SIO 224

Thermal convection

1. Conservation of Mass and Momentum

We derived these earlier in class

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

or, equivalently

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{2}$$

Note that if $\nabla \cdot \mathbf{v} = 0$, it immediately follows that $D\rho/Dt = 0$ so that the density of a particle does not change with time. This states that the medium is *incompressible* and is a commonly used approximation in fluid mechanics.

Conservation of linear momentum can be written

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{g} \tag{3}$$

which are Cauchy's equations of motion and they apply to the current deformed configuration. We have not made any approximation about the constitutive relationship or the size of the deformation. The cauchy stress tensor is related to the surface tractions on the body by

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \mathbf{T} \tag{4}$$

where $\mathbf{t}$ is the traction acting on the surface with normal $\hat{\mathbf{n}}$.

To discuss thermal convection in the Earth we must also use another equation which we get from conservation of energy

2. Conservation of energy

Body forces and surface forces do work on a parcel of fluid and change the internal energy and the kinetic energy. The rate at which work is done (the input power) is

$$P_{input} = \int_S \mathbf{t} \cdot \mathbf{v} dS + \int_V \rho \mathbf{g} \cdot \mathbf{v} dV \tag{5}$$

This can be separated into two contributions; mechanical work performed in deforming the body and work done in changing the kinetic energy of the body. The mathematical development is as follows (remember that $A \cdot\cdot B = A_{ij} B_{ji}$ and $A : B = A_{ij} B_{ij}$ in a Cartesian coordinate system):

$$
\begin{aligned}
P_{input} &= \int_S \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \mathbf{v} \, dS + \int_V \rho \mathbf{g} \cdot \mathbf{v} \, dV \\
&= \int_V [\nabla \cdot (\mathbf{T} \cdot \mathbf{v}) + \rho \mathbf{g} \cdot \mathbf{v}] \, dV \qquad \text{(using Gauss' theorem)} \\
&= \int_V [(\nabla \cdot \mathbf{T}) \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v} + \rho \mathbf{g} \cdot \mathbf{v}] \, dV \\
&= \int_V [(\nabla \cdot \mathbf{T} + \rho \mathbf{g}) \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v}] \, dV \\
&= \int_V [\rho \frac{D\mathbf{v}}{Dt} \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v}] \, dV \\
&= \int_V \tfrac{1}{2} \rho \frac{D}{Dt} (\mathbf{v} \cdot \mathbf{v}) \, dV + \int_V \mathbf{T} \cdot\cdot \nabla \mathbf{v} \, dV \\
&= \frac{D}{Dt} \int_V \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dV + \int_V \mathbf{T} \cdot\cdot \nabla \mathbf{v} \, dV \\
&= \frac{D}{Dt} \int_V \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dV + \int_V \mathbf{T} \cdot\cdot (\dot{\boldsymbol{\epsilon}} + \dot{\boldsymbol{\Omega}}) \, dV \\
&= \frac{D}{Dt} \int_V \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dV + \int_V \mathbf{T} \cdot\cdot \dot{\boldsymbol{\epsilon}} \, dV \qquad \begin{array}{l} \text{because } \mathbf{T} \text{ is symmetric} \\ \text{and } \boldsymbol{\Omega} \text{ is antisymmetric} \end{array}
\end{aligned}
$$

Note that we separated the gradient of velocity tensor into a symetric and antisymmetric part: $\nabla \mathbf{v} = \dot{\boldsymbol{\epsilon}} + \dot{\boldsymbol{\Omega}}$) where $\dot{\boldsymbol{\epsilon}}$ is the symmetric strain rate tensor. Finally we get

$$
P_{input} = \frac{D}{Dt} \int_V \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dV + \int_V \mathbf{T} : \dot{\boldsymbol{\epsilon}} \, dV \qquad \text{because } \mathbf{T} \text{ is symmetric} \tag{6}
$$

which clearly shows the separation into mechanical work and kinetic energy. The mechanical work contributes to the change in internal energy (from the first law of thermodynamics).

There will be other sources (and sinks) of energy input which we denote Q_{input}. In mantle convection, we will have conduction and radioactive heat generation:

$$
Q_{input} = -\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, dS + \int_V \rho h \, dV
$$

where h is the rate of heat generation per unit mass, $\mathbf{q} = -k \nabla T$ is the heat flux, T is temperature, and k is the thermal conductivity. If U is the total energy of the volume then

$$
\dot{U} = P_{input} + Q_{input} \tag{7}
$$

This is a statement of the first law of thermodynamics. U consists of the change of kinetic energy plus the change of internal energy of the volume so

$$
\dot{U} = \frac{D}{Dt} \int_V [\tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \rho E] \, dV
$$

where E is the internal energy per unit mass. As we have already separated out the change of kinetic energy in P_{input} we can get an expression for the change of internal energy. Combining 6 and 7 gives

$$\frac{D}{Dt}\int_V \rho E\, dV = \int_V \mathbf{T} : \dot{\boldsymbol{\epsilon}}\, dV - \int_S \mathbf{q}\cdot\hat{\mathbf{n}}\, dS + \int_V \rho h\, dV$$

or (using Gauss' theorem and the Reynolds mass transport theorem)

$$\rho\frac{DE}{Dt} = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T} : \dot{\boldsymbol{\epsilon}} \tag{8}$$

For a homogeneous material, $dE = TdS - PdV = TdS + Pd\rho/\rho^2$ and

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP = \frac{C_p}{T}dT - \frac{\alpha}{\rho}dP$$

so

$$\rho\frac{DE}{Dt} = \rho C_p\frac{DT}{Dt} - \alpha T\frac{DP}{Dt} + \frac{P}{\rho}\frac{D\rho}{Dt} = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T} : \dot{\boldsymbol{\epsilon}}$$

Finally we write $\mathbf{T}$ as a deviatoric and isotropic part (where the isotropic part is, by definition, the pressure)

$$\mathbf{T} = \mathbf{T}' - P\mathbf{I} \quad \text{whence} \quad \mathbf{T} : \dot{\boldsymbol{\epsilon}} = \mathbf{T}' : \dot{\boldsymbol{\epsilon}} - P\nabla\cdot\mathbf{v} = \mathbf{T}' : \dot{\boldsymbol{\epsilon}} + \frac{P}{\rho}\frac{D\rho}{Dt}$$

and so we end up with

$$\rho C_p\frac{DT}{Dt} - \alpha T\frac{DP}{Dt} = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T}' : \dot{\boldsymbol{\epsilon}} \tag{9}$$

or equivalently

$$\rho C_p\left[\frac{DT}{Dt} - \left(\frac{\partial T}{\partial P}\right)_S \frac{DP}{Dt}\right] = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T}' : \dot{\boldsymbol{\epsilon}} \tag{10}$$

The term in square brackets is the change in temperature in excess of that which comes from adiabatic compression.

