) + h(\frac{h}{a})^{l+1}\sigma_l\right] \tag{53}$$

$$\delta\Phi(c) = \frac{4\pi G}{2l+1}\left[a(\frac{c}{a})^l\rho_a\delta r(a) + c(\frac{c}{c})^{l+1}\Delta\rho\delta r(c) + h(\frac{c}{h})^{l}\sigma_l\right] \tag{54}$$

$$\frac{d}{dr}\delta\Phi(a) = \frac{4\pi G}{2l+1}\left[l\rho_a\delta r(a) - (l+1)(\frac{c}{a})^{l+2}\Delta\rho\delta r(c) - (l+1)(\frac{h}{a})^{l+2}\sigma_l\right] \tag{55}$$

$$\frac{d}{dr}\delta\Phi(c) = \frac{4\pi G}{2l+1}\left[l(\frac{c}{a})^{l-1}\rho_a\delta r(a) - (l+1)\Delta\rho\delta r(c) + l(\frac{c}{h})^{l-1}\sigma_l\right] \tag{56}$$

The unknowns now are $v_\theta(a), v_\theta(c), \delta r(a), \delta r(c) = \mathbf{x}$ say . The six equations corresponding to equation 48 can be reduced to 4 and we can solve them for $\mathbf{x}$. We can then construct $\mathbf{y}(c)$ and solve 47 for $\mathbf{y}$ everywhere. To see how this is done, define a transformation such that

$$\mathbf{y}(a) = \mathbf{T}_a \cdot \mathbf{x} + \mathbf{g}_a\sigma_l \tag{57}$$

Define $N_a = 4\pi Ga/(2l+1)$ then $\mathbf{T}_a$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\rho_a g_a a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & N_a a \rho_a & N_a c \left(\frac{c}{a}\right)^{l+1} \Delta\rho \\ 0 & 0 & N_a l a \rho_a & -N_a(l+1)c\left(\frac{c}{a}\right)^{l+1} \Delta\rho \end{bmatrix} \tag{58}$$

and $\mathbf{g}_a = [0, 0, 0, 0, N_a h_i (h_i/a)^{l+1}, -N_a(l+1)h_i(h_i/a)^{l+1}]$. Also define a transformation such that

$$\mathbf{y}(c) = \mathbf{T}_c \cdot \mathbf{x} + \mathbf{g}_c \sigma_l \tag{59}$$

Define $N_c = 4\pi Gc/(2l+1)$ then $\mathbf{T}_c$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -N_c \rho_c a \left(\frac{c}{a}\right)^l \rho_a & -c\Delta\rho(g_c + \rho_c N_c) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & N_c a \rho_a \left(\frac{c}{a}\right)^l & N_c c \Delta\rho \\ 0 & 0 & N_c l a \rho_a \left(\frac{c}{a}\right)^l & -N_c(l+1)c\Delta\rho \end{bmatrix} \tag{60}$$

and $\mathbf{g}_c = [0, 0, -\rho_c N_c h_i (c/h_i)^l, 0, N_c h_i (c/h_i)^l, N_c l h_i (c/h_i)^l]$. With these definitions, equation 48 becomes

$$(\mathbf{T}_a - \mathbf{P}(a,c) \cdot \mathbf{T}_c) \cdot \mathbf{x} = \mathbf{P}(a,c) \cdot \mathbf{g}_c \sigma_i - \mathbf{g}_a \sigma_i + \mathbf{P}(a, h_i) \mathbf{f}_i \tag{61}$$

where $\mathbf{f}_i$ is proportional to σ_i (see equation 46). Note that σ_i is arbitrary and so can be taken equal to one, so this equation can now be solved for $\mathbf{x}$

5. Fitting the geoid

Equation 48 gives the response at the surface $(\mathbf{y}(a))$ due to a sheet mass anomaly at a radius h_i. Note that the propagators depend upon harmonic degree (eqn 26) but not on azimuthal order. Equation 53 can be used to compute the potential perturbation at the surface for a single sheet mass anomaly. Typically, we might parameterize the mantle in shells with 3d density expanded in spherical harmonics for each shell. The equivalent sheet mass anomaly would then be

$$\sigma_l^m = \int_{r_1}^{r_2} \delta\rho_l^m(r) \tag{62}$$

where r_1 and r_2 are the lower and upper radii of the shell. In the absence of dynamic boundary deformation (i.e., $\delta r(a) = \delta r(c) = 0$ then, on summing over all shells, Equation 53 would become

$$\delta \Phi_l^m(a) = \frac{4\pi G a}{2l+1} \int_c^a G_l(r) \delta \rho_l^m(r)\, dr \tag{63}$$

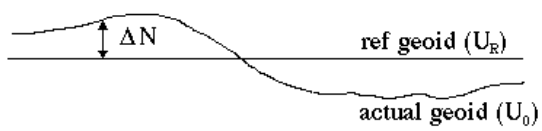
where

$$G_l(r) = \left(\frac{r}{a}\right)^{l+1} \tag{64}$$

Note that the geoid is the equipotential surface which most closely coincides with sea level and a geoid anomaly, ΔN is the distance between the reference ellipsoid and the actual geoid (see figure below). Since the acceleration due to gravity on the reference ellipsoid g_R is, to a very good approximation, just the radial gradient of the potential we find that

$$\Delta N_l^m = -\frac{\delta \Phi_l^m(a)}{g_R} \tag{65}$$

where $\Delta N = \sum \Delta N_l^m Y_l^m$.



It turns out that the dynamic topography terms in equation 53 are very important, and depending on viscosity structure, can actually change the sign of the resulting geoid anomaly. The solution described at the end of the last section gives $\delta r(a)$ and $\delta r(c)$ in terms of the σ_l (see equation 61) so equation 63 remains valid but with two additonal terms contributing to $G_l(r)$.

