**SIO 224 Homework 7**

1) A simple 1-D model of asthenospheric counterflow can be used to deduce some interesting results about the mantle viscosity and upper mantle return flow. Assume a lithospheric plate that is 100 km thick overlies an upper mantle with viscosity $\eta$ that extends down to 600 km depth. Assume the mantle below 600 km depth is "rigid". The plate has length $L$ and moves from ridge to trench with velocity $V_0$.

As was shown in class, there are two components to the flow in the asthenosphere: one is a Couette flow driven by the viscous drag of the moving plate above and the other is the return flow in the opposite direction as the plate motion (from trench to ridge). The continuity equation prescribes that these two flows must be equivalent and in opposite directions.

a) The return flow is a channel flow driven by the pressure gradient established by the shear flow above it. What is the total pressure drop from trench to ridge?

b) Assuming the plate speed $V$ =8 cm/yr and $L$ = 8000 km (something like the Pacific plate) and assuming an upper bound of 1 km for the "observed" dynamic topography in the ocean basins, find an upper bound for $\eta$. How does this compare to estimates from post-glacial rebound studies? What have we assumed about this problem that could cause these two values to be different?

2a) In the class notes for fluid-dynamics, a derivation for the terminal velocity for Stokes flow past a rigid sphere with no slip boundary conditions is presented. Show that the terminal velocity for an inviscid sphere rising through a fluid of viscosity $\eta_f$ is given by

$$v_{term} = \frac{a^2 g(\rho_f - \rho_s)}{3\eta_f}$$

where $\rho_f$ and $\rho_s$ are the densities of the surrounding fluid and the sphere, respectively, and $a$ is the radius of the sphere. An inviscid sphere is defined as having no viscosity which approximates the case for very low viscosity fluid bubbles. This results in "free-slip" boundary conditions meaning no shear stress is supported along the boundary.

b) Use an integral to show that the inviscid sphere remains spherical while rising through the surrounding fluid. [Hint: Show that the total normal stress acting on the surface of the sphere is constant.]

c) How fast will a 1 $cm^3$ air bubble rise through water? What is the Reynold's number?

3) Estimate the size of the plume head that would have initiated the Hawaiian hotspot. Approximate the plume head as a spherical diapir (with radius $R_p$, viscosity $\eta_p$, and density contrast $\Delta\rho$) rising through the mantle of viscosity $\eta_m$. Write the expression for its terminal velocity, $V_\infty$, (hint: you derived this solution using a free slip boundary on a previous question). Imagine the trailing plume conduit that gets established by the rising plume head is a cylindrical pipe (i.e. Poiseuille flow) and has the same viscosity as the plume head ($\eta_p = \eta_m$).

Recall that the total volumetric flux of fluid in the conduit is given by: $Q_v = -dP/dz(\pi {R_c}^4)/(8\eta_p$ where $R_c$ is the radius of the pipe and in this case, the pressure gradient $-dP/dz = \Delta\rho g$. Write an expression for volumetric flow in terms of the cross-sectional area of the conduit, $A = \pi {R_c}^2$, i.e. $Q_v = f(A)$. You can now eliminate $A$ from this expression by using the definition of $Q_v = A\bar{v_c}$, where $\bar{v_c}$ is the mean velocity (averaged over the cross-sectional area). This allows you to derive an expression for $\bar{v_c}$ as a function of $Q_v$, i.e. $\bar{v_c} = f(Q_v)$.

Now assume the buoyancy of the rising plume head established the pressure gradient for flow in the conduit. This condition would mean that flow in the conduit maintains a mean velocity equal to the terminal velocity of the diapir (i.e. $V_\infty = \bar{v_c}$). You can now derive a final expression for $R_p$ in terms of $Q_v$ and independent of $A$. Given that the estimate for $Q_v$ under Hawaii is presently 13.7 $km^3$/yr, what is your estimate for $R_p$ of the Hawaiian plume head? Assume a single viscosity for the mantle.