3. Incompressible convection in the Boussinesq approximation

As noted above, in incompressible convection, $\nabla\cdot\mathbf{v} = 0$, and we have been playing with a simplified version of the momentum equation where we have a Newtonian fluid with a constant viscosity. In the limit of small Reynolds number, we have

$$0 = -\nabla P + \eta\nabla^2\mathbf{v} + \delta\rho\mathbf{g} \tag{11}$$

where P is the pressure due to flow. Note that we have included a source of buoyancy to drive flow where, for thermal convection,

$$\delta\rho \simeq -\rho\alpha\delta T \tag{12}$$

α is the coefficient of thermal volume expansion and δT is the perturbation in the temperature. Note that assuming incompressibility but allowing a variation in density only in the terms involving gravity is called the Boussinesq approximation.

In an incompressible material, there is no change in density due to movement in a hydrostatic pressure gradient so the second term in the square brackets in equation (10) can be neglected (essentially we are assuming the bulk modulus becomes infinite in which limit the adiabatic temperature gradient is zero) so giving

$$\rho C_p \frac{DT}{Dt} = \rho h - \nabla \cdot \mathbf{q} + \mathbf{T}' : \dot{\boldsymbol{\epsilon}} \tag{13}$$

Now divide through by ρC_p and assume that the thermal conductivity in $\mathbf{q} = -k\nabla T$ is independent of position and we end up with

$$\frac{DT}{Dt} = \frac{h}{C_p} + \kappa \nabla^2 T + \frac{1}{\rho C_p} \mathbf{T}' : \dot{\boldsymbol{\epsilon}} \tag{14}$$

where $\kappa = k/(\rho C_p)$ is the thermal diffusivity. In what immediately follows, we will be ignoring radioactivity and just consider heating from below. We will also ignore the viscous heating associated with the flow (we will check the validity of this assumption later). We now end up with

$$\frac{DT}{Dt} = \kappa \nabla^2 T \tag{15}$$

4. Linear Stability Analysis

Consider a layer of fluid that has depth b and is heated from below (the Benard problem), such that the temperatures at the top and bottom are T_0 and T_1, respectively. The fluid is stationary, with heat conducting across it having established a linear temperature gradient according to the steady-state energy equation (pure diffusion and no advection). We can assume the temperature distribution (T_c) within the fluid is

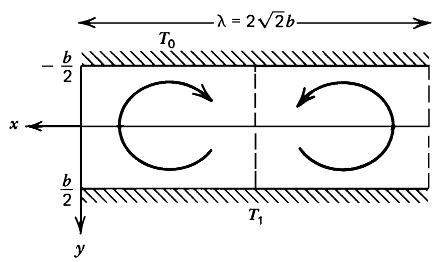


Figure 6.38 Two-dimensional cellular convection in a fluid layer heated from below.

$$T_c = \frac{T_1 + T_0}{2} + \frac{T_1 - T_0}{b} y \tag{16}$$

However, the temperature of the bottom boundary (T_1) is sufficient to drive convection, so we are actually interested in the temperature in the fluid as it first exceeds this conductive temperature profile. This can be considered an arbitrarily small perturbation to the background profile which will be denoted as

$$T' \equiv T - T_c = T - \frac{T_1 + T_0}{2} - \frac{T_1 - T_0}{b} y \tag{17}$$

We begin with the conservation equations for mass, momentum, and energy in 2-D cartesian geometry assuming buoyancy forces driven by thermal expansion but that the fluid is still incompressible (Boussinesq). Writing the above equations incomponent form (y is downward) we have

$$
\begin{aligned}
0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\
0 &= -\frac{\partial P}{\partial x} + \eta \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
0 &= -\frac{\partial P}{\partial y} + \eta \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho_0 \alpha g \Delta T \\
\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
\end{aligned}
\tag{18}
$$

Notice that these equations are coupled with temperature appearing in the momentum equation and velocities appearing in the energy equation. It also has non-linear terms which contain both temperature and velocity.

When the fluid is stable it is motionless throughout, $u = v = 0$ everywhere. Since we are interested in describing the system when it first begins to overturn, we would be looking at very small perturbations to velocity, such as u' and v'. The same is true for the pressure.

We can rewrite T as a function of T'

$$
T = T' + \frac{T_1 + T_0}{2} + \frac{T_1 - T_0}{b} y \tag{19}
$$

Derivatives in t, x, and y components would correspond to

$$
\frac{\partial T}{\partial t} = \frac{\partial T'}{\partial t} \tag{20}
$$

$$
\frac{\partial T}{\partial x} = \frac{\partial T'}{\partial x} \tag{21}
$$

$$
\frac{\partial T}{\partial y} = \frac{\partial T'}{\partial y} + \frac{T_1 - T_0}{b} \tag{22}
$$

The equations can be recast entirely in terms of the small perturbations which gives us the equations of convection at the first onset. We should distinguish that ΔT is substituted for T' as the temperature perturbation is providing the driving force, however the finite temperature difference describing the background temperature profile $(T_1 - T_0)/b$ remains in the energy equation:

$$
\begin{aligned}
0 &= \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \\
0 &= -\frac{\partial P'}{\partial x} + \eta \left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} \right) \\
0 &= -\frac{\partial P'}{\partial y} + \eta \left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} \right) - \rho_0 \alpha g T' \\
\frac{\partial T'}{\partial t} + u'\frac{\partial T'}{\partial x} + v'\left(\frac{\partial T'}{\partial y} + \frac{T_1 - T_0}{b} \right) &= \kappa \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} \right)
\end{aligned}
\tag{23}
$$

To linearize the system, we can assume that terms with two small perturbations multiplied can be neglected, however we must distinguish between higher order derivatives on a perturbation variable (and keep them). Only the energy equation is effected and reduces to:

$$
\frac{\partial T'}{\partial t} + v'\frac{T_1 - T_0}{b} = \kappa \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} \right) \tag{24}
$$

The momentum equations can be combined by using the standard trick of taking the $\partial/\partial y$ on the x-momentum equation and $\partial/\partial x$ on the y-momentum equation, then subtracting one from the other. Further, we can substitute in for the stream function ψ' where possible. This gives us now just two equations since the continuity equation is automatically satisfied with the stream function:

$$0 = \eta \left(\frac{\partial^4 \psi'}{\partial x^4} + 2 \frac{\partial^4 \psi'}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi'}{\partial y^4} \right) - \rho_0 \alpha g T' = \eta \nabla^4 \psi' - \rho_0 \alpha g \frac{\partial T'}{\partial x} \tag{25}$$

Substituting the stream function where possible into the energy equation gives:

$$\frac{\partial T'}{\partial t} + \frac{\Delta T}{b} \frac{\partial \psi'}{\partial x} = \kappa \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} \right) \tag{26}$$

