The Anelastic Approximation in Linear and Finite-Amplitude Models of Mantle Convection

Gary T. Jarvis
Associate Dean, Faculty of Graduate Studies
York University, Toronto

J. Fluid Mech. (1980), *vol.* 96, *part* 3, *pp.* 515–583
Printed in Great Britain

Convection in a compressible fluid with infinite Prandtl number

By GARY T. JARVIS† AND DAN P. McKENZIE
Department of Geodesy and Geophysics, University of Cambridge, Madingley Rise, Madingley Road, Cambridge CB3 0EZ

(Received 22 August 1978 and in revised form 6 April 1979)

The Anelastic Approximation

- The an-elastic (or non-elastic) approximation $\leftrightarrow$ $\partial\rho/\partial t = 0$

  i.e., elastic waves (acoustic modes) do not exist

  Condition:
- $M^2 << 1$ [Gough, 1969. J. Atmos. Sci., **26**, 448-456.]

  M: Mach number (ratio of convective velocity to sound velocity):

  $M^2 << 1 \longrightarrow$ a separation of time scales $\longrightarrow$ elastic vibrations irrelevant on convective time scales *(and vice-versa)*.

For gases, $\alpha T = 1$. For liquids, $\alpha T << 1$. <= Anelastic Liquid Approx.]

Conservation of Mass, Momentum and Energy

$$\partial\rho/\partial t + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho(\partial \mathbf{v}/\partial t + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla P + \rho \mathbf{g} + \partial \tau_{ij}/\partial x_j$$

$$\rho C_p[\partial T/\partial t - (\alpha T/\rho C_p)\partial P/\partial t + \mathbf{v} \cdot (\nabla T - \nabla T_s)]$$

$$= \nabla \cdot (K \nabla T) + H + \tau_{ij}(\partial v_i/\partial x_j)$$

Non Boussinesq Terms

Incompressible Conditions

$$\partial\rho/\partial t + \nabla\bullet\rho\mathbf{v} = 0$$

Anelastic:

$$\nabla\bullet\rho\mathbf{v} = 0$$

or

$$\rho\nabla\bullet\mathbf{v} + \mathbf{v}\bullet\nabla\boldsymbol{\rho} = 0$$

Incompressible ($\nabla\rho = 0$):

$\rho\nabla\bullet\mathbf{v} = 0 \Rightarrow \nabla\bullet\mathbf{v} = 0$ Solenoidal **v** field.

Aside: in 2D:

$\exists\ \boldsymbol{\Psi} = (0, \psi, 0)$ s.t. $\mathbf{v} = (u, 0, w) = (-\partial\psi/\partial z, 0, \partial\psi/\partial x)$

Reasonable for shallow layers, i.e. $d << H_T$

$H_T = C_p/g\alpha$ <= Temperature scale height

[Boussinesq approximation requires $d/H_T << 1$.]

Deep Layers - d = O(H_T)

The Hydrostatic, Isentropic, Stratified **Reference State**:

Hydrostatic Pressure, $P_H(z)$
Adiabatic Temperature Distribution, $T_s(z)$
Hydrostatically Compressed Density, $\rho_r(z) = \rho_r(S,P_H)$

$$\nabla\rho_r\,(S, P_H) = (\partial\rho_r/\partial S)_P\,\nabla S + (\partial\rho_r/\partial P_H)_s\,\nabla P_H$$

Isentropic $\Rightarrow$ $\nabla S = 0$,
Hydrostatic $\Rightarrow$ $\nabla P_H = \rho_r \mathbf{g}$,

$$\partial\rho_r/\partial z = -\,\rho_r g\,(\partial\rho_r/\partial P_H)_s = -\,\rho_r^2 g/K_s$$

Gruneisen's Parameter,

$$\Gamma = \alpha K_s/\rho C_p = \alpha K_T/\rho C_v \approx \text{constant} \approx 1.1$$

=> Adams-Williamson relation:

$$\partial\rho_r/\partial z = -\rho_r g\alpha/\Gamma C_p \;=\; -\rho_r/H_T\Gamma$$

$$\Rightarrow \rho_r(z) = \rho_d \exp\,[\,(d-z)/H_T\Gamma\,]$$

At bottom: $\rho_r(z{=}0) = \rho_d \exp\,[\,d/H_T\Gamma\,]$,

At top: $\rho_r(z{=}d) = \rho_d$

and

$$\rho_r(z{=}0)/\rho_r(z{=}d) = \exp[\,d/H_T\Gamma\,]$$

$\therefore \Delta\rho$ Negligible iff $(d/H_T) \ll 1$ $[\because \Gamma \approx 1]$

Expand density about the reference state:

$$\rho(T,P) = \rho_r [1 - \alpha T_1 + K_T^{-1} P_1]$$

where $T_1 = T - T_s$, and $P_1 = P - P_H$

$\because K_T^{-1} = \alpha / \Gamma \rho_r C_v$ and $C_v = C_p / [1 + \alpha \Gamma T_s]$

$$\rho = \rho_r [1 - \alpha T_1 + \alpha \{(1 + \alpha \Gamma T_s)/(\Gamma \rho_r C_p)\} P_1]$$

In Dimensionless form:

$$\rho = \rho_r \left[1 - \mu T_1 + \mu D \left\{(1+ \mu\Gamma (T_s+T_d))/(\Gamma \rho_r)\right\} P_1\right]$$

where

$$\rho_r = e^{(1-z)D/\Gamma}, \quad \mu = \alpha\Delta T \quad \text{and} \quad D = d/H_T.$$

For liquids: $\mu \ll 1$

For shallow layers: $D \ll 1$

(For dilute gases: $\rho = P/RT \Rightarrow \alpha = T^{-1} \Rightarrow \mu \approx 1$)

Expanding the P_1 term:

$$\rho = \rho_r[1 - \mu T_1 + \mu D\{1/(\Gamma \rho_r)\}P_1 + \mu^2 D\{\Gamma (T_s + T_d)/(\Gamma \rho_r)\}P_1]$$

For the Boussinesq Approximation [μ , $D << 1$]; neglecting terms of second order in small parameters yields,

$$\rho = \rho_r [1 - \mu T_1].$$

For the Anelastic-Liquid Approximation [$\mu << 1$, $D = O(1)$]; neglecting terms of second order in small parameters yields,

$$\rho = \rho_r [1 - \mu T_1 + \mu D/(\Gamma \rho_r)P_1].$$

[Both $\partial T_1/\partial x$ and $\partial P_1/\partial x$ contribute to driving forces]

Conservation of Mass (dimensionless):

$$\nabla\bullet\rho\mathbf{v} = \nabla\bullet\rho_r[1+O(\mu)]\mathbf{v} = 0$$

Neglecting terms of first order in μ,

$$\rho_r\nabla\bullet\mathbf{v} + \mathbf{v}\bullet\nabla\rho_r = 0$$

⇒ $$\nabla\bullet\mathbf{v} = -(1/\rho_r)\,\mathbf{v}\bullet\nabla\rho_r$$

⇒ $$\nabla\bullet\mathbf{v} = w(D/\Gamma)$$

Stream function - vorticity relation:

$$\nabla \bullet \mathbf{v} = w(D/\Gamma)$$

v is not solenoidal, but $\rho_r \mathbf{v}$ is, and

$$\mathbf{V} = e^{-D/\Gamma} \rho_r \mathbf{v} = \mathbf{v}\, e^{-zD/\Gamma}$$

is solenoidal.