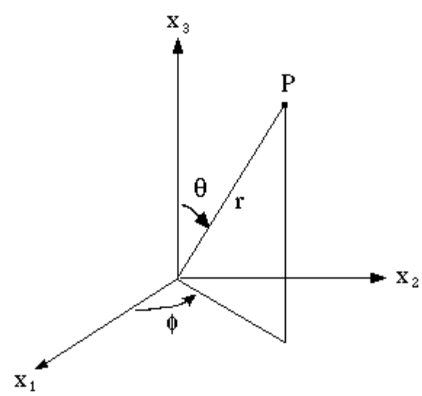
APPENDIX A

Spherical Geometry, Spherical Harmonics and Tensor Calculus

Introduction

As the Earth is almost a spherical body, many problems in geophysics require the use of spherical coordinates rather than Cartesian coordinates. Furthermore we often need to be able to specify scalar, vector and tensor fields anywhere within this body and this is most commonly done using expansions in spherical harmonics. Unfortunately, tensor calculus in anything but Cartesian coordinates is algebraically tedious and many techniques have been devised to alleviate the labor. You may be familiar with the use of Christoffel symbols in covariant differentiation of tensors. Another technique, familiar in quantum mechanics, is expansion in generalized spherical harmonics (GSH). This technique was made popular in the geophysical literature in a paper by Phinney and Burridge (1973) and some detail can also be found in the book by Edmonds (1960).

We use standard spherical polar coordinates :



$$x_1 = r \sin\theta \cos\phi \quad \text{and} \quad x_2 = r \sin\theta \sin\phi \quad \text{and} \quad x_3 = r\cos\theta$$

where θ is colatitude and ϕ is east longitude.

The spherical harmonics in most common use in the geophysical literature are those defined by Edmonds (1960), *i.e.*,

$$Y_l^m(\theta,\phi) = (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} P_l^m(\cos\theta)e^{im\phi}$$

where the P_l^m are associated Legendre functions which are progressively wigglier functions of θ as l increases. We often need to compute Y_l^m's and their derivatives with respect to θ. This can easily be done using the recursion relationships given by Edmonds (1960) for the P_l^m.

Some properties of the Y_l^m's are:

$$Y_l^{-m} = (-1)^m Y_l^{m^*} \quad \text{where } * \text{ denotes complex conjugation}$$

$$\nabla_1^2 Y_l^m = -l(l+1)Y_l^m \quad \text{where} \quad \nabla_1^2 = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \operatorname{cosec}^2\theta\frac{\partial^2}{\partial\phi^2}$$

This relationship allows us to cast higher order θ derivatives in terms of Y_l^m and $\partial Y_l^m/\partial\theta$. Note that ϕ derivatives are trivial, *i.e.*,

$$\frac{\partial}{\partial\phi}Y_l^m = imY_l^m \quad \textit{etc.}$$

Finally the Y_l^m's are fully normalized, *i.e.*,

$$\int_S Y_{l'}^{m'*} Y_l^m dS = \delta_{mm'}\delta_{ll'} \quad \text{where} \quad dS = \sin\theta d\theta d\phi$$

If we wish to specify an appropriately smooth *scalar* function of position inside the Earth, $\rho(r,\theta,\phi)$ say, then we can make an expansion in spherical harmonics, e.g.,

$$\rho(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \rho_l^m(r)Y_l^m(\theta,\phi) \tag{A.1}$$

and the field is completely specified by the coefficients, $\rho_l^m(r)$.

The situation is a little more complicated if we have a vector field and it turns out that it is not useful to expand the r,θ,ϕ components in a way similar to the scalar expansion. The usual way to represent a vector field in spherical geometry is to use the form (Morse and Feshback, 1953):

$$\mathbf{v}(r,\theta,\phi) = \hat{\mathbf{r}}U(r,\theta,\phi) + \nabla_1 V(r,\theta,\phi) - \hat{\mathbf{r}} \times (\nabla_1 W(r,\theta,\phi)) \tag{A.2}$$

where U, V, and W are scalars, ∇_1 is the surface gradient operator defined by

$$\nabla_1 = \hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \operatorname{cosec}\theta\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\phi}$$

and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are unit vectors in the r,θ,ϕ directions. Note that $\hat{\mathbf{r}} \times \nabla_1$ ($\times$ is vector cross product) is

$$\hat{\mathbf{r}} \times \nabla_1 = -\hat{\boldsymbol{\theta}}\operatorname{cosec}\theta\frac{\partial}{\partial\phi} + \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}$$

The scalars U, V and W can now be expanded as in A1.

Canonical components and generalized spherical harmonics

When we come to tensor fields, the algebra gets a little more awkward and it turns out that things simplify if we abandon the r,θ,ϕ coordinates and introduce new ones. We label these new directions $-, 0, +$. If $\mathbf{v}(r,\theta,\phi)$ has components v_r, v_θ, v_ϕ then the new directions are defined as

$$v^- = \frac{1}{\sqrt{2}}(v_\theta + iv_\phi)$$

$$v^0 = v_r$$

$$v^+ = \frac{1}{\sqrt{2}}(-v_\theta + iv_\phi)$$

It is convenient to represent this coordinate transformation as a matrix operation. Suppose we let

$$v_1 = v_\theta, \quad v_2 = v_\phi \quad \text{and} \quad v_3 = v_r$$

then

$$v^\alpha = \sum_{i=1}^{3} C^\dagger_{\alpha i} v_i \qquad \alpha = -, 0, + \tag{A.3}$$

where

$$C^\dagger_{\alpha i} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{bmatrix} \quad \text{with Hermitian conjugate} \quad C_{i\alpha} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \tag{A.4}$$

A tensor of any order can be put into the new coordinate system by repetition of the operation A3, *i.e.*,

$$\mathrm{m}^{\alpha\beta\gamma\ldots} = C^\dagger_{\alpha i} C^\dagger_{\beta j} C^\dagger_{\gamma k} \cdots \mathrm{m}_{ijk\ldots} \tag{A.5}$$

where summation over repeated indices is implied. Remember that i, j, k, etc. go from 1 to 3 corresponding to θ, ϕ, r directions respectively. $\mathrm{m}^{\alpha\beta\gamma\cdots}$ can be returned to the original coordinate system by the operation

$$\mathrm{m}_{ijk\ldots} = C_{i\alpha} C_{j\beta} C_{k\gamma} \cdots \mathrm{m}^{\alpha\beta\gamma\cdots} \quad \text{(summation implied)} \tag{A.6}$$