We also need to consider the new boundary conditions, as now the equations are for perturbations (u', v', T') rather than (u, v, T). Since the boundary conditions for the momentum equation are $v = 0$ on $y = \pm b/2$, these must also be enforced when there are only small deviations of velocity being considered. So even if small perturbations of v can occur throughout the fluid, at the boundary there cannot be any deviations as $v = 0$, so as a consequence, $v' = 0$ on $y = \pm b/2$. The same is true for the temperature boundary conditions, so $T' = 0$ on both top and bottom boundaries as the specified temperatures at those boundaries still hold. Because these equations are linear with constant coefficients, we can solve them with separation of variables. The boundary conditions are automatically satisfied by solutions of the form:

$$\psi' = \psi'_0 \cos\left(\frac{\pi y}{b}\right) \sin(kx) e^{\beta t} \tag{27}$$

and

$$T' = T'_0 \cos\left(\frac{\pi y}{b}\right) \cos(kx) e^{\beta t} \tag{28}$$

where the wavenumber $k = 2\pi/\lambda$. These solutions have already taken into account the boundary conditions, which must be symmetric about y with $v' = 0$ on both top and bottom boundaries which requires a cosine. We will refer to these equations in shorthand as $\psi' = \psi'_0 CSE$ and $T' = T'_0 CCE$. These can be plugged into the energy and momentum equations to get:

$$0 = \eta \left(\psi'_0 k^4 CSE + 2\psi'_0 k^2 \left(\frac{\pi}{b}\right)^2 CSE + \psi'_0 \left(\frac{\pi}{b}\right)^4 CSE \right) + \rho_0 g \alpha T'_0 k CSE \tag{29}$$

$$T'_0 \beta CCE + (\Delta T/b) k \psi'_0 CCE = \kappa \left(-T'_0 k^2 CCE - T'_0 \left(\frac{\pi}{b}\right)^2 CCE \right) \tag{30}$$

After eliminating the common terms on both sides of each equations (CCE and CSE), the remaining terms can be grouped with T'_0 and ψ'_0 as follows:

$$-T'_0 \rho_0 g \alpha k = \eta \psi'_0 \left(k^2 + \left(\frac{\pi}{b}\right)^2 \right)^2 \tag{31}$$

$$T'_0 \left(\beta + \kappa \left(k^2 + \left(\frac{\pi}{b}\right)^2 \right) \right) = -\psi'_0 (\Delta T/b) k \tag{32}$$

Now T'_0 and ψ'_0 can be eliminated from the equations as the amplitudes of the perturbation turn out to be both arbitrary and unimportant to the conditions at the very onset of convection.

$$-\left(\frac{\psi'_0}{T'_0}\right) = \frac{\rho_0 g \alpha k}{\eta} \frac{1}{\left(k^2 + \left(\frac{\pi}{b}\right)^2\right)^2} = \frac{\beta + \kappa \left(k^2 + \left(\frac{\pi}{b}\right)^2\right)}{(\Delta T/b) k} \tag{33}$$

The system has been reduced to a single expression, which can be rearranged into a characteristic equation for the growth rate, β,

$$\beta + \kappa \left(k^2 + \left(\frac{\pi}{b}\right)^2\right) = \frac{\rho_0 g \alpha \Delta T k^2}{\eta} \frac{1}{b\left(k^2 + \left(\frac{\pi}{b}\right)^2\right)^2} \tag{34}$$

Now we will multiply both sides through by a b^2 and divide both sides by κ giving

$$\frac{b^2}{\kappa}\beta + (k^2 b^2 + \pi^2) = \frac{\rho_0 g \alpha \Delta T k^2 b^2}{\eta \kappa} \frac{1}{b\left(k^2 + \left(\frac{\pi}{b}\right)^2\right)^2} \tag{35}$$

Multiplying the left hand side by b on the top and bottom will give first us a Rayleigh number, and then again by b^2 will give:

$$\frac{b^2}{\kappa}\beta + (k^2 b^2 + \pi^2) = \frac{Ra\, k^2}{b^2\left(k^2 + \left(\frac{\pi}{b}\right)^2\right)^2} = \frac{Ra\, b^2 k^2}{b^4\left(k^2 + \left(\frac{\pi}{b}\right)^2\right)^2} \tag{36}$$

where

$$Ra = \frac{\rho_0 g \alpha \Delta T b^3}{\eta \kappa} \tag{37}$$

This is the appropriate form for the Rayleigh number for bottom heating and balances the buoyancy force on the top with the dissipative forces on the bottom. We can factor the denominator on the LHS,

$$\frac{b^2}{\kappa}\beta + (k^2 b^2 + \pi^2) = \frac{Ra\, k^2 b^2}{(k^2 b^2 + \pi^2)^2} \tag{38}$$

and then substitute $\gamma = kb$,

$$\frac{b^2}{\kappa}\beta + (\gamma^2 + \pi^2) = \frac{Ra\, \gamma^2}{(\gamma^2 + \pi^2)^2} \tag{39}$$

We can isolate β onto the RHS:

$$\frac{b^2}{\kappa}\beta = \frac{Ra\, \gamma^2 - (\gamma^2 + \pi^2)^3}{(\gamma^2 + \pi^2)^2} \tag{40}$$

At the point of marginal stability, the system is neither stable nor unstable, and thus β is exactly zero. For this to be true, it means:

$$Ra\, \gamma^2 - (\gamma^2 + \pi^2)^3 = 0 \tag{41}$$

and solving for Ra we get the critical Rayleigh number, Ra_c

$$Ra_c = \frac{(\gamma^2 + \pi^2)^3}{\gamma^2} \tag{42}$$

and the system will be unstable if β is positive, or if $Ra > Ra_c$. Note that the critical Rayleigh number is dependent on frequency – see figure below

We can find the wavelength where the critical Rayleigh number is an absolute minimum by taking the derivative with respect to γ

$$\frac{dRa}{d\gamma} = \frac{3 \cdot 2\gamma\, (\gamma^2 + \pi^2)^2}{\gamma^2} - \frac{2\,(\gamma^2 + \pi^2)^3}{\gamma^3} \tag{43}$$

Setting equal to zero, we get, after a little algebra,

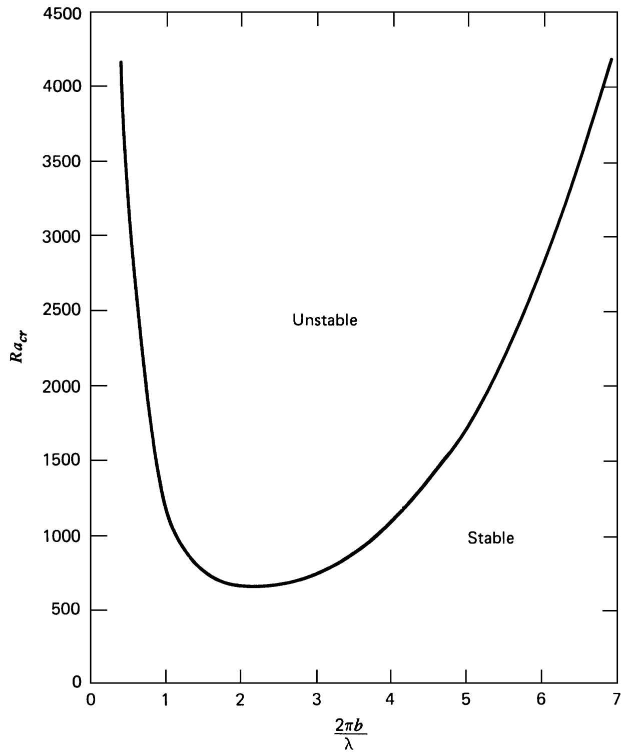


Figure 6.39 Critical Rayleigh number $\mathrm{Ra_{cr}}$ for the onset of convection in a layer heated from below with stress-free boundaries as a function of dimensionless wave number $2\pi b/\lambda$.