∴ **V** can be described by a stream function, ψ, as

$$\mathbf{V} = [\, -\partial\psi / \partial z,\ 0,\ \partial\psi / \partial x],$$

and

$$\mathbf{v} = e^{zD/\Gamma}\, \mathbf{V}.$$

Vorticity:

$$\mathbf{\Omega} = (0, \omega, 0) = \nabla \times \mathbf{v} = \nabla \times e^{zD/\Gamma}\, \mathbf{V},$$

$$\Rightarrow \qquad \nabla^2 \psi + (D/\Gamma)\ \partial\psi / \partial z = -\omega\, e^{-zD/\Gamma}.$$

Temperature derives from the Entropy Equation

$$\rho T\, Ds/Dt = \nabla \bullet k \nabla T + H + \Phi$$

where

T is absolute temperature, s is entropy, H is the volumetric rate of internal heating,

$\Phi = \tau_{ij}\, (\partial v_i / \partial x_j)$,
is the rate of heating from the dissipation of mechanical energy and

$\tau_{ij} = \eta(\partial v_i / \partial x_j + \partial v_j / \partial x_i - 2/3\delta_{ij} \nabla \bullet \mathbf{v})$
is the deviatoric stress tensor.

Temperature (continued) ..

Since $$T ds = C_p dT - (\alpha T/\rho)\, dP,$$

$$\rho C_p\, DT/Dt - \alpha T\, DP/Dt = \nabla \bullet k \nabla T + H + \Phi.$$

$$\Rightarrow \rho C_p[\partial T/\partial t - (\alpha T/\rho C_p)\partial P/\partial t + \mathbf{v} \bullet (\nabla T - (\alpha T/\rho C_p)\nabla P)]$$

$$= \nabla \bullet k \nabla T + H + \Phi$$

$$\Rightarrow [\partial T/\partial t - (\alpha T/\rho C_p)\partial P/\partial t] = - \mathbf{v} \bullet (\nabla T - \nabla T_s) + \nabla \bullet k \nabla T + H + \Phi$$

where ∇T_s, refers to the adiabatic temperature gradient, defined as: $\nabla T_s = (\alpha T/\rho C_p)\nabla P \approx -(g\alpha/C_p)T$

[N.B. ∇T_s is proportional to T]

and

$|\nabla T_s|$ *decreases with height if* $(g\alpha/C_p)$ *is constant.*

Dimensionless Temperature Equation

$$\rho \frac{\partial T}{\partial t} - \mu D T \frac{\partial P_1}{\partial t} = -\nabla . T\rho\mathbf{u} - \rho D(T+T_0)\, w + \mu D T \mathbf{u} . \nabla P_1 + \kappa_0 \nabla^2 T + \epsilon_0 + D\, e^{2zD/\Gamma} \times \left[\left(2\frac{\partial^2 \psi}{\partial x \partial z} + \frac{D}{\Gamma}\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} - \frac{D}{\Gamma}\frac{\partial \psi}{\partial z} \right)^2 + \frac{1}{3}\left(\frac{D}{\Gamma}\frac{\partial \psi}{\partial x} \right)^2 \right]$$

Neglecting terms of order μ:

$$e^{(1-z)D/\Gamma} \frac{\partial T}{\partial t} = -\nabla . [T\, e^{(1-z)D/\Gamma} \mathbf{u}] - D\, e^{(1-z)D/\Gamma}(T+T_0)\, w + \kappa_0 \nabla^2 T + \epsilon_0 + D\, e^{2zD/\Gamma} \times \left[\left(2\frac{\partial^2 \psi}{\partial x \partial z} + \frac{D}{\Gamma}\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} - \frac{D}{\Gamma}\frac{\partial \psi}{\partial z} \right)^2 + \frac{1}{3}\left(\frac{D}{\Gamma}\frac{\partial \psi}{\partial x} \right)^2 \right].$$

Momentum equation.

$$\mathbf{0} = -\nabla P + \rho \mathbf{g} + \nabla \bullet \tau_{ij}$$

For constant η:

$$\mathbf{0} = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{v} + \eta \nabla(\nabla \bullet \mathbf{v})/3$$

Taking the **Curl**, (∇ **x** .), of momentum equation eliminates both **Grad** terms:

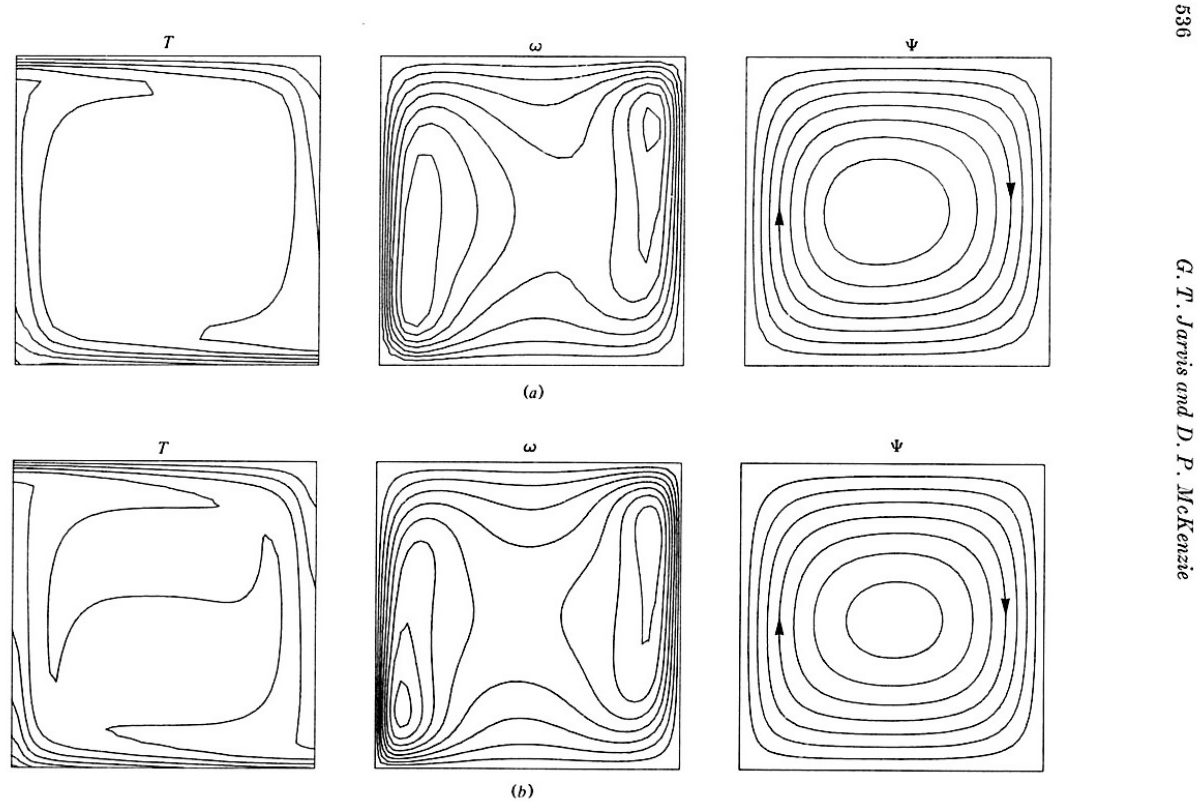
$$\nabla^2 \mathbf{\Omega} = [0, \nabla^2 \omega, 0] = [0, -(g/\eta)\partial\rho/\partial x, 0]$$

