CHAPTER 2

Continuum mechanics and the equations of motion

2.1 Forces acting on a body . When dealing with the seismic motion of the Earth, it turns out that both body forces (gravity) and surface forces are important. We consider a body with volume V and surface S and introduce the body force density $\mathbf{b}$ and the *traction* vector $\mathbf{t}$ such that the total body force acting on the body is

$$\int_V \rho \mathbf{b}\, dV \tag{2.1}$$

and the total surface force acting on the body is

$$\int_S \mathbf{t}\, dS \tag{2.2}$$

Note that $\mathbf{b}$ is reckoned per unit mass and $\mathbf{t}$ is reckoned per unit area. $\mathbf{t}$ is most conveniently specified by introducing the stress tensor. If $\hat{\mathbf{n}}$ is the normal to a surface then the traction acting on the plane with that normal is defined by

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \mathbf{T} \tag{2.3}$$

This equation defines the *Cauchy stress tensor*, $\mathbf{T}$, which is the linear vector function which associates with each unit normal $\hat{\mathbf{n}}$ the traction vector $\mathbf{t}$ acting at the point across the surface whose normal is $\hat{\mathbf{n}}$. $\mathbf{T}$ is a tensor and so transforms under a rotation of the coordinate system as

$$\mathbf{T}' = \mathbf{A}^{-\mathbf{1}}\mathbf{T}\,\mathbf{A} \tag{2.4}$$

where $\mathbf{T}'$ is the stress tensor in the rotated system and $\mathbf{A}$ is the orthogonal rotation matrix *i.e.*, $\mathbf{A}^{-\mathbf{1}} = \mathbf{A}^T$. We shall find out later that $\mathbf{T}$ is symmetric.

We can always find a coordinate system in which $\mathbf{T}$ is diagonal. Such a coordinate system is called the *principal axes* system and the diagonal elements of $\mathbf{T}$ are then called *principal stresses*. One finds the principal stresses by performing an eigenvector-eigenvalue decomposition of $\mathbf{T}$. The eigenvalues of $\mathbf{T}$ do not change under a change of coordinates so the coefficients of the cubic polynomial defining the characteristic equation for the eigenvalues are invariants. In particular, $T_{kk} = \text{Trace}(\mathbf{T})$ is invariant.

The mean normal pressure is defined as

$$p = -\tfrac{1}{3}\,(T_{kk}) \tag{2.5}$$

The minus sign arises because of our sign convention. T_{ij} describes the surface force acting in the j'th direction on the surface with normal in the i'th direction. Thus T_{11} acts in the 1 direction on a plane with normal in the 1 direction and is a tensile stress (fig 2.1). A positive pressure is usually taken to be a compressive stress and so we have the minus sign in equation 2.5.
Sometimes it is convenient to use the *stress deviator* which is the deviatoric part of $\mathbf{T}$, *i.e.*,

$$\mathbf{T}^{\mathbf{D}} = \mathbf{T} + p\mathbf{I} \tag{2.6}$$

where $\mathbf{I}$ is the unit tensor.

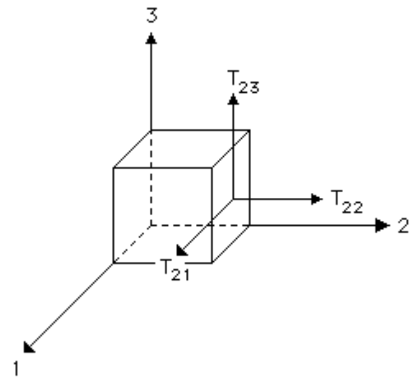

Fig 2.1 Stresses on a face of a cube of material

2.2 Strain and deformation . Consider an infinitesimal line element $d\mathbf{X}$ joining points P and Q in a material (fig 2.2). After deformation, the particle at point P has moved to p and the particle at point Q has moved to q.

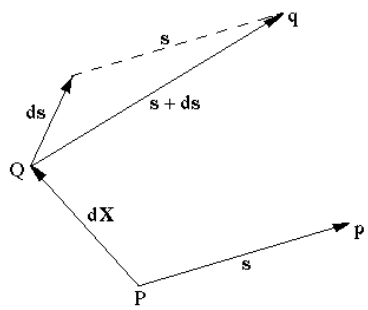

Fig 2.2 Relative displacement of two points in a continuum

The relative displacement is given by

$$d\mathbf{s} = d\mathbf{X} \cdot \nabla \mathbf{s}$$

where, in a Cartesian coordinate system, the elements of ∇s are given by $\partial s_i/\partial X_k$. To make clear what this notation means, think of a line in the 1 direction of initial length ΔX_1 being extended by an amount Δs_1 also in the 1 direction. The ratio $\Delta s_1/\Delta X_1$ is a measure of the deformation which the body is undergoing. We can consider smaller and smaller line segments but the amount of extension will get correspondingly smaller and the ratio has a finite limit as ΔX_1 tends to zero. We write this limit as $\partial s_1/\partial X_1$. The other elements of ∇s are defined similarly.