$$0 = (4\gamma^2 - 2\pi^2)\left(\gamma^2 + \pi^2\right)^2 \tag{44}$$

which means the only real roots that exist are for when

$$\gamma^2 = \frac{\pi^2}{2} \quad \text{or} \quad \gamma = \pi/\sqrt{2}] \tag{45}$$

This gives us a wavenumber of $k = \pi/(b\sqrt{2})$ and we get a wavelength of $\lambda = 2\sqrt{2}b$ which is the wavelength that will occur at the onset of convection. We can now go back and substitute the result $\gamma^2 = \pi^2/2$ into our expression for critical Rayleigh number,

$$Ra_c = \frac{\left(\frac{3\pi^2}{2}\right)^3}{\frac{\pi^2}{2}} = \frac{27\pi^4}{4} \tag{46}$$

This value is about 660. This analysis can be repeated for internal heating and a slightly different definition of the Rayleigh number is appropriate

$$Ra_h = \frac{\rho_0^2 g\alpha h b^5}{k\eta\kappa} \tag{47}$$

where h is the internal heat generation per unit mass. Using reasonable values of the parameters gives a Rayleigh number of 2×10^9 for whole mantle convection. This is a large value and leads to vigorous rapidly varying convection. Of course, the b^5 term is very important and restricting convection to the upper mantle results in a Rayleigh number of about 2×10^6

5. Boundary layer theory

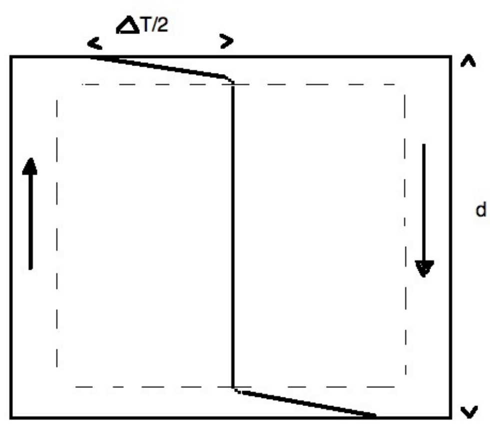
These notes are based on a presentation of Bruce Buffett at a CIDER workshop. For the purposes of this section, we continue to consider an incompressible flow in 2d geometry. The main consequence of incompressibility is that the interior of a convective flow tends to be isothermal (when averaged over horizontal surfaces) rather than adiabatic. We shall mainly use the energy equation (without viscous heating):

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + \rho h \tag{48}$$

which represents the balance between the heat released by cooling with the heat lost by conduction and the heat input by radioactive heat generation. Assuming k is independent of position then

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T + \frac{h}{C_p} \tag{49}$$

where κ is the thermal diffusivity. If we assume that we have a steady state so that $\partial T/\partial t = 0$, the second term on the left represents convection of heat, the first term on the right is conduction and the second term on the right is heat production by radioactive elements. To keep life simple, we shall also neglect the radioactive heat production term and just have heating from below. In this case, we expect the horizontally averaged temperature profile to look something like:



If convection dominates, equation 49 becomes:

$$\mathbf{v} \cdot \nabla T \simeq 0$$

so the interior of the flow becomes isothermal. In the boundary layers, conduction dominates, and equation 49 just becomes the diffusion equation:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \tag{50}$$

Let the thickness of the upper boundary layer be δ , then

$$\nabla T \simeq \frac{\Delta T}{2\delta} \quad \text{and} \quad \nabla^2 T = \nabla \cdot (\nabla T) \simeq \frac{\Delta T}{2\delta^2}$$

Now think of fresh material coming to the surface and beginning to cool (as in a tectonic plate). What are the characteristic times and distances for development of the boundary layer. Let us suppose the characteristic time for conduction is τ then

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \rightarrow \frac{\Delta T}{2\tau} = \kappa \frac{\Delta T}{2\delta^2} \tag{51}$$

from which

$$\tau = \frac{\delta^2}{\kappa} \quad \text{and} \quad \delta = \sqrt{\kappa \tau}$$

We now think of the top boundary layer cooling by conduction and thickening until it reaches a critical time, t_c, at which it becomes unstable and sinks into the interior (kind of like subduction – but not quite, since the location of subduction is not just controlled by the place where the upper boundary layer becomes unstable). The heat conducted out of the surface as a function of time is

$$q(t) = k\frac{\Delta T}{2\delta(t)} = k\frac{\Delta T}{2\sqrt{\kappa t}}$$

Now suppose we time-average to get the mean heat flux out as time evolves from $t = 0$ to $t = t_c$. i.e., the time taken for the boundary layer to evolve to instability. Integrating the above equation gives

$$\bar{q} = \frac{1}{t_c}\int_0^{t_c} q(t)\,dt = \frac{k\Delta T}{\sqrt{\kappa t_c}} = k\frac{\Delta T}{\delta_c} \tag{52}$$

Now we shall make a quick digression to discuss the Rayleigh number which, for our case of bottom-heated convection, is

$$\text{Ra} = \frac{\alpha g \Delta T d^3}{\kappa \nu} \tag{53}$$

where $\nu = \eta/\rho$ is the kinematic viscosity. The Rayleigh number is the ratio of buoyancy forces, in this case $\alpha g \Delta T$, to forces that oppose convection (viscous and conductive). As shown in the previous section, if the Rayleigh number exceeds a certain amount (the critical Rayleigh number) then the system will convect. For spherical shells, the critical Rayleigh number (Ra_c) is about 1000. The larger the Rayleigh number, the more vigorous is the convection. Note that d is the thickness of the convecting system and the Rayleigh number for bottom heating is proportional to d^3.

If the system is initially not convecting then the solution to the diffusion equation gives a linear temperature profile across the whole layer of thickness d:

$$q = \frac{k\Delta T}{d} \tag{54}$$

When the system convects, the heat flow will be larger than this. We measure this effect using the Nusselt number which is defined as the total heat flow out of the system divided by the heat flow if only conduction is operating. The Nusselt number plays a key role in thermal histories of the Earth.