$$\Rightarrow \quad \nabla^2 \omega = (g\alpha/\eta)\,[\,\rho_r \partial T_1/\partial x \;-\; \{(1+\alpha\Gamma T_s)/(\Gamma C_p)\}\partial P_1/\partial x\,],$$

where

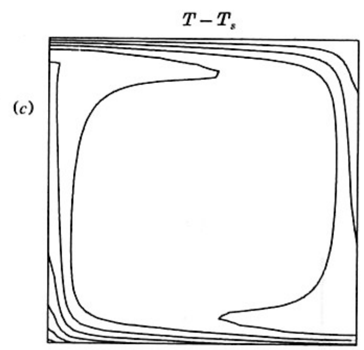
$$\partial P_1/\partial x = \eta \nabla^2 u + (\eta/3H_T\Gamma)\,\partial w/\partial x$$

Boussinesq



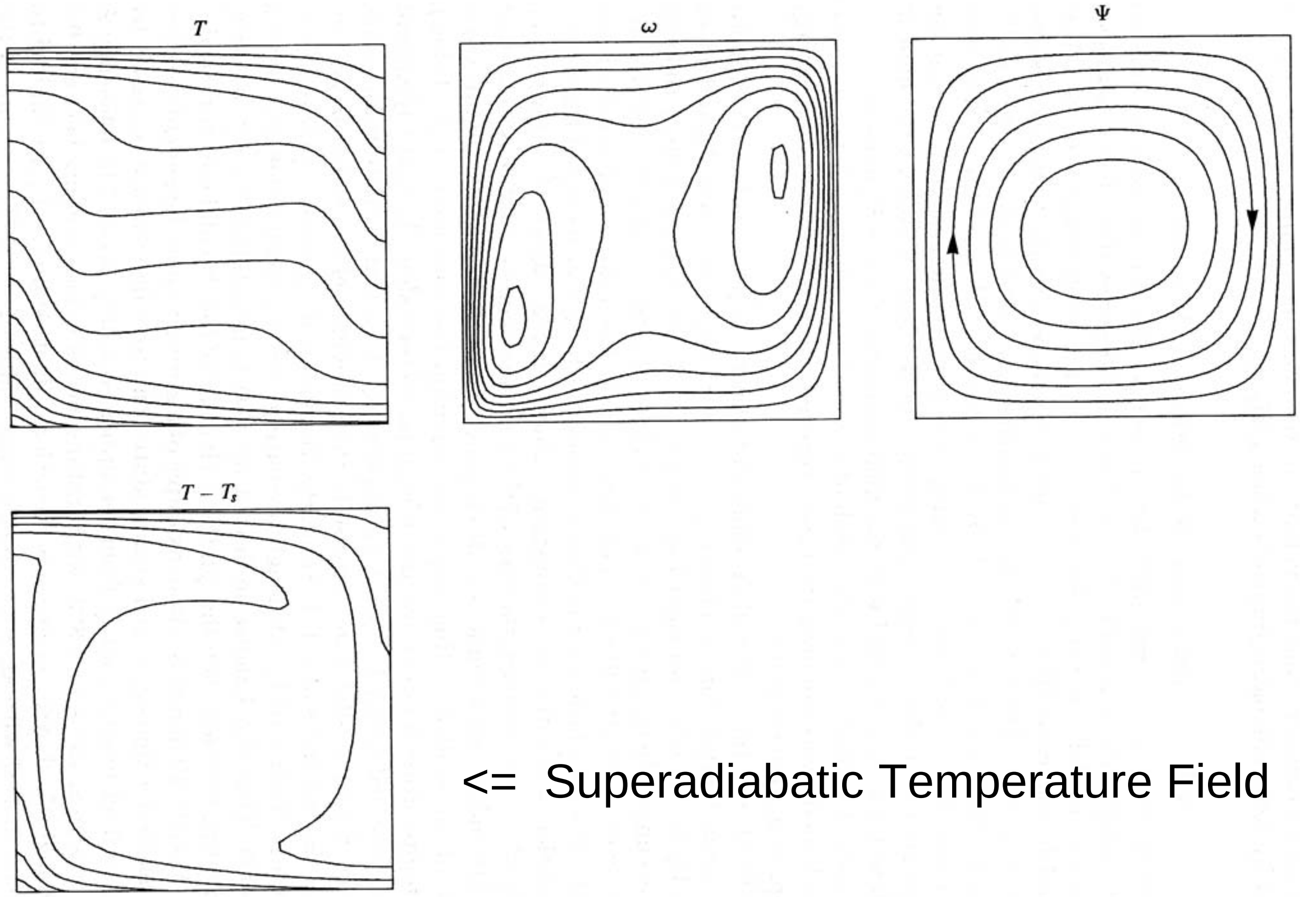
Anelastic

Anelastic

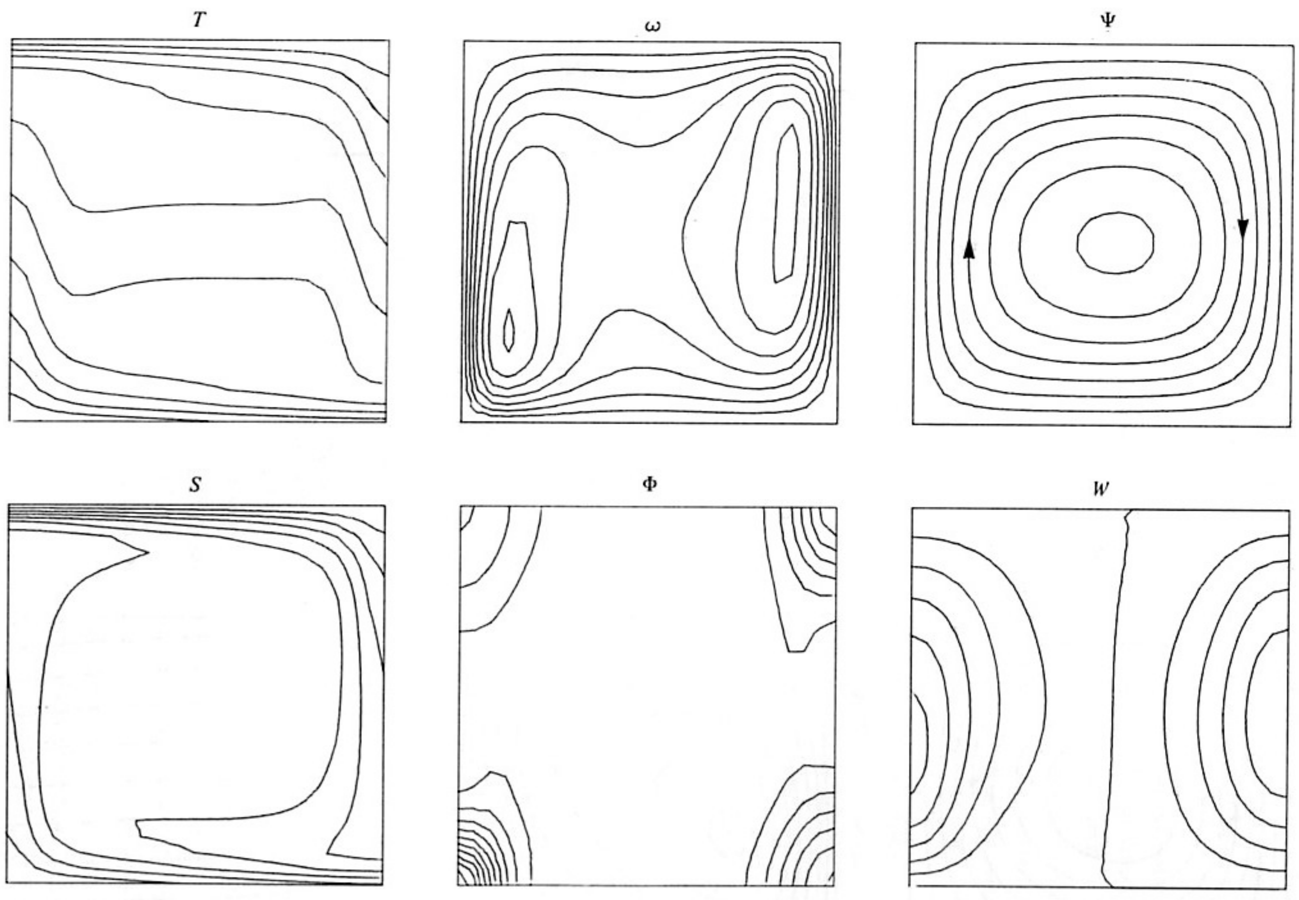


Superadiabatic temperature in Anelastic Liquid model is similar to Boussinesq model.

D = 0.117, Boussinesq Limit

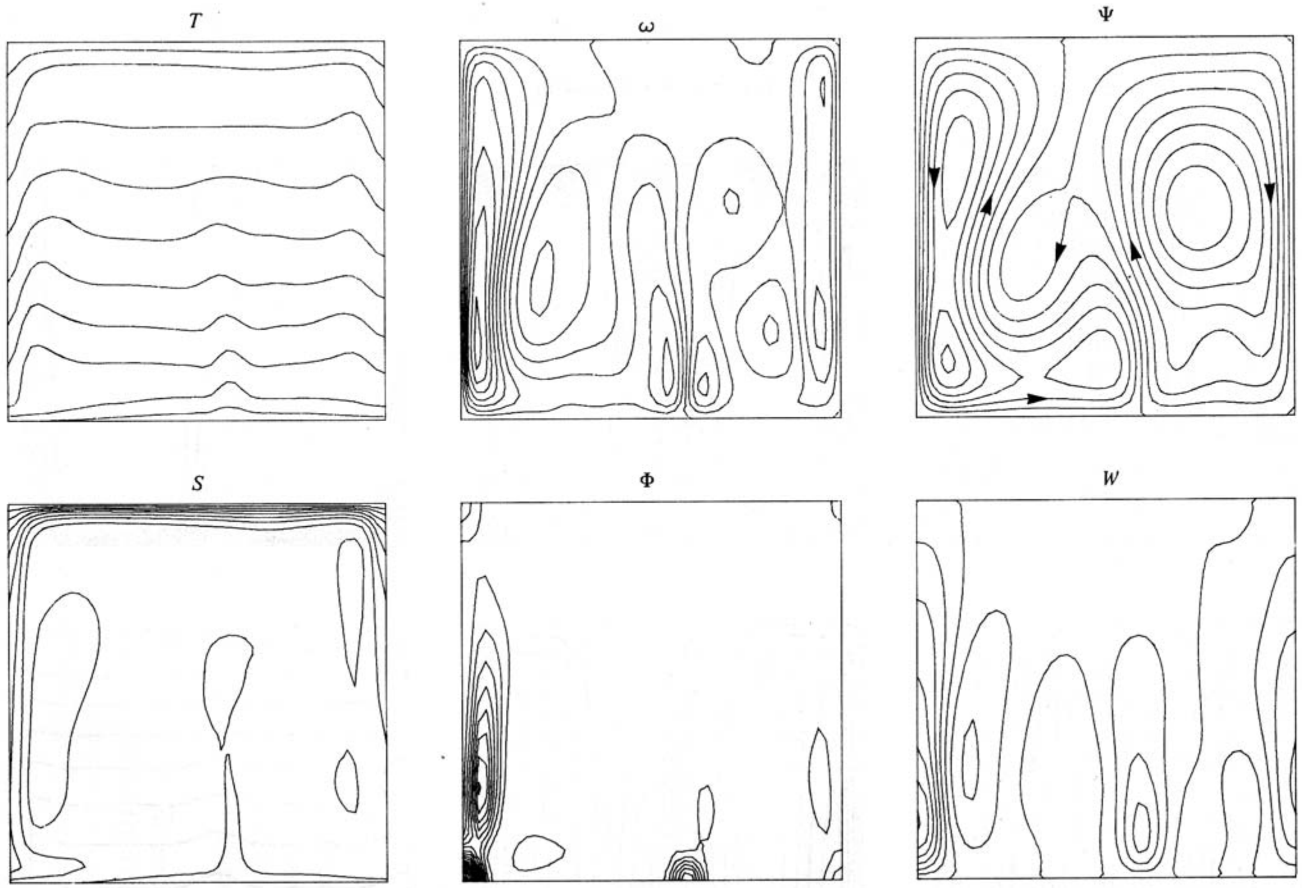


<= Superadiabatic Temperature Field



(b)