Tensor calculus becomes much simpler if we expand the contravariant canonical components in *generalized spherical harmonics* rather than the Y_l^m's defined earler, *i.e.*, we set

$$\mathrm{m}^{\alpha\beta\gamma\cdots}(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} M_l^{\alpha\beta\gamma\ldots,m}(r) Y_l^{N,m}(\theta,\phi) \tag{A.7}$$

where $N = \alpha + \beta + \gamma + \ldots$. For the contravariant canonical components of the velocity field we have

$$v^\alpha(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_l^{\alpha,m} Y_l^{\alpha,m}(\theta,\phi) \quad \text{where} \quad \alpha = -, 0, + \tag{A.8}$$

The generalized spherical harmonics are given by

$$Y_l^{N,m} = P_l^{N,m}(\cos\theta) e^{im\phi}$$

where the $P_l^{N,m}$ are defined by Phinney and Burridge who also give recursion relations for their calculation. We note two of their formulae here:

$$\left.\begin{aligned} \Omega^l_{N+1} Y_l^{N+1,m} + \Omega^l_N Y_l^{N-1,m} &= \sqrt{2}[N\cot\theta - m\,\mathrm{cosec}\,\theta] Y_l^{N,m} \\ \Omega^l_N Y_l^{N-1,m} - \Omega^l_{N+1} Y_l^{N+1,m} &= \sqrt{2}\frac{dY_l^{N,m}}{d\theta} \end{aligned}\right\} \tag{A.9}$$

where $\Omega^l_N = [\frac{1}{2}(l+N)(l-N+1)]^{\frac{1}{2}}$. Note that $\Omega^l_{-N+1} = \Omega^l_N$ so that $\Omega^l_0 = \Omega^l_1$, $\Omega^l_{-1} = \Omega^l_2$, etc.

It is convenient to write down some relationships between the GSH and the ordinary Y_l^m's. Let $\gamma_l = \sqrt{\left(\frac{2l+1}{4\pi}\right)}$. Then

$$\left.\begin{aligned}
\gamma_l Y_l^{0,m} &= Y_l^m \\
\frac{1}{\sqrt{2}}\gamma_l \Omega_1^l [Y_l^{-1,m} + Y_l^{+1,m}] &= -m \operatorname{cosec}\theta Y_l^m \\
\frac{1}{\sqrt{2}}\gamma_l \Omega_1^l [Y_l^{-1,m} - Y_l^{+1,m}] &= \frac{\partial Y_l^m}{\partial\theta} \\
\tfrac{1}{2}\gamma_l \Omega_1^l \Omega_2^l [Y_l^{-2,m} + Y_l^{+2,m}] &= [m^2 \operatorname{cosec}^2\theta - \frac{l(l+1)}{2}]Y_l^m - \cot\theta \frac{\partial Y_l^m}{\partial\theta} \\
\tfrac{1}{2}\gamma_l \Omega_1^l \Omega_2^l [Y_l^{-2,m} - Y_l^{+2,m}] &= m \operatorname{cosec}\theta \left[\cot\theta Y_l^m - \frac{\partial Y_l^m}{\partial\theta}\right]
\end{aligned}\right\} \tag{A.10}$$

Relationships for higher N can be found from the recursion formulae A9. These relationships are useful if one wants to convert from the canonical components back to expressions in r, θ, ϕ involving ordinary spherical harmonics.

Equation A5 defines the contravariant canonical component of $\mathrm{m}_{ijk\ldots}$ and we can also define covarient components of $\mathrm{m}_{ijk\ldots}$ (though these are less useful), *i.e.*,

$$\mathrm{m}_{\alpha\beta\gamma} = C_{i\alpha}C_{j\beta}C_{k\gamma}\cdots\mathrm{m}_{ijk\ldots} \quad \text{(summation implied)} \tag{A.11}$$

We will need the covarient components of the tensor δ_{ij}. We define these as

$$\Delta_{\alpha\beta} = C_{i\alpha}C_{j\beta}\delta_{ij} = C_{i\alpha}C_{i\beta} = C_{1\alpha}C_{1\beta} + C_{2\alpha}C_{2\beta} + C_{3\alpha}C_{3\beta}$$

From A4 we find that

$$\Delta_{00} = 1 \quad \text{and} \quad \Delta_{-+} = \Delta_{+-} = -1 \quad \text{and all other components are zero} \tag{A.12}$$

As an example of the use of Δ, we consider the trace of a second-order tensor, *i.e.*,

$$\mathrm{Tr}\,(\mathrm{m}_{ij}) = \mathrm{m}_{ij}\delta_{ij} = \mathrm{m}_{ii} = \mathrm{m}_{11} + \mathrm{m}_{22} + \mathrm{m}_{33}$$

m_{ii} is a scalar and is equivalent to

$$\begin{aligned}
\mathrm{m}_{ii} &= \Delta_{\alpha\beta}\mathrm{m}^{\alpha\beta} \quad \text{(summation implied)} \\
&= \mathrm{m}^{00} - \mathrm{m}^{+-} - \mathrm{m}^{-+} \quad \text{(as all other components of } \Delta_{\alpha\beta} \text{ are zero)}
\end{aligned}$$

We have to worry about the derivatives of vectors and tensors. Differentiation is straightforward in the new coordinate system. For definiteness we consider a second-order tensor m_{ij}, differentiation gives a third-order tensor which we write as $\mathrm{m}_{ij,k}$.