So, why did we have this short digression on the Rayleigh number? We shall suppose that the upper boundary layer becomes unstable when its Rayleigh number exceeds its critical value. This Rayleigh number would be

$$\text{Ra}_\delta = \frac{\alpha g (\Delta T/2)}{\kappa \nu}\delta^3(t)$$

The upper boundary layer will become unstable when this number exceeds Ra_c, ie when $\delta = \delta_c$. Thus

$$\delta_c = \text{Ra}_c^{\frac{1}{3}} \left(\frac{2\nu\kappa}{\alpha g \Delta T}\right)^{\frac{1}{3}}$$

We can simplify this by noting the definition of the Rayleigh number (equation 53) so that

$$\frac{\delta_c}{d} = \left(\frac{2\mathrm{Ra}_c}{\mathrm{Ra}}\right)^{\frac{1}{3}} \tag{55}$$

Now consider the (time-averaged) Nusselt number, which, by definition, is:

$$\mathrm{Nu} = \frac{\bar{q}}{q_{cond}}$$

where $\bar{q}$ is given by equation 52 and q_{cond} is given by equation 54. Thus

$$\mathrm{Nu} = \frac{k\Delta T}{\delta_c} \cdot \frac{d}{k\Delta T} = \frac{d}{\delta_c}$$

So, from equation 55, we have

$$\mathrm{Nu} = \left(\frac{\mathrm{Ra}}{2\mathrm{Ra}_c}\right)^{\frac{1}{3}} \tag{56}$$

Note that this form is valid for vigorous convection when Ra$\gg$ Ra_c. For low Rayleigh number, we expect the Nusselt number to tend to one as Ra tends to Ra_c. Numerical calculations and lab experiments indicate that the power in this equation should be closer to 0.3 rather than 1/3 and the relation is usually written in a more general form:

$$\mathrm{Nu} = a\mathrm{Ra}^{\beta} \tag{57}$$

This form has been used for calculating thermal histories of the earth. We also note that this boundary layer analysis can also be developed for internally heated flows where slightly different results are obtained.

Note that nothing in this development yet implies laminar flow and simple Rayleigh-Bernard convection (as implied by the above figure) which is the usual case in the development of boundary layer theory. The convection can be time-dependent and our heat flow estimates are time averaged over the time scale of an instability developing in the upper boundary layer. If we want to estimate a velocity of convection from these scaling arguments, we have to use the energy equation in approximate steady state:

$$\mathbf{v} \cdot \nabla T = \kappa \nabla^2 T$$

Consider the picture above. Since we are dealing with unit aspect ratio convection, in the top boundary layer, we have

$$v_x \frac{\partial T}{\partial x} \simeq \kappa \frac{\partial^2 T}{\partial z^2} \rightarrow \frac{v\Delta T}{d} = \kappa \frac{\Delta T}{2\delta^2}$$

where we have used $\partial T/\partial x \simeq \Delta T/d$ so

$$v = \frac{\kappa}{2d}\left(\frac{d}{\delta}\right)^2 = \frac{\kappa}{2d}\mathrm{Nu}^2 \tag{58}$$

Equations 55, 56, and 58 are our main results though equation 58 is really a low Ra approximation. It is instructive to consider some numbers. Ra_c is about 10^3 and using standard numbers for whole mantle convection ($d \simeq 3000$km) gives Ra about 10^7. The Nusselt number is about 17 and the thickness of the boundary layer is about 170 km. A typical velocity is only about 2mm/yr. If Ra is 10^8, the Nusselt number is about 37 and the thickness of the boundary layer is about 80km and a typical velocity is 1 cm/yr. Clearly our velocities are a little lower than plate speeds but velocities in the lower mantle may well be much slower than the surface plate values so 1 cm/yr may not be unreasonable for an average value.

6. Compressible convection

We have already derived the governing equations which are conservation of mass (equations 1 and 2), conservation of momentum which we write in terms of the deviatoric stress as

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \nabla \cdot \mathbf{T}' + \rho \mathbf{g} \tag{59}$$

and equation 10 which we rewrite here:

$$\rho C_p \frac{DT}{Dt} - \alpha T \frac{DP}{Dt} = \rho h - \nabla \cdot \mathbf{q} + \phi \tag{60}$$

where ϕ is the viscous dissipation. When allowing for compressibility, we need to be a little careful with the constitutive relation. We write

$$T'_{ij} = 2\eta[\dot{\epsilon}_{ij} - \tfrac{1}{3}\,\dot{\epsilon}_{kk}\delta_{ij}] \tag{61}$$

where η is the effective viscosity which can be a function of stress, temperature, etc. The term in square brackets is called the "strain rate deviator" and we use this because the deviatoric stress is trace-free by definition. It is conventional to ignore "bulk" viscosity which could arise during compression and expansion of a material.

In compressible convection, a background reference state needs to be chosen. A natural choice is one of hydrostatic pressure and an adiabatic thermal state. We let

$$T = \bar{T} + \theta \quad P = \bar{P} + P', \quad \text{and} \quad \rho = \bar{\rho}(\bar{T}, \bar{P}) + \rho' \tag{62}$$

where the barred quantities are independent of time. Note that ρ' and P' are small relative to the background state but θ may not be. The background state has $\mathbf{g} = -g_r\hat{\mathbf{r}}$, and in what follows, we shall neglect the effect of temperature variations associated with convection on $\mathbf{g}$. Furthermore, we shall neglect the effect of convection on background thermodynamic variables such as C_p and α, *etc.* . Consequently,

$$\left.\begin{aligned} \nabla \bar{P} &= \bar{\rho}\mathbf{g} = -\bar{\rho} g_r \hat{\mathbf{r}} \\ \nabla \bar{\rho} &= \nabla \bar{P} \left(\frac{\partial \rho}{\partial P}\right)_S = \nabla \bar{P} \frac{\bar{\rho}}{K_s} \\ \nabla \bar{T} &= \nabla \bar{P} \left(\frac{\partial T}{\partial P}\right)_S = \nabla \bar{P} \frac{\alpha \bar{T}}{\bar{\rho} Cp} \end{aligned}\right\} \tag{63}$$

People who do mantle convection tend to be a little cavalier about integrating this background state. For example, the last of these can be written

$$\frac{d\bar{T}}{dr} = -\frac{\alpha g_r}{C_p}\bar{T} \tag{64}$$

It is often assumed that $\alpha g_r / C_p$ is a constant giving

$$\bar{T}(r) = \bar{T}_a \exp\left(\frac{\alpha g_r}{C_p}(a - r)\right) \tag{65}$$

where a refers to some reference radius where $\bar{T} = \bar{T}_a$. While C_p and g_r are roughly constant in the mantle, α is not. Similarly, they are cavalier about the integration for $\bar{\rho}$. Sometimes, they use the Gruneisen ratio γ to write

$$\frac{\bar{\rho}}{K_s} = \frac{\alpha}{\gamma C_p} \tag{66}$$

so the second of equation 63 becomes

$$\frac{d\bar{\rho}}{dr} = -\frac{\alpha g_r}{\gamma C_p}\bar{\rho} \tag{67}$$

which they integrate to give

$$\bar{\rho} = \rho_r \exp\left(\frac{\alpha g_r}{\gamma C_p}(a-r)\right) \tag{68}$$

where ρ_r is a reference density at $r = a$ which, again,may not be the surface. This form actually has the wrong curvature – but we will follow this convention for now. Finally, we relate the perturbation in density ρ' to perturbations in pressure and temperature using the following approximate equation of state:

$$d\rho = \left(\frac{\partial\rho}{\partial P}\right)_T dP + \left(\frac{\partial\rho}{\partial T}\right)_P dT \tag{69}$$

so

$$\rho' = \frac{\bar{\rho}}{K_T}P' - \alpha\bar{\rho}\theta = \frac{\bar{\rho}}{K_S}\frac{K_s}{K_t}P' - \alpha\bar{\rho}\theta = \frac{C_p}{C_v}\frac{\alpha}{\gamma C_p}P' - \alpha\bar{\rho}\theta \tag{70}$$