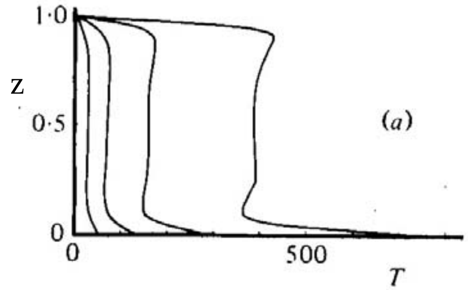
D = 0.50, Entropy, S, looks like T in Boussinesq models.

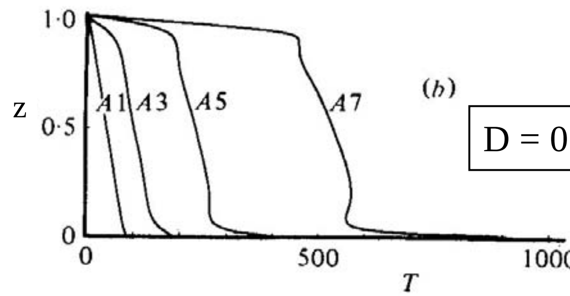


D = 1.00   Time dependent solution - S mimics T-T_s ,   Φ affects T (and S)

Temperature Profiles

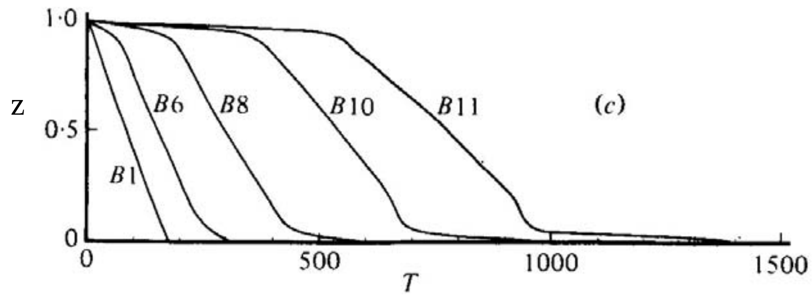
D = 0.0





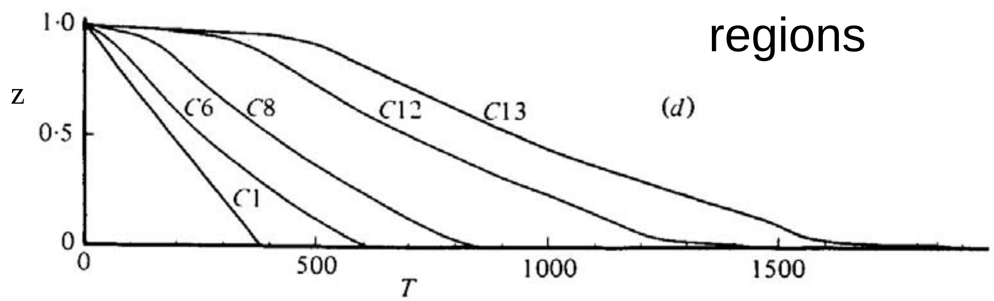
D = 0.25

D = 0.5



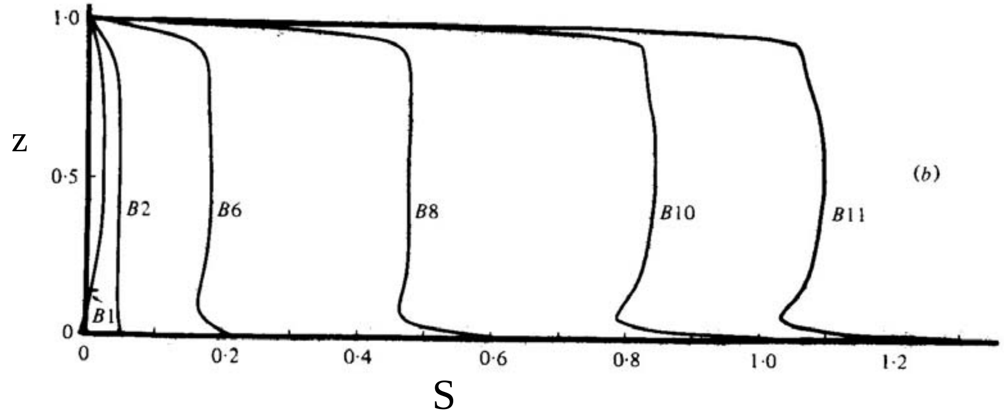
Adiabatic vs Isothermal central regions

D = 1.0

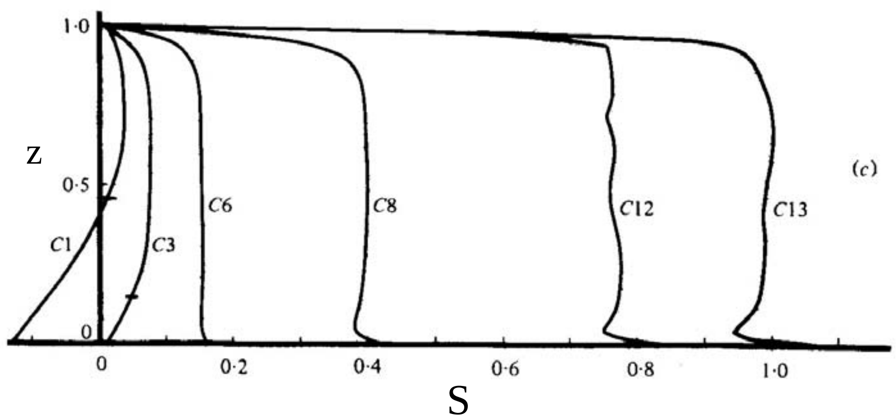


Entropy Profiles

D = 0.5



D = 1.0



Note: Isentropic central regions

Thermodynamic Efficiency: $E = \Phi / F_{surface}$

Global integration of Energy equation in Steady State:

$$\Phi = D \langle F \rangle,$$

$\langle F \rangle$ = mean convected heat flux,

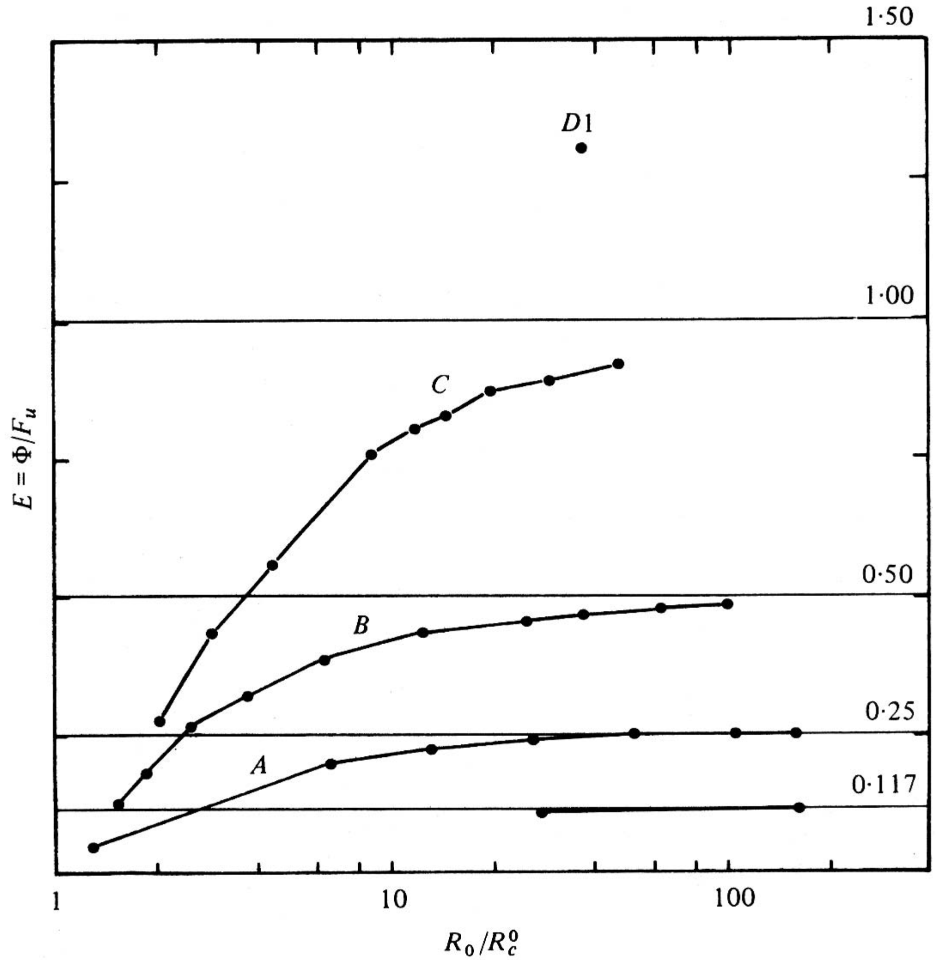
or

$$\Phi / \langle F \rangle = D.$$

For vigorous convection (Nu >>1) , $\langle F \rangle \approx F_{surface}$

$$\therefore \quad E = \Phi / F_{surface} \approx \Phi / \langle F \rangle = D,$$

in the high Rayleigh number limit.


1·50
D1
1·00
C
$E = \Phi/F_u$
0·50
B
0·25
A
0·117
1
10
100
R_0/R_c^0

... Thermodynamic Efficiency

For low Rayleigh numbers,

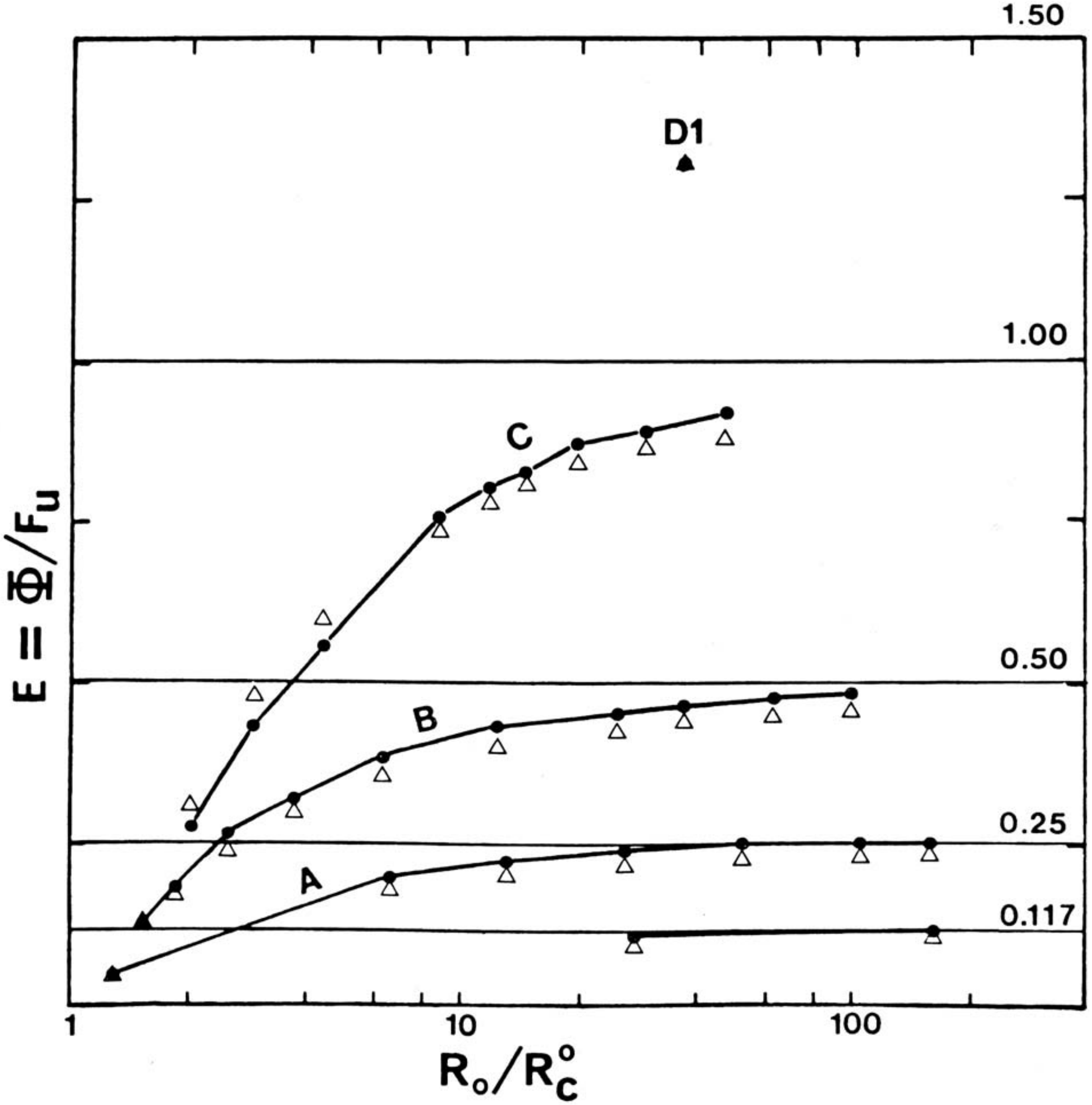
$$\langle F \rangle = pF_{surface}, \quad \text{where } p < 1$$