From equation A7 we have the expansion

$$\mathrm{m}^{\alpha\beta}(r,\theta,\phi) = \sum_{l=N}^{\infty}\sum_{m=-l}^{l} M_l^{\alpha\beta,m}(r) Y_l^{(\alpha+\beta),m}(\theta,\phi)$$

To reduce the number of indices we have to write out, we consider a single l, m component of $\mathrm{m}^{\alpha\beta}$ and write this as

$$\mathrm{m}^{\alpha\beta} = M^{\alpha\beta}\, Y^{\alpha+\beta} \quad (\, l \text{ and } m \text{ understood})$$

Differentiation results in a higher rank tensor which we write as

$$\mathrm{m}^{\alpha\beta,\gamma} = M^{\alpha\beta|\gamma}\, Y^{\alpha+\beta+\gamma}$$

The coefficients $M^{\alpha\beta|\gamma}$ can be found in terms of the $M^{\alpha\beta}$ by using the following recipe which we give for a tensor of any order. Consider $\gamma = -, 0, +$ separately, then we have

$$\left.\begin{aligned} M^{\alpha\beta\dots|-} &= \frac{1}{r}\left[\,\Omega^l_N\, M^{\alpha\beta\dots} \;-\left\{\begin{matrix}\text{terms obtained from } M^{\alpha\beta\cdots} \text{ by changing}\\ + \text{ into } 0, 0 \text{ into } - \text{ one at a time}\end{matrix}\right\}\right] \\ M^{\alpha\beta\dots|0} &= \frac{dM^{\alpha\beta}}{dr} \\ M^{\alpha\beta\dots|+} &= \frac{1}{r}\left[\Omega^l_{N+1} M^{\alpha\beta\dots} -\left\{\begin{matrix}\text{terms obtained from } M^{\alpha\beta\cdots} \text{ by changing}\\ - \text{ into } 0, 0 \text{ into } + \text{ one at a time}\end{matrix}\right\}\right] \end{aligned}\right\} \tag{A.13}$$

Here $N = \alpha + \beta + ...$ and when calculating N, regard $-, 0, +$ as shorthand for $-1, 0$ and $+1$.
The use of A13 is best illustrated by example. For a second rank tensor we consider the $+0$ component only so $N = 1$. Thus

$$\begin{aligned} M^{+0|-} &= \frac{1}{r}\left[\Omega^l_1 M^{+0} - M^{00} - M^{+-}\right] \\ M^{+0|0} &= \frac{dM^{+0}}{dr} \\ M^{+0|-} &= \frac{1}{r}\left[\Omega^l_2 M^{+0} - M^{++}\right] \end{aligned}$$

As another example of the use of $\Delta_{\alpha\beta}$, we consider the divergence of a second rank tensor which would be written $\mathrm{m}_{ij,j}$. This is equivalent to

$$\Delta_{\beta\gamma}\mathrm{m}^{\alpha\beta,\gamma} = \mathrm{m}^{\alpha 0,0} - \mathrm{m}^{\alpha +,-} - \mathrm{m}^{\alpha -,+}$$

Another example of the use of A13 is given by the computation of the Laplacian of a scalar field. A single l, m component of the expansion of a scalar field ϕ can be written

$$\phi = \Phi\, Y^{0,m}$$

where Φ is an expansion coefficient. Differentiation gives

$$\phi^{,\alpha} = \Phi^{|\alpha}\, Y^{\alpha,m}$$

which using A13 becomes

$$\begin{aligned} \phi^{,-} &= \frac{1}{r}\Omega^l_0 \Phi\, Y^{-1,m} \\ \phi^{,0} &= \frac{d}{dr}\Phi\, Y^{0,m} \\ \phi^{,+} &= \frac{1}{r}\Omega^l_1 \Phi\, Y^{+1,m} \end{aligned}$$

The Laplacian is given by

$$\Delta_{\alpha\beta}\phi^{,\alpha\beta} = \phi^{,00} - \phi^{,-+} - \phi^{,+-}$$

but using A13 again gives

$$\begin{aligned}
\phi^{,00} &= \frac{d^2}{dr^2}\Phi\, Y^{0,m} \\
\phi^{,-+} &= \frac{1}{r}\left[\Omega_0^l \frac{1}{r}\Omega_0^1 \Phi - \frac{d}{dr}\Phi\right]\, Y^{0,m} \\
\phi^{,+-} &= \frac{1}{r}\left[\Omega_1^l \frac{1}{r}\Omega_1^l \Phi - \frac{d}{dr}\Phi\right]\, Y^{0,m}
\end{aligned}$$

so the Laplacian of a single l, m component of ϕ is

$$\nabla^2\phi = \nabla^2(\Phi\, Y^{0,m}) = \left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right]\Phi\, Y^{0,m} \tag{A.14}$$

The canonical components of velocity

These definitions are sufficient for our purposes so now we illustrate how to use these formulae with the momentum equation. Suppose we have represented the velocity field by the expression in equation A2, *i.e.*,

$$\begin{aligned}
\mathbf{v}(r,\theta,\phi) = \hat{\mathbf{r}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} U_l^m(r) Y_l^m(\theta,\phi) + \hat{\boldsymbol{\theta}}\sum_{l}^{\infty}\sum_{m=-l}^{l} V_l^m(r)\frac{\partial Y_l^m}{\partial\theta}(\theta,\phi) \\
+ i\operatorname{cosec}\theta\hat{\boldsymbol{\phi}}\sum_{l}\sum_{m=-l}^{l} mV_l^m(r)Y_l^m(\theta,\phi) + i\hat{\boldsymbol{\theta}}\operatorname{cosec}\theta\sum_{l}\sum_{m=-l}^{l} mW_l^m(r)Y_l^m(\theta,\phi) \\
- \hat{\boldsymbol{\phi}}\sum_{l}\sum_{m=-l}^{l} W_l^m(r)\frac{\partial Y_l^m}{\partial\theta}(\theta,\phi)
\end{aligned}$$