(remember $K_S/K_T = C_p/C_v = 1 + \alpha T\gamma$ where the last term is on the order of 0.05). With this background state, the momentum equation looks like

$$\left.\begin{aligned} \rho\frac{D\mathbf{v}}{Dt} &= \nabla\cdot\mathbf{T}' - \nabla P + \rho\mathbf{g} \\ &= \nabla\cdot\mathbf{T}' - \nabla P' + \rho'\mathbf{g} \\ &= \nabla\cdot\mathbf{T}' - \nabla P' + \frac{C_p}{C_v}\frac{\alpha\mathbf{g}P'}{C_p\gamma} - \alpha\bar{\rho}\mathbf{g}\theta \end{aligned}\right\} \tag{71}$$

Consider the entropy equation (equation 60). Here we use the fact that $P' \ll P$ and the local time derivatives of $\bar{T}$ and $\bar{P}$ are zero, so

$$\bar{\rho}C_p\frac{D\theta}{Dt} = \bar{\rho}h + \nabla\cdot\left(k\nabla(\bar{T}+\theta)\right) + \phi - \rho C_p\mathbf{v}\cdot\nabla\bar{T} + \alpha(\bar{T}+\theta)\mathbf{v}\cdot\nabla\bar{P} \tag{72}$$

Using the fact that $\bar{\rho}C_p\mathbf{v}\cdot\nabla\bar{T} = \alpha\bar{T}\mathbf{v}\cdot\nabla\bar{P}$, the definition of $\nabla\bar{P}$, and neglecting products of small quantities gives

$$\bar{\rho}C_p\frac{D\theta}{Dt} = \bar{\rho}h + \nabla\cdot\left(k\nabla(\bar{T}+\theta)\right) + \phi - \bar{\rho}g_r\alpha\theta(\mathbf{v}\cdot\hat{\mathbf{r}}) \tag{73}$$

7. Scaling the equations and the anelastic liquid approximation

We have already chosen a reference value for density (ρ_r) and for C_p so choosing a reference value for the thermal conductivity k_r gives us a reference value for thermal diffusivity: $\kappa_r = k_r/\rho_r C_p$. It is conventional to scale time using the diffusion time: L^2/κ_r where L is a charcteristic length scale (e.g. the depth of the mantle). Velocity scales as κ_r/L and stress scales as $\eta_r\kappa_r/L^2$ where η_r is a reference viscosity. We also need to choose a reference value for thermal expansion α_r and bulk modulus K_{Tr}, and finally, we normalize temperature with a characteristic temperature difference driving convection, ΔT_r.

At this point, reconsider our "equation of state" for density, equation (70), and, using $\rho = \bar{\rho} + \rho'$ we have

$$\rho = \bar{\rho}\left[1 + \frac{P'}{K_T} - \alpha\theta\right] \tag{74}$$

Now non-dimensionalize and using an asterisk to denote scaled quantities gives

$$\frac{\rho *}{\bar{\rho} *} = 1 + \frac{P' *}{K_T^*} M^2 Pr - \alpha^* \theta^* \epsilon \tag{75}$$

where M is the Mach Number and Pr is the Prandtl number given by

$$M^2 = \frac{\kappa_r^2 \rho_r}{K_{Tr} L^2} \quad \text{and} \quad Pr = \frac{\eta_r}{\rho_r \kappa_r} \quad \text{and} \quad \epsilon = \alpha_r \Delta T_r \tag{76}$$

The Mach number is a characteristic velocity of flow divided by the velocity of sound. To see this, recognise that $K_T = \rho c^2$ where c is a sound velocty (in this case, the bulk sound speed). The velocity scaling for convection is κ_r/L and, while this is a diffusive time scale, we know from discussion of the Peclet number, that convective time scales are ony 3 orders of magnitide larger.

The anelastic liquid approximation is valid if $M^2 Pr \ll 1$ and $\epsilon \ll 1$. The Prandtl number is about 10^{23} in the mantle, but the Mach number is tiny and $M^2 \simeq 10^{-33}$. Even if a typical mantle convective velocity is used to compute the Mach number, $M^2 Pr \ll 1$.

This analysis tells us that $\rho'/\bar{\rho} \ll 1$. In the limit that $\epsilon \to 0$ and $M^2 Pr \to 0$, the equation for conservation of mass just reads $\partial\rho/\partial t = 0$, (since $\bar{\rho}$ is independent of time and ρ' is small). From equation (1) we get that

$$\nabla \cdot (\rho \mathbf{v}) = 0 = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho$$

Using the reference state, mass conservation becomes:

$$\nabla \cdot \mathbf{v} - \frac{\alpha g_r}{\gamma C_p} \mathbf{v} \cdot \hat{\mathbf{r}} = 0 \tag{77}$$

We can now non-dimensionalize equations 71, 73, and 77. First, we define the "Dissipation number"

$$Di = \frac{\alpha_r g_r L}{C_p} \tag{78}$$

and equation 77 becomes

$$\nabla \cdot \mathbf{v}^* - \frac{Di}{\gamma} \mathbf{v}^* \cdot \hat{\mathbf{r}} = 0 \tag{79}$$

and the Navier-Stokes equation becomes

$$\frac{\rho^*}{Pr} \frac{D\mathbf{v}^*}{Dt^*} = \nabla \cdot \bar{\mathbf{T}}'^* - \nabla P'^* - \frac{Di}{\gamma} \frac{C_p}{C_v} \rho^* P'^* \hat{\mathbf{r}} - Ra\, \rho^* \alpha^* \theta^* \hat{\mathbf{r}} \tag{80}$$

Clearly the first term can be neglected: inertial terms are irrelevant which means that if we stopped driving mantle convection the motion would stop instantaneously. Ra is the Rayleigh number, which, for this type of temperature normalization is

$$Ra = \frac{\alpha_r \rho_r^2 g_r \Delta T_r L^3}{\eta_r \kappa_r} \tag{81}$$

The energy equation becomes (note that ϕ has units of a stress times a velocity)

$$\rho^* \frac{D\bar{\theta}^*}{Dt^*} + Di\, \rho^* \alpha^* \theta^* \mathbf{v}^* \cdot \hat{\mathbf{r}} = \nabla \cdot [k^* (\nabla \theta^*)] + \frac{Di}{Ra} \phi^* + \rho^* h^* + Di^2 \bar{T} \tag{82}$$

and the non dimensional scaling for h is given by $L^2 h_r / k_r \Delta T_r$. The second order term at the end of the equation comes from using equation 65 in the equation $\nabla \cdot (k \nabla \bar{T})$ since $\nabla \bar{T} = -Di \bar{T}/L$.