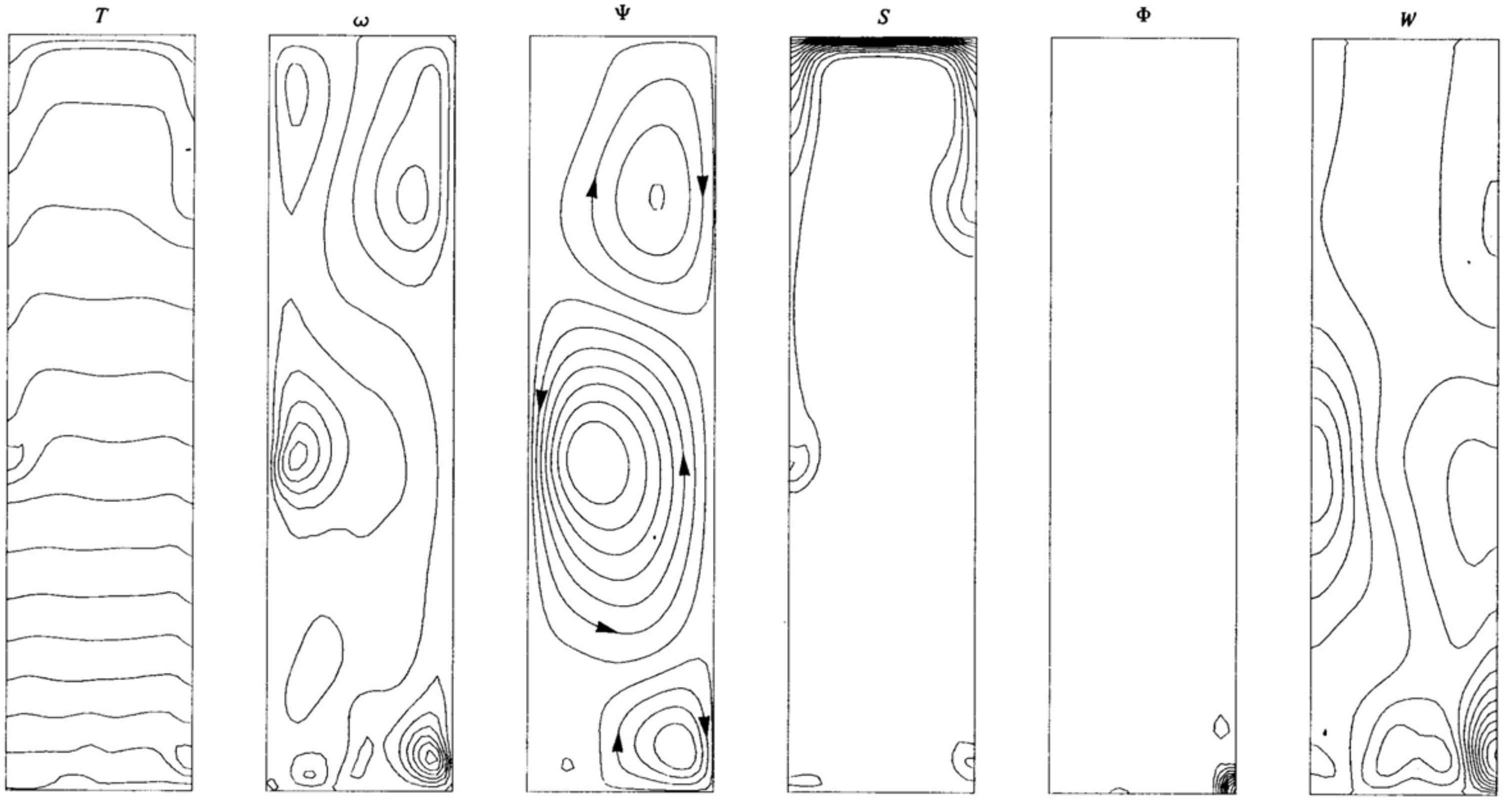
Estimate : $p \approx [1 - Nu^{-1}]$

With $F_{surface} = \langle F \rangle / p$,

$$E = \Phi / F_{surface} = p\, \Phi / \langle F \rangle = pD \approx [1 - Nu^{-1}]D$$

$\therefore$ $$E \approx [1 - Nu^{-1}]D$$


1.50
D1
1.00
C
0.50
B
0.25
A
0.117
E = Φ/Fu
1
10
100
Ro/Rc°



Deep Layer Model on 24 x 96 grid for D = 1.50

Depth Dependence of Adiabatic Temperature Gradients

Adiabatic Gradient: $-g\alpha T/C_p$ is proportional to T in our models.

This stabilizes the lower regions relative to the upper.

In the Earth α decreases by a factor of ~3 with depth across the mantle, while T increases by ~3 with depth from the base of the lithosphere to the CMB.

So the product αT remains ~ constant.

Thus many features in our models do not apply to the Earth.

Adding a constant adiabatic temperature gradient to incompressible model temperature fields may suffice.