For simplicity we consider a single l, m component so that, on collecting terms, we obtain

$$\mathbf{v}(r,\theta,\phi) = \hat{\mathbf{r}}UY_l^m + \hat{\boldsymbol{\theta}}\left(V\frac{\partial Y_l^m}{\partial\theta} + im\operatorname{cosec}\theta WY_l^m\right) + \hat{\boldsymbol{\phi}}\left(im\operatorname{cosec}\theta VY_l^m - W\frac{\partial Y_l^m}{\partial\theta}\right) \tag{A.15}$$

where the indices l and m on U, V, and W are understood. Applying equation A3 gives

$$\begin{aligned}
v^- &= \frac{1}{\sqrt{2}}\left[(V - iW)\left(\frac{\partial Y_l^m}{\partial\theta} - m\operatorname{cosec}\theta Y_l^m\right)\right] \\
v^0 &= UY_l^m \\
v^+ &= \frac{1}{\sqrt{2}}\left[(V + iW)\left(-\frac{\partial Y_l^m}{\partial\theta} - m\operatorname{cosec}\theta Y_l^m\right)\right]
\end{aligned} \tag{A.16}$$

Equation A8 for a single l, m component gives

$$\begin{aligned} v^{-} &= V^{-} Y_l^{-1,m} \\ v^{0} &= V^{0} Y_l^{0,m} \\ v^{+} &= V^{+} Y_l^{+1,m} \end{aligned} \tag{A.17}$$

Comparison of A16 and A17 and use of the relationships between $Y_l^{N,m}$ and Y_l^m gives

$$\begin{aligned} V^{-} &= (V - iW)\gamma_l \Omega_0^l \\ V^{0} &= U\gamma_l \\ V^{+} &= (V + iW)\gamma_l \Omega_0^l \end{aligned} \tag{A.18}$$

We now have the relationship between the new and old coordinate system and so can turn to giving examples of tensor differentiation for a single l, m component.

The Strain Rate Tensor

In Cartesian coordinates the strain rate tensor is given by

$$\epsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \tfrac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

In spherical polar coordinates, the answer is not so obvious. In our canonical coordinate system we have

$$\epsilon^{\alpha\beta} = \tfrac{1}{2}\left(v^{\alpha,\beta} + v^{\beta,\alpha}\right)$$

where

$$v^{\alpha,\beta} = V^{\alpha|\beta}\, Y^{\alpha+\beta}$$

Using equation A13 gives

$$\begin{array}{lll} V^{-|-} = \tfrac{1}{r}\Omega_{-1}^l V^{-} & V^{-|0} = \tfrac{dV^{-}}{dr} & V^{-|+} = \tfrac{1}{r}[\Omega_0^l V^{-} - V^{0}] \\ V^{0|-} = \tfrac{1}{r}[\Omega_0^l V^{0} - V^{-}] & V^{0|0} = \tfrac{dV^{0}}{dr} & V^{0|+} = \tfrac{1}{r}[\Omega_1^l V^{0} - V^{+}] \\ V^{+|-} = \tfrac{1}{r}[\Omega_1^l V^{+} - V^{0}] & V^{+|0} = \tfrac{dV^{+}}{dr} & V^{+|+} = \tfrac{1}{r}\Omega_2^l V^{+} \end{array}$$

Thus

$$\epsilon^{\alpha\beta} = E^{\alpha\beta}\, Y^{\alpha+\beta} \quad \text{where} \quad E^{\alpha\beta} = \tfrac{1}{2}\left(V^{\alpha|\beta} + V^{\beta|\alpha}\right)$$

and we obtain for the components of $\epsilon^{\alpha\beta}$

$$\begin{array}{ll} \epsilon^{--} = \dfrac{1}{r}\Omega_{-1}^l V^{-} Y_l^{-2,m} & \epsilon^{-0} = \epsilon^{0-} = \tfrac{1}{2}\left[\dfrac{dV^{-}}{dr} - \dfrac{V^{-}}{r} + \dfrac{\Omega_0^l V^{0}}{r}\right] Y_l^{-1,m} \\ \epsilon^{00} = \dfrac{dV^{0}}{dr} Y_l^{0,m} & \epsilon^{+0} = \epsilon^{0+} = \tfrac{1}{2}\left[\dfrac{dV^{+}}{dr} - \dfrac{V^{+}}{r} + \dfrac{\Omega_1^l V^{0}}{r}\right] Y_l^{+1,m} \\ \epsilon^{++} = \dfrac{1}{r}\Omega_2^l V^{+} Y_l^{2,m} & \epsilon^{+-} = \epsilon^{-+} = \tfrac{1}{2}\dfrac{1}{r}[\Omega_0^l V^{-} + \Omega_1^l V^{+} - 2V^{0}] Y_l^{0,m} \end{array}$$

If we use the fact that $\Omega^l_{-1} = \Omega^l_2$ and $\Omega^l_0 = \Omega^l_1$ (as pointed out earlier) and we substitute for U^-, U^0 and U^+ from equation A18, we finally obtain

$$\left.\begin{aligned}
\epsilon^{00} &= U'\gamma_l Y_l^{0,m} \\
\epsilon^{\pm\pm} &= \frac{1}{r}(V \pm iW)\gamma_l \Omega^l_1 \Omega^l_2 Y_l^{\pm 2,m} \\
\epsilon^{\pm 0} &= \epsilon^{0\pm} = \tfrac{1}{2}\,[X \pm iZ]\gamma_l \Omega^l_1 Y_l^{\pm 1,m} \\
\epsilon^{\pm\mp} &= -\tfrac{1}{2}\,F\gamma_l Y_l^{0,m}
\end{aligned}\right\} \tag{A.19}$$

where prime ($'$) denotes differentiation with respect to radius and

$$X = V' + \frac{U-V}{r}, \quad Z = W' - \frac{W}{r} \quad \text{and} \quad F = \frac{1}{r}[2U - l(l+1)V]$$

For reference, we note that inspection of A19 gives the expansion coefficients, $E^{\alpha\beta}$:

$$\left.\begin{aligned}
E^{00} &= U'\gamma_l \\
E^{\pm\pm} &= \frac{1}{r}(V \pm iW)\gamma_l \Omega^l_1 \Omega^l_2 \\
E^{\pm 0} &= E^{0\pm} = \tfrac{1}{2}\,[X \pm iZ]\gamma_l \Omega^l_1 \\
E^{\pm\mp} &= -\tfrac{1}{2}\,F\gamma_l
\end{aligned}\right\} \tag{A.20}$$