The Boussinesq approximation (where the only place departures from the reference density state are included is in the thermal buoyancy term and the material is otherwise deemed incompressible) is obtained by setting $Di = 0$. Other slightly different forms of the equations are possible (see Tackley 1996).

A more general form for the Rayleigh number is given by

$$Ra = \frac{g_r \alpha_r F L^4}{\kappa_r^2 \nu_r} \quad \text{where} \quad \nu_r = \frac{\eta_r}{\rho_r} \tag{83}$$

where F is related to the heat flux that would be carried in the absence of heating: $|q| = \rho C_p F$. For internal heating only

$$F = \frac{|q|}{\rho C_p} = \frac{\rho h L}{\rho C_p} = \frac{hL}{C_p}$$

and Ra is

$$Ra = \frac{\alpha_r \rho_r^2 g_r h_r L^5}{k_r \eta_r \kappa_r}$$

For bottom heating which induces a difference in "potential" temperature across the system of ΔT then $F = \kappa \Delta T / L$ and Ra is defined as in equation 81. (The potential temperature measures the temperature difference available to drive convection and does not include any adiabatic temperature rise). The Rayleigh number measures the strength of buoyancy forces relative to the strength of dissipative forces. The critical Rayleigh number that needs to be exceeded to get convection is on the order of 1000 (see Schubert et al, 2001, Chapter 7). For whole mantle convection Ra is on the order of 10^9. In this case, mantle convection is vigorous and turbulent. High viscosities at the base of the mantle may mean that the effective Rayleigh number is depth dependent and that the critical Rayleigh number near the base of the mante is barely exceeded. Flow in such a region may be large-scale and laminar.

8. Other approximations to the system of equations

The paper by King et al (2010) considers some other approximations to the equations. Some solvers have difficulty with the P' buoyancy term in equation 80 and truncate to give

$$0 = \nabla \cdot \bar{\mathbf{T}}'^{*} - \nabla P'^{*} - Ra\, \rho^* \alpha^* \theta^* \hat{\mathbf{r}} \tag{84}$$

This is called the "truncated anelastic liquid approximation" but has some negative numerical consequences. In particular, there are some volume-averaged quantities which should be identical (see below) but this is not achieved in this approximation.

The extended Boussinesq approximation keeps incompressibility but includes the term in D_i on the LHS of equation 82 and the viscous dissipation term on the RHS. The reference state is changed to $\bar{\rho} = 1, \bar{T} = 0, \alpha = 1, k = 1$, and $C_p = 1$.

9. Convective efficiency for the mantle

A treatment of the global entropy and energy equations leads to the concept of convective efficiency. The energy equation can be manipulated to give the desired result. For simplicity, we assume a steady state and that the boundaries of the mantle are not moving radially ($\mathbf{v} \cdot \hat{\mathbf{r}} = 0$). A steady state implies that $\partial\rho/\partial t = 0$ so conservation of mass gives $\nabla \cdot (\rho \mathbf{v}) = 0$ – this is the same as the anelastic liquid approximation. Integrating the energy equation over the whole mantle gives the rather obvious result (for steady state)

$$Q = \int_S \mathbf{q} \cdot \mathbf{dS} = \int_V \rho h dV \tag{85}$$

where Q (the net heat flux out of the mantle) is just balanced by internal radioactive heat production. To look at dissipation, we must use the entropy equation again

$$\rho T \frac{Ds}{Dt} = \rho C_p \left[\frac{DT}{Dt} - \left(\frac{\partial T}{\partial P} \right)_S \frac{DP}{Dt} \right] = \rho h - \nabla \cdot \mathbf{q} + \mathbf{T}' : \dot{\boldsymbol{\epsilon}} \tag{86}$$

Integrating this equation over the mantle gives (assuming steady state)

$$\int_V \rho C_p \mathbf{v}\cdot\nabla T dV - \int_V \alpha T\mathbf{v}\cdot\nabla P dV + \int_S \mathbf{q}\cdot\mathbf{dS} - \int_V \rho h dV = \int_V \mathbf{T}' : \dot{\boldsymbol{\epsilon}} dV = \Phi \tag{87}$$

where Φ is the global rate of viscous heating. To a good approximation, C_p is a constant in the mantle then we can write

$$\int_V \rho C_p \mathbf{v}\cdot\nabla T dV = \int_V C_p \nabla\cdot(\rho\mathbf{v}T) dV = \int_S \rho T C_p \mathbf{v}\cdot\mathbf{dS} = 0 \tag{88}$$

where we have assumed the mantle is neither expanding or contracting and we have used $\nabla\cdot(\rho\mathbf{v}) = 0$. If we now use the global conservation of energy, we have

$$\Phi = -\int_V \alpha T\mathbf{v}\cdot\nabla P dV \tag{89}$$

This equation implies that the global rate of dissipative heating is exactly cancelled by the work done against the adiabatic gradient. It turns out that the pressure gradient is dominated by the hydrostatic background term so that $\nabla P \simeq -\rho g\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ points in the upward radial direction. Then

$$\Phi = \int_V \frac{g\alpha}{C_p} \rho C_p T V_r dV \tag{90}$$

where V_r is the radial velocity. If we average over horizontal surfaces, we see that $< \rho C_p T V_r >$ is the horizontally averaged convective heat flux, and using the definition of the Dissipation number (assumed constant) we find that

$$\Phi = Di Q_{conv} \leq Di Q_s \tag{91}$$

where Q_s is the total heat flux out of the top surface and Q_{conv} is the convected heat flux. When the convection is vigorous (i.e. large Rayleigh number), the Nusselt number is large and $Q_{conv} \simeq Q_s$ so we can define an "efficiency" as

$$\frac{\Phi}{Q_s} \leq Di \tag{92}$$

Clearly, when the dissipation number is small and the Boussinesq approximation is valid, the global rate of viscous dissipation is small and can be neglected. The dissipation number depends on the depth scale of convection and for whole-mantle convection, $Di \simeq 0.5$ so that it is possible that viscous dissipation is important on a global scale.

For internal heating, the efficiency must be modified a bit as the convective heat flux now changes as a function of depth and we find that viscous dissipation can be reduced.

This global analysis says nothing about the local importance of dissipative heating which can localize deformation when one has temperature dependent viscosity.

10. Thermal history for the mantle

The Nusselt number - Rayleigh number relationship derived above and verified in numerical experiments on isoviscous fluids can be used in thermal history calculations:

$$Nu = aRa^{\beta} \tag{93}$$

with $\beta \simeq 0.3$. One might think that assuming a constant viscosity is not a particularly good approximation for the Earth and it is certainly true that including a temperature dependent viscosity can dramatically change the exponent. Typically what happens is that the cold upper boundary layer becomes highly viscous and forms a "stagnant lid" which strongly reduces the efficiency of convective heat flow and β can become 0.1 or smaller. Such a relationship would probably be appropriate to use on Venus but Earth has plate tectonics and the upper boundary layer gets recycled in a fashion more similar to the isoviscous calculations. In fact, numerical calculations with weak zones in the upper plate to allow initiation of subduction lead to values of β of about 0.3.

