SIO 224 Homework 9
Thermal history

As noted in the class notes (see also chapter 13 of STO), the equation that governs the mean temperature of the mantle over time is given by

$$MC_p \frac{d\bar{T}}{dt} = -Q_{out} + Q_{in} + MH \tag{1}$$

H is the radioactive heat generation in the mantle and can be roughly approximated by a single exponential $H_0 \exp(-\lambda t)$ where λ is about 1.42×10^{-17} s^{-1} . M is the mass of the mantle and C_p is the heat capacity. Also noted in the notes is the relationship between Nusselt number and Rayleigh number:

$$Nu = aRa^{\beta} \tag{2}$$

with $\beta \simeq 0.3$. From the notes, we have

$$Nu = \left(\frac{Ra}{2Ra_c}\right)^{\beta} \simeq 0.8 \left(\frac{Ra}{Ra_c}\right)^{\beta} \tag{3}$$

(this is valid for Ra much greater than Ra_c) Ra_c is typically about 1000. The Rayleigh number has the usual form given by:

$$Ra = \frac{\alpha g \rho \bar{T} d^3}{\kappa \eta} \tag{4}$$

where we have normalized the temperature such that the surface temperature is zero and $\bar{T}$ is the mean interior temperature. The Nusselt number can be written as

$$Nu = \frac{Q_{out}}{Q_{cond}(\bar{T})} \tag{5}$$

where $Q_{cond}(\bar{T})$ is the hypothetical heat flow that would emerge with the given average temperature, $\bar{T}$ and when only conduction operates. Generally, we can write $Q_{cond}(\bar{T}) \simeq c\bar{T}$ where $c \simeq k/d$ so that

$$Q_{out} = c\bar{T} \left(\frac{Ra}{Ra_c}\right)^{\beta} \tag{6}$$

The only term that is likely to be a strong function of time in the Rayleigh number (apart from temperature itself) is the viscosity η because the viscosity is a strong function of temperature. In the past, when temperature was high, the viscosity would be low and convection would be more vigorous. This would lower the mean temperature and increase viscosity so slowing down convection. In this sense, the temperature is self-regulating. We approximate the temperature dependence of viscosity as

$$\eta = \eta_0 \exp\left(\frac{A_0}{T}\right) \tag{7}$$

Choosing the constants in this viscosity law and a mean mantle temperature now (say about 2500K) allows you to calculate η and thus the Rayleigh number. Note that, given a value for A_0, you should adjust η_0 so that the current mean mantle viscosity is between $10^{21} \rightarrow 10^{22}$ Pa s. Given the current day heat flux (corrected for radioactive heating in the continental crust) you can calculate c from equation 6 (assuming a value for the critical Rayleigh number). Q_{in} is the heat from the core into the mantle and you can choose a suitable value based on your reading of the notes. If the whole core is convecting, Q_{in} will be proportional to the temperature at the top of the core and so should probably slowly decrease with time – but this is an optional complication you might want to consider. You will also need an estimate of H_0, M and C_p. According to

STO, H_0 is about 35×10^{-12} W/kg. This gives a current day radioactive heat production of about 18TW which might be a bit large. 28×10^{-12} W/kg gives a current radioactive heat production of about 15TW which is probably more reasonable. If you want to use the more complete treatment of radioactive decay given in the McNamara paper, be my guest!

You can now integrate equation 1 back in time (numerically) – you might want to consult Numerical Recipes for a good algorithm for this – or trust MATLAB – at each time step you would update the viscosity and the Rayleigh number based on the new mean mantle temperature and you would compute the radioactive heat production and the heat flow out.

I think most of the numbers you need are fairly robust but there are four that are uncertain: β could be quite a bit lower that 0.3, A_0 could be between 3×10^4 and 9×10^4 K. Note that setting $A_0 = 0$ results in an isoviscous calculation which would be fun to explore. H_0 could be off – (you might want to see what happens if there is no radioactivity – also fun to explore). Q_{in} is probably between 8 and 12TW – recent results have made it unlikely that this number is lower than this (an option in the past), but unlikely to be higher.

1) Justify your choice of parameters needed for the calculation

2) Experiment with the four in particular that are uncertain – which affect the outcome of the thermal history the most?

3) What are the ranges of parameters that give "successful" thermal histories – i.e., finite positive temperatures in the past, reasonable reductions in viscosity, reasonable heat flows in the past. Note that temperatures should not exceed about 4000K or you will start to melt the mantle. Similarly, temperatures should not get too cold in the past – McNamara and van-Keken say that 1500K is probably a minimum.

4) Compute the Urey numbers for the successful calculations (radioactive heat/conducted heat out)

5) What is your best estimate of the cooling rate of the mantle – at least over the last 2–3 billion years

6) Is there any independent information that can support your estimate of the cooling rate

7) Would you expect the core to be cooling at the same rate or a greater or lower rate?