Often we need go no further as the canonical components can be used throughout the calculations. If we wish to obtain expressions for ϵ in the original r, θ, ϕ coordinate system we must use equation A6, *i.e.*,

$$\begin{aligned}
\epsilon_{ij} &= C_{i\alpha} C_{j\beta} \epsilon^{\alpha\beta} \\
&= C_{i-}C_{j-}\epsilon^{--} + C_{i-}C_{j0}\epsilon^{-0} + C_{i-}C_{j+}\epsilon^{-+} \\
&\quad + C_{i0}C_{j-}\epsilon^{0-} + C_{i0}C_{j0}\epsilon^{00} + C_{i0}C_{j+}\epsilon^{0+} \\
&\quad + C_{i+}C_{j-}\epsilon^{+-} + C_{i+}C_{j0}\epsilon^{+0} + C_{i+}C_{j+}\epsilon^{++}
\end{aligned}$$

Remember that i and j go from 1 to 3 and $1 \equiv \theta$, $2 \equiv \phi$, $3 \equiv r$ directions respectively. As an example, consider the $\epsilon_{11} = \epsilon_{\theta\theta}$ component. We have from A4 that

$$C_{1-} = \frac{1}{\sqrt{2}}, \quad C_{10} = 0, \quad \text{and} \quad C_{1+} = -\frac{1}{\sqrt{2}}$$

Substitution gives

$$\begin{aligned}
\epsilon_{\theta\theta} &= \tfrac{1}{2}\,\epsilon^{--} - \tfrac{1}{2}\,\epsilon^{-+} - \tfrac{1}{2}\,\epsilon^{+-} + \tfrac{1}{2}\,\epsilon^{++} \\
&= \frac{V}{2r}\gamma_l \Omega^l_0 \Omega^l_2 \left[Y_l^{+2,m} + Y_l^{-2,m}\right] + \frac{iW}{2r}\gamma_l \Omega^l_0 \Omega^l_2 \left[Y_l^{+2,m} - Y_l^{-2,m}\right] + \tfrac{1}{2}\,F\gamma_l Y_l^{0,m}
\end{aligned}$$

where we have made use of equation A19. A complete list of strain-rate components is

$$\left.\begin{aligned}
\epsilon_{rr} &= U' K_0 \\
\epsilon_{\theta\theta} &= \frac{V}{r} K_2^+ - \frac{iW}{r} K_2^- + \frac{F}{2} K_0 \\
\epsilon_{\phi\phi} &= -\frac{V}{r} K_2^+ + \frac{iW}{r} K_2^- + \frac{F}{2} K_0 \\
2\epsilon_{r\theta} &= X K_1^- - iZ K_1^+ \\
2\epsilon_{r\phi} &= -iX K_1^+ - Z K_1^- \\
2\epsilon_{\theta\phi} &= -\frac{i2V}{r} K_2^- - \frac{2W}{r} K_2^+
\end{aligned}\right\} \tag{A.21}$$

where

$$\begin{aligned}
K_0 &= \gamma_l Y_l^{0,m} \\
K_1^{\pm} &= \frac{1}{\sqrt{2}} \gamma_l \Omega_1^l (Y_l^{-1,m} \pm Y_l^{+1,m}) \\
K_2^{\pm} &= \frac{1}{2} \gamma_l \Omega_1^l \Omega_2^l (Y_l^{-2,m} \pm Y_l^{+2,m}) \\
K_3^{\pm} &= \frac{1}{2\sqrt{2}} \gamma_l \Omega_1^l \Omega_2^l \Omega_3^l (Y_l^{-3,m} \pm Y_l^{+3,m})
\end{aligned}$$

Equation A21 can be cast in terms of ordinary spherical harmonics using the relationships in A10. The result is:

$$\left.\begin{aligned}
\epsilon_{rr} &= U' Y_l^m \\
\epsilon_{\theta\theta} &= \frac{1}{r}\left[U Y_l^m + V \left(m^2 \operatorname{cosec}^2\theta Y_l^m - l(l+1) Y_l^m - \cot\theta \frac{\partial Y_l^m}{\partial\theta} \right) \right] \\
&\quad + i\frac{W}{r} m \operatorname{cosec}\theta \left(\frac{\partial Y_l^m}{\partial\theta} - \cot\theta Y_l^m \right) \\
\epsilon_{\phi\phi} &= \frac{1}{r}\left[U Y_l^m + V \left(\cot\theta \frac{\partial Y_l^m}{\partial\theta} - m^2 \operatorname{cosec}^2\theta Y_l^m \right) \right] \\
&\quad - i\frac{W}{r} m \operatorname{cosec}\theta \left(\frac{\partial Y_l^m}{\partial\theta} - \cot\theta Y_l^m \right) \\
2\epsilon_{r\theta} &= X \frac{\partial Y_l^m}{\partial\theta} + im \operatorname{cosec}\theta Z Y_l^m \\
2\epsilon_{r\phi} &= im \operatorname{cosec}\theta X Y_l^m - Z \frac{\partial Y_l^m}{\partial\theta} \\
2\epsilon_{\phi\phi} &= i\frac{2mV}{r} \operatorname{cosec}\theta \left(\frac{\partial Y_l^m}{\partial\theta} - \cot\theta Y_l^m \right) \\
&\quad + \frac{W}{r} \left(2\cot\theta \frac{\partial Y_l^m}{\partial\theta} - 2m^2 \operatorname{cosec}^2\theta Y_l^m + l(l+1) Y_l^m \right)
\end{aligned}\right\} \tag{A.22}$$

where X and Z are defined in equation A19.

The Divergence of the Velocity Vector $\nabla \cdot \mathbf{v}$

In Cartesian coordinates $\nabla \cdot \mathbf{v} = \partial v_i/\partial x_i = \epsilon_{ii}$ (summation implied). In canonical components we have $\epsilon_{ii} = \epsilon_{ij}\delta_{ij} \equiv \Delta_{\alpha\beta}\epsilon^{\alpha\beta}$ and from equation A12 this becomes

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \epsilon^{00} - \epsilon^{+-} - \epsilon^{-+} \\ &= (U' + F)\,\gamma_l Y_l^{0,m} = (U' + F)\,Y_l^m \end{aligned} \tag{A.23}$$

where we have used equation A19. This is easily verified by summing $\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi}$ in equation A22.