To do thermal history calculations, we use the energy equation integrated over the volume of the mantle. The mantle is allowed to cool and, in most cases, the adiabatic heating term and viscous heating terms are neglected. As shown above, these terms cancel globally if the mantle neither expands or contracts though this is unlikely to be a good approximation over the age of the Earth. We write the simplified energy equation as

$$\int_V \rho C_p \frac{\partial T}{\partial t} dV = -Q_{out} + Q_{in} + \int_V \rho h dV \tag{94}$$

We now write $d\bar{T}/dt$ as the mean cooling rate of the mantle giving

$$MC_p \frac{d\bar{T}}{dt} = -Q_{out} + Q_{in} + E \tag{95}$$

where E is the radioactive heat generation in the mantle and M is the mass of the mantle. Note that E is a function of time and is roughly exponentially decreasing. We are going to use the form for the Nusselt number defined above but we need to be a little careful about our definition of Rayleigh number. The usual form is

$$Ra = \frac{\alpha g \rho \bar{T} d^3}{\kappa \eta} \tag{96}$$

where we have normalized the temperature such that the surface temperature is zero and $\bar{T}$ is the mean interior temperature. The Nusselt number can be written as

$$Nu = \frac{Q_{out}}{Q_{cond}(\bar{T})} \tag{97}$$

where $Q_{cond}(\bar{T})$ is the hypothetical heat flow that would emerge with the given average temperature, $\bar{T}$ and when only conduction operates. Generally, we can write $Q_{cond}(\bar{T}) = c\bar{T}$ so that

$$Q_{out} = ac\bar{T}Ra^{\beta} \tag{98}$$

Incorporating this with the definition of the Rayleigh number and treating everything as a constant except for η which may be a function of temperature and so will change with time, we have

$$MC_p \frac{d\bar{T}}{dt} = Q_{in}(t) + E(t) - a' \frac{\bar{T}^{1+\beta}}{[\eta(\bar{T})]^{\beta}} \tag{99}$$

We now have to specify a viscosity law, e.g.

$$\eta = \eta_0 \exp\left(\frac{gT_m}{\bar{T}}\right) \tag{100}$$

This is the form we would use (or something similar) when we integrate equation 100 back in time numerically. In order to get make some analytic progress, we can specify the viscosity relative to a reference temperature and write an approximate form:

$$\eta = \eta_0 \left(\frac{\bar{T}}{T_0} \right)^{-n} \tag{101}$$

The reference temperature might be the present mean temperature and, provided $\bar{T}$ doesn't change dramatically with time, this equation represents the exponential behavior reasonably well. Note that n is probably in the range 30 to 40.

The energy equation can now be integrated backward in time (using the present conditions as initial conditions) or forward in time using some guess of the initial conditions. The constant a' can be estimated from numerical calculations or it can be fudged out by normalizing Q_{out} to be some specific value at a particular value of $\bar{T}$. A natural choice is at $t = t_0$ (at present time) set Q_{out} to Q_0 and $\bar{T} = T_0$. Q_0 is the current day heat loss from the mantle and is thought to be about 80% of the total surface heat loss(after correction for continental heat production). Combining these results together gives

$$MC_p \frac{d\bar{T}}{dt} = Q_{in}(t) + E(t) - Q_0 \left(\frac{\bar{T}}{T_0} \right)^{1+\beta+n\beta} \tag{102}$$

This form is convenient as it allows the thermal response of the Earth to be analytically investigated for some simple cases. Consider the case when we have no heat sources:

$$MC_p \frac{d\bar{T}}{dt} = -Q_0 \left(\frac{\bar{T}}{T_0} \right)^{m} \tag{103}$$

where $m = 1 + \beta + n\beta$. For $n = 30 \rightarrow 40$ and $\beta \simeq .3$ we find that $m = 12$. If convective heat transport is ignored and conduction dominates ($\beta = 0$) then $m = 1$. (Actually m might be larger than 1 for conduction because of contributions of radiative heat transfer which would lead to a temperature dependent thermal conductivity and could give an m of 2 to 4.) When $m = 1$, the solution to the above equation is

$$\bar{T} = T_0 \exp \left[-\frac{Q_0}{T_0 MC_p}(t - t_0) \right] \tag{104}$$

which gives a conductive time scale of cooling of $T_0 MC_p / Q_0$ of approximate 8By. If m is greater than 1 then the solution looks like

$$\left(\frac{\bar{T}}{T_0} \right)^{m-1} = 1 + \frac{Q_0}{T_0 MC_p}(t - t_0)(m - 1) \tag{105}$$

It is interesting that this equation can lead to infinite temperatures in the past – this happens in the last 4By if m is greater than 2. Both of these results suggest that the assumption of no internal heat sources is inconsistent with the present day heat flow.

When we have internal heat sources, it is possible to ask what the thermal response time is if, at some time, we increase stepwise the amount of heating. The solution has a decay constant τ where

$$\tau = \frac{MC_p T_0}{m Q_0} \tag{106}$$

which is the conductive time constant divided by m. For $m = 12$, $\tau \simeq$ 700my so there is time for thermal impulses to decay. For smaller values of β associated with stagnant lid convection, $\beta = .1$ so $m \simeq 5$ and $\tau \simeq$1.5By which is a significant fraction of the age of the Earth. Numerical integration of the equations leads to some general results which are characterized by the "Urey ratio" which is the ratio of heat produced to heat lost as a function of time. The variation with time of heat production by radioactive elements is usually characterized by some average half life of the major heat producing elements assuming that their relative abundances has remained constant over geological time. With β in the range 0.2 to 0.35, we arrive at the following conclusions:

1) The internal heat production is severely constrained to prevent models ending up with $\bar{T}$ either going to infinity or to zero as we go back in time. The present Urey ratio is in the range 0.7 to 0.9 and stays constant over roughly the last 2.5By. This result is consistent with the self-regulation hypothesis where, for efficient convection, the Earth basically manages to lose all the heat that is produced and there is a rough balance between heat loss and heat production.
2) The mantle has forgotten its initial condition, i.e., it arrives at the same temperature after about 1By whether it has a cold or a hot origin
3) The temperature drop over the last 3By is between 150K and 250K
4) Heat flow was higher in the past so plate velocities were correspondingly higher. In a cooling plate model, heat flow is proportional to the square root of the spreading rate so, if the heat flow is 2 to 5 times higher in the past, spreading rates could have been 4 to 25 times higher.

There is some evidence that the mean temperature of the mantle was higher in the Archean. This comes from the presence of komatiites which are igneous rocks with a high MgO content and require temperatures which are 200K to 400K higher than present. On the other hand, Archean geotherms estimated from metamorphic mineral assemblages seem to be similar to the present day geotherm. There is also little evidence for plate rates being significantly higher than at present.