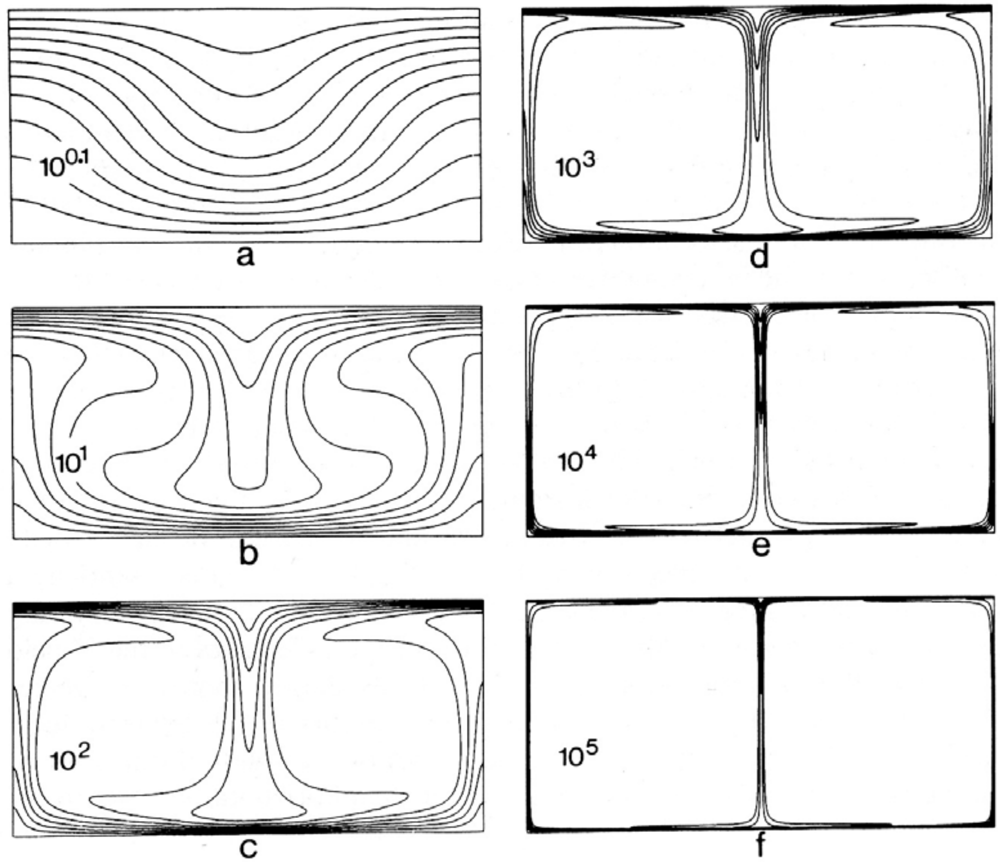


FIG. 8. Steady two-dimensional temperature fields for pairs of counter-rotating convection rolls. Contours of temperature are plotted with a constant interval of $\Delta T/11$ in each frame (where ΔT is the temperature difference across the respective layers). Each frame is labelled with the corresponding value of R_B/R_c. Solutions were obtained on the following numerical meshes: a and b, 24×24; c, 80×80; d, e, and f, 200×200.

Basic Equations

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\rho(\partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla P + \rho\mathbf{g} + \partial\tau_{ij}/\partial x_j$$

$$\rho C_p[\partial T/\partial t - (\alpha T/\rho C_p)\,\partial P/\partial t + \mathbf{v}\cdot(\nabla T - \nabla T_s)]$$

$$= \nabla\cdot(K\nabla T) + H + \tau_{ij}(\partial v_i/\partial x_j)$$

$$\rho = \rho_r[1 - \mu T_1 + \mu\mathbf{D}\{1/(\Gamma\,\rho_r)\}P_1 + \mu^2\mathbf{D}\{\Gamma\,(T_s + T_d)/(\Gamma\,\rho_r)\}P_1]$$

Anelastic Equations

$$\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{v}) = 0$$

$$\rho(\partial\mathbf{v}/\partial t + \mathbf{v} \cdot \nabla\mathbf{v}) = -\nabla P + \rho\mathbf{g} + \partial\tau_{ij}/\partial x_j$$

$$\rho C_p[\partial T/\partial t - (\alpha T/\rho C_p)\,\partial P/\partial t + \mathbf{v} \cdot (\nabla T - \nabla T_s)]$$

$$= \nabla \cdot (K\nabla T) + H + \tau_{ij}(\partial v_i/\partial x_j)$$

$$\rho = \rho_r[1 - \boldsymbol{\mu} T_1 + \boldsymbol{\mu}\mathbf{D}\{1/(\Gamma\rho_r)\}P_1 + \boldsymbol{\mu}^2\mathbf{D}\{\Gamma(T_s + T_d)/(\Gamma\rho_r)\}P_1]$$

Anelastic Liquid Equations

$$\partial\rho/\partial t + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho(\partial \mathbf{v}/\partial t + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla P + \rho \mathbf{g} + \partial \tau_{ij}/\partial x_j$$

$$\rho C_p[\partial T/\partial t - (\alpha T/\rho C_p)\partial P/\partial t + \mathbf{v} \cdot (\nabla T - \nabla T_s)]$$

$$= \nabla \cdot (K \nabla T) + H + \tau_{ij}(\partial v_i/\partial x_j)$$

$$\rho = \rho_r[1 - \mu T_1 + \mu \mathbf{D}\{1/(\Gamma \rho_r)\}P_1 + \mu^2 \mathbf{D}\{\Gamma (T_s + T_d)/(\Gamma \rho_r)\}P_1]$$

Truncated Anelastic Liquid Equations

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\rho(\partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla P + \rho\mathbf{g} + \partial\tau_{ij}/\partial x_j$$

$$\rho C_p[\partial T/\partial t - (\alpha T/\rho C_p)\,\partial P/\partial t + \mathbf{v}\cdot(\nabla T - \nabla T_s)]$$

$$= \nabla\cdot(K\nabla T) + H + \tau_{ij}(\partial v_i/\partial x_j)$$

$$\rho = \rho_r[1 - \mu T_1 + \mu\mathbf{D}\{1/(\Gamma\rho_r)\}P_1 + \mu^2\mathbf{D}\{\Gamma(T_s+T_d)/(\Gamma\rho_r)\}P_1]$$

Extended Boussinesq Equations

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\rho(\partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla P + \rho\mathbf{g} + \partial\tau_{ij}/\partial x_j$$

$$\rho C_p[\partial T/\partial t - (\alpha T/\rho C_p)\,\partial P/\partial t + \mathbf{v}\cdot(\nabla T - \nabla T_s)]$$

$$= \nabla\cdot(K\nabla T) + H + \tau_{ij}(\partial v_i/\partial x_j)$$

$$\rho = \rho_r[1 - \mu T_1 + \mu\mathbf{D}\{1/(\Gamma\rho_r)\}P_1 + \mu^2\mathbf{D}\{\Gamma(T_s+T_d)/(\Gamma\rho_r)\}P_1]$$

$\rho_r = \rho_{surface}$ = a constant

Boussinesq Equations

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\rho(\partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla P + \rho\mathbf{g} + \partial\tau_{ij}/\partial x_j$$

$$\rho C_p[\partial T/\partial t - (\alpha T/\rho C_p)\,\partial P/\partial t + \mathbf{v}\cdot(\nabla T - \nabla T_s)]$$
$$= \nabla\cdot(K\nabla T) + H + \tau_{ij}(\partial v_i/\partial x_j)$$

$$\rho = \rho_r[1 - \mu T_1 + \mu\mathbf{D}\{1/(\Gamma\rho_r)\}P_1 + \mu^2\mathbf{D}\{\Gamma(T_s+T_d)/(\Gamma\rho_r)\}P_1]$$

$\rho_r = \rho_{surface}$ = a constant