The Divergence of the Stress Tensor

In the equations of motion we require $\nabla \cdot \tau$. As pointed out before, in canonical components this can be written

$$\nabla \cdot \tau \equiv \Delta_{\beta\gamma}\tau^{\alpha\beta,\gamma} = \tau^{\alpha 0,0} - \tau^{\alpha +,-} - \tau^{\alpha -,+} \equiv P^\alpha \quad \text{say} \tag{A.24}$$

Now suppose that the stress tensor has been expanded in generalised spherical harmonics as in A7 with a single l, m component being

$$\tau^{\alpha\beta} = T^{\alpha\beta}\,Y_l^{\alpha+\beta} \tag{A.25}$$

The covariant derivative is therefore

$$\tau^{\alpha\beta,\gamma} = T^{\alpha\beta|\gamma} Y_l^{\alpha+\beta+\gamma}$$

where the coefficients $T^{\alpha\beta|\gamma}$ can be determined from the $T^{\alpha\beta}$ using the recipe of equation A13. Substituting this last equation into A24 gives

$$\begin{aligned} P^- &= (T^{-0|0} - T^{-+|-} - T^{--|+})Y_l^{-1,m} \\ P^0 &= (T^{00|0} - T^{0+|-} - T^{0-|+})Y_l^{0,m} \\ P^+ &= (T^{+0|0} - T^{++|-} - T^{+-|+})Y_l^{+1,m} \end{aligned}$$

and using equation A13 gives

$$\begin{array}{lll} T^{-+|-} = \frac{1}{r}[\Omega_0^l T^{-+} - T^{-0}] & T^{-0|0} = \frac{d}{dr}T^{-0} & T^{--|+} = \frac{1}{r}[\Omega_{-1}^l T^{--} - T^{-0} - T^{0-}] \\ T^{0+|-} = \frac{1}{r}[\Omega_1^l T^{0+} - T^{00} - T^{-+}] & T^{00|0} = \frac{d}{dr}T^{00} & T^{0-|+} = \frac{1}{r}[\Omega_0^l T^{0-} - T^{00} - T^{+-}] \\ T^{++|-} = \frac{1}{r}[\Omega_2^l T^{++} - T^{0+} - T^{+0}] & T^{+0|0} = \frac{d}{dr}T^{+0} & T^{+-|+} = \frac{1}{r}[\Omega_1^l T^{+-} - T^{+0}] \end{array}$$

Combining these and remembering that the stress tensor is symmetric (so $T^{\alpha\beta} = T^{\beta\alpha}$) gives

$$\left.\begin{aligned} P^- &= \left[\frac{d}{dr}T^{-0} - \frac{1}{r}(\Omega_0^l T^{-+} - 3T^{-0} + \Omega_2^l T^{--})\right] Y_l^{-1,m} \\ P^0 &= \left[\frac{d}{dr}T^{00} - \frac{1}{r}(\Omega_0^l T^{0+} - 2T^{00} - 2T^{-+} + \Omega_0^l T^{0-})\right] Y_l^{0,m} \\ P^+ &= \left[\frac{d}{dr}T^{+0} - \frac{1}{r}(\Omega_0^l T^{-+} - 3T^{+0} + \Omega_2^l T^{++})\right] Y_l^{+1,m} \end{aligned}\right\} \tag{A.26}$$

Finally, the r, θ, ϕ components can be recovered from the $-, 0, +$ components by applying equation A6. Thus

$$
\begin{aligned}
P_i &= C_{i\alpha} P^\alpha = C_{i-} P^- + C_{i0} P^0 + C_{i+} P^+ \\
\text{so} \quad P_\theta &= P_1 = \frac{1}{\sqrt{2}} (P^- - P^+) \\
P_\phi &= P_2 = -\frac{i}{\sqrt{2}} (P^- + P^+) \\
P_r &= P_3 = P^0
\end{aligned}
$$

To proceed further we must specify the form of the stress-strain-rate relation. The case of an isotropic Newtonian fluid is particularly simple as the canonical components of stress are simply proportional to the canonical components of strain except for the terms involving the dilatation which require a little care. We have by analogy with an elastic solid:

$$\boldsymbol{\tau} = 2\mu \boldsymbol{\epsilon} + \lambda (\nabla \cdot \mathbf{v}) \, \mathbf{I}$$

where $\mathbf{I}$ is the identity tensor. In an incompressible fluid we let λ go to infinity as $\nabla \cdot \mathbf{v}$ goes to zero so that we keep pressure variations due to the flow in the equations. Thus

$$\boldsymbol{\tau} = 2\mu \boldsymbol{\epsilon} - \delta P \, \mathbf{I}$$

In canonical components the identity tensor is given by

$$\Delta^{\alpha\beta} = C^\dagger_{\alpha i} C^\dagger_{\beta j} \delta_{ij} = C^\dagger_{\alpha i} C^\dagger_{\beta i}$$

so that using equation A3 gives

$$\Delta^{00} = 1, \quad \Delta^{-+} = \Delta^{+-} = -1 \quad \text{and all other elements are zero}$$

(Note that $\Delta^{\alpha\beta} = \Delta_{\alpha\beta}$.) We can now write out the canonical components of τ, *i.e.*,

$$\tau^{\alpha\beta} = 2\mu\epsilon^{\alpha\beta} - \delta P \, \Delta^{\alpha\beta}$$

Thus

$$
\left.
\begin{aligned}
\tau^{00} &= [2\mu U' - \delta P] \, \gamma_l Y_l^{0,m} \\
\tau^{\pm\pm} &= \frac{2\mu}{r} (V \pm iW) \gamma_l \Omega_0^l \Omega_2^l Y_l^{\pm 2,m} \\
\tau^{\pm 0} &= \mu [X \pm iZ] \gamma_l \Omega_0^l Y_l^{\pm 1,m} \\
\tau^{\pm\mp} &= [-\mu F + \delta P] \, \gamma_l Y_l^{0,m}
\end{aligned}
\right\} \tag{A.27}
$$

