SIO 224 Homework 2

1) By using basic thermodynamic relationships (see over), show that

$$K_S = K_T(1 + \alpha T \gamma)$$

and

$$C_P = C_V(1 + \alpha T \gamma)$$

and

$$\left(\frac{\partial \alpha}{\partial P}\right)_T = -\frac{\alpha \delta_T}{K_T} \quad \text{where} \quad \delta_T = -\frac{1}{\alpha K_T}\left(\frac{\partial K_T}{\partial T}\right)_P$$

Use this latter equation to estimate how much α changes across the mantle.

2) Graph g, the acceleration due to gravity, as a function of radius in the Moon assuming it has a constant density, $\overline{\rho}$, of $3350\,\mathrm{kg\,m^{-3}}$ and a radius, R, of 1738 km? Graph the pressure as a function of radius. A multi-anvil press can now reproduce the conditions at a depth of 700 km inside the Earth; could this apparatus reproduce conditions at the center of the Moon?

3a) What is the "Peclet number" and what is its value for the mantle? Make a guess for what it might be in the core?

b) Estimate the adiabatic temperature rise across the lower mantle and across the outer core (you can take the temperature to be about 1850K at the 660km discontinuity and about 5200K at the ICB – we will be discussing these estimates in more detail later in the course). Comment on the implications of your results for the temperature behavior at the CMB.

4) You have almost no information about the inner core of the Earth. From seismology you have found that the seismic parameter is a function of radius given by the equation $\phi(r) = \phi_0 + Br^2$ where ϕ_0 and B are measured constants. ($\phi = V_p^2 - 4/3V_s^2$). You also know that density must decrease slightly with radius in the inner core (because the pressure is decreasing) so you approximate $\rho(r)$ by $\rho(r) = \rho_0 + Cr^2$ where ρ_0 and C are unknown constants.
Assuming that the inner core is adiabatic and homogeneous, compute ρ_0 and C given that $\phi_0 = 110\mathrm{km^2\,s^{-2}}$, $B = -3 \times 10^{-6}$ and the gravitational constant, $G = 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$.

If $\phi(r)$ is only determined with a precision of $\sim$1%, estimate by how much your density profile can vary and so determine if this is a good way to estimate $\rho(r)$ in the inner core. Furthermore, you have essentially assumed that the inner core is convecting. Do you think this is likely?
[Hint – you may assume that $Cr^2/\rho_0 \ll 1$]

1. Some useful relationships

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_z} \tag{1}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x = -1 \tag{2}$$

$$\left(\frac{\partial x}{\partial y}\right)_q = \left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_q \tag{3}$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial q}\right)_z \left(\frac{\partial q}{\partial y}\right)_z \tag{4}$$

$$\left(\frac{\partial}{\partial z}\left(\frac{\partial x}{\partial y}\right)_z\right)_y = \left(\frac{\partial}{\partial y}\left(\frac{\partial x}{\partial z}\right)_y\right)_z \tag{5}$$

2. Some definitions of thermodynamic properties

Thermal expansion and isothermal compressibility (isothermal bulk modulus):

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \quad \text{and} \quad \beta_T = \frac{1}{K_T} = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \tag{6}$$

The specific heat at constant pressure and volume, and Gruneisen's ratio:

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_P \quad \text{and} \quad C_v = T\left(\frac{\partial S}{\partial T}\right)_V \quad \text{and} \quad \gamma = \frac{\alpha K_T V}{C_v} \tag{7}$$

3. Maxwell's relations

$$-\left(\frac{\partial P}{\partial S}\right)_V = \left(\frac{\partial T}{\partial V}\right)_S \tag{8}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T \tag{9}$$

$$\left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial T}{\partial P}\right)_S \tag{10}$$

$$\left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial S}{\partial P}\right)_T \tag{11}$$