Comparison with equation A25 gives the expansion coefficients, $T^{\alpha\beta}$ *i.e.*,

$$
\left.
\begin{aligned}
T^{00} &= [2\mu U' - \delta P] \, \gamma_l \\
T^{\pm\pm} &= \frac{2\mu}{r} (V \pm iW) \gamma_l \Omega_0^l \Omega_2^l \\
T^{\pm 0} &= T^{0\pm} = \mu [X \pm iZ] \gamma_l \Omega_0^l \\
T^{\pm\mp} &= - [\mu F - \delta P] \, \gamma_l
\end{aligned}
\right\} \tag{A.28}
$$

Substituting these into equation A26 gives

$$\left.\begin{aligned}
P^- &= \left\{ \frac{d}{dr}(\mu X) - i\frac{d}{dr}(\mu Z) \right.\\
&\quad \left. +\frac{1}{r}\left[\mu F - \delta P + 3\mu(X - iZ) - \frac{2\mu}{r}(V - iW)\Omega_2^l\Omega_2^l\right]\right\}\gamma_l\Omega_0^l Y_l^{-1,m}\\
P^0 &= \left\{ \frac{d}{dr}\left(2\mu U' - \delta P\right) \right.\\
&\quad \left. -\frac{1}{r}\left[2\mu X\Omega_0^l\Omega_0^l - 4\mu U' + 2\mu F\right]\right\}\gamma_l Y_l^{0,m}\\
P^+ &= \left\{ \frac{d}{dr}(\mu X) + i\frac{d}{dr}(\mu Z) \right.\\
&\quad \left. +\frac{1}{r}\left[\mu F - \delta P + 3\mu(X + iZ) - \frac{2\mu}{r}(V + iW)\Omega_2^l\Omega_2^l\right]\right\}\gamma_l\Omega_0^l Y_l^{+1,m}
\end{aligned}\right\} \tag{A.29}$$

so that the (r, θ, ϕ) components are

$$\left.\begin{aligned}
P_r &= \left\{ \frac{d}{dr}\left(2\mu U' - \delta P\right) + \frac{\mu}{r}\left[4U' - 2F - l(l+1)X\right]\right\} Y_l^m\\
P_\theta &= \left\{ \frac{d}{dr}(\mu X) + \frac{1}{r}\left[\mu F - \delta P + 3\mu X - \frac{\mu V}{r}(l+2)(l-1)\right]\right\}\frac{\partial Y_l^m}{\partial\theta}\\
&\quad + \left\{ \frac{d}{dr}(\mu Z) + \frac{\mu}{r}\left[3Z - \frac{W}{r}(l+2)(l-1)\right]\right\} im\,\mathrm{cosec}\,\theta Y_l^m\\
P_\phi &= \left\{ \frac{d}{dr}(\mu X) + \frac{1}{r}\left[\mu F - \delta P + 3\mu X - \frac{\mu V}{r}(l+2)(l-1)\right]\right\} im\,\mathrm{cosec}\,\theta Y_l^m\\
&\quad - \left\{ \frac{d}{dr}(\mu Z) + \frac{\mu}{r}\left[3Z - \frac{W}{r}(l+2)(l-1)\right]\right\}\frac{\partial Y_l^m}{\partial\theta}
\end{aligned}\right\} \tag{A.30}$$

The Traction Vector

The traction vector $\mathbf{t} = \hat{\mathbf{r}}\tau_{rr} + \hat{\boldsymbol{\theta}}\tau_{r\theta} + \hat{\boldsymbol{\phi}}\tau_{r\phi}$ can also be expanded in vector spherical harmonics (as we have implicity showed in the last section). We can recover the usual expression for a single l, m component of the elements of $\mathbf{t}$ from equation A25 along with equations A3 and A5, *i.e.*,

$$\tau_{ij} = C_{i\alpha}C_{j\beta}\tau^{\alpha\beta}$$

Thus

$$\begin{aligned}
\tau_{rr} &= \tau_{33} = C_{3\alpha}C_{3\beta}\tau^{\alpha\beta} = \tau^{00}\\
\tau_{r\theta} &= \tau_{31} = C_{3\alpha}C_{1\beta}\tau^{\alpha\beta} = \frac{1}{\sqrt{2}}(\tau^{0-} - \tau^{0+})\\
\tau_{r\phi} &= \tau_{32} = C_{3\alpha}C_{2\beta}\tau^{\alpha\beta} = -\frac{i}{\sqrt{2}}(\tau^{0-} + \tau^{0+})
\end{aligned}$$

For an isotropic Newtoninan fluid we get

$$
\begin{aligned}
\tau_{rr} &= [2\mu U' - \delta P]\, K_0 \\
\tau_{r\theta} &= \mu X K_1^- - i\mu Z K_1^+ \\
\tau_{r\phi} &= -i\mu X K_1^+ - \mu Z K_1^-
\end{aligned}
$$

so

$$
\left.
\begin{aligned}
\tau_{rr} &= [2\mu U' - \delta P]\, Y_l^m \\
\tau_{r\theta} &= \mu X \frac{\partial Y_l^m}{\partial\theta} + im \operatorname{cosec}\theta \mu Z Y_l^m \\
\tau_{r\phi} &= im \operatorname{cosec}\theta \mu X Y_l^m - \mu Z \frac{\partial Y_l^m}{\partial\theta}
\end{aligned}
\right\} \tag{A.31}
$$

Thus, defining

$$
R = 2\mu U' - \delta P \quad \text{and} \quad S = \mu X \quad \text{and} \quad T = \mu Z \tag{A.32}
$$

gives the usual vector spherical harmonic expansion of $\mathbf{t}$:

$$
\mathbf{t} = \hat{\mathbf{r}} R(r) Y_l^m + S(r) \nabla_1 Y_l^m - T(r) \hat{\mathbf{r}} \times \nabla_1 Y_l^m \tag{A.33}
$$