One way of thinking about deformation of a material is to think of it as a collection of particles which have some natural reference configuration at some time (which we shall take to be $t = 0$). Referring to figure 2.2, we might say that the particle at P is at position $\mathbf{X}$ in the reference configuration and at time t moves to $\mathbf{X} + \mathrm{s}(t)$ where $\mathrm{s}(t)$ is the displacement of the particle at time t. Similarly the particle at Q is at $\mathbf{X} + d\mathbf{X}$ in the reference state and moves to $\mathbf{X} + d\mathbf{X} + \mathrm{s} + d\mathrm{s}$ at time t. Define x to be the position at time t of the particle originally at $\mathbf{X}$ then we simply have

$$\mathbf{x} = \mathbf{X} + \mathbf{s} \tag{2.7}$$

or, since s is a function of $\mathbf{X}$ and time, we have

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

(Note that we employ a common short cut in continuum mechanics and use $\mathbf{x}$ twice in this equation to mean different things. On the right, $\mathbf{x}$ is a function and on the left $\mathbf{x}$ is the value of the function.) This equation represents the material (or "Lagrangian") description of motion. It essentially labels particles and is often the most natural description in seismology because the relevant conservation laws apply to particles rather than to some fixed region of space.

An alternative way of looking at things is the spatial (or Eulerian) description, $\mathbf{x}$ and t are taken to be the independent variables and we have

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$$

This description gives the initial position of the particles now (at time t) occupying position $\mathbf{x}$. The velocity field would be written as

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$$

but the velocity is still the velocity of a particle *i.e.*,

$$\mathbf{v} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}} = \frac{D\mathbf{x}}{Dt} \tag{2.8}$$

where we have used a capital D to emphasize that this is a material derivative.
The *particle* acceleration is

$$\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{X}} = \frac{D\mathbf{v}}{Dt} = \frac{D^2\mathbf{x}}{Dt^2} \tag{2.9}$$

This is not the same as the *local* time derivative which is

$$\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}}$$

We can find a relationship connecting these two derivatives using the following formula from calculus:

$$\left(\frac{\partial a}{\partial y}\right)_z = \left(\frac{\partial a}{\partial y}\right)_p + \left(\frac{\partial a}{\partial p}\right)_y \left(\frac{\partial p}{\partial y}\right)_z \tag{2.10}$$

so

$$\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_t \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}}$$

or

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_x \mathbf{v} \tag{2.11}$$

where ∇_x here denotes the gradient with respect to spatial coordinates. This is different from the gradient operator we introduced at the beginning of this section but the difference is unimportant if we are considering small deformations. To see this, consider the gradient of displacement (at fixed time). We have two possibilities: $\nabla \mathbf{s}$ or $\nabla_x \mathbf{s}$. Referring to figure 2.2, $d\mathbf{x} = d\mathbf{s} + d\mathbf{X}$ and goes from p to q. We have

$$\left(\frac{\partial \mathbf{s}}{\partial \mathbf{X}}\right)_t = \left(\frac{\partial \mathbf{s}}{\partial \mathbf{x}}\right)_t \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)_t \tag{2.12}$$

where the tensor with elements $\partial x_i / \partial X_j$ is known as the deformation gradient tensor (usually given the symbol $\mathbf{F}$) and is discussed in detail by Malvern in the context of finite deformations. Taking the gradient of 2.7 (with respect to material coordinates) gives

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{s}$$

and substitution into 2.12 gives

$$\nabla \mathbf{s} = \nabla_x \mathbf{s}(\mathbf{I} + \nabla \mathbf{s})$$

which, given that deformation is small, reduces to

$$\nabla \mathbf{s} \equiv \nabla_x \mathbf{s}$$

so we are able to ignore the distinction between the two gradient definitions. We can use a scalar field in 2.10 such as density, $\rho(\mathbf{x}, t)$ *i.e.*,

$$\left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \rho}{\partial \mathbf{x}}\right)_{t} \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}}$$

or

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \tag{2.13}$$

and, in general, we have the following relationship between the material and the local derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{2.14}$$

The gradient tensor, ∇s of the displacement vector is a 3×3 tensor whose transpose is sometimes written as s∇. Note that ∇s can be written as the sum of a symmetric tensor and an antisymmetric tensor:

$$\nabla \mathbf{s} = \tfrac{1}{2}\left(\nabla \mathbf{s} + \mathbf{s}\nabla\right) + \tfrac{1}{2}\left(\nabla \mathbf{s} - \mathbf{s}\nabla\right) \tag{2.15}$$

or

$$\nabla \mathbf{s} = \boldsymbol{\epsilon} + \boldsymbol{\Omega} \tag{2.16}$$

$\boldsymbol{\epsilon}$ is the symmetric strain tensor and $\boldsymbol{\Omega}$ is the tensor which describes rigid body rotations when ∇s is small. ($\boldsymbol{\Omega}$ does not describe rigid body rotation during *finite* deformation of a body but we won't have to worry about this here. For infinitesimal deformation, note that both s and ∇s must be small).

Because ∇s is a tensor, both $\boldsymbol{\epsilon}$ and $\boldsymbol{\Omega}$ are tensors and so $\boldsymbol{\epsilon}$ changes under a coordinate system rotation as

$$\boldsymbol{\epsilon}' = \mathbf{A}^T \boldsymbol{\epsilon}\, \mathbf{A} \tag{2.17}$$

We can define principal axes of strain and principal strains by analogy with the stress tensor definitions. Similarly, Trace($\boldsymbol{\epsilon}$) is an invariant and is called the *dilatation*, *i.e.*, neglecting second order terms, we have

$$\epsilon_{ii} = \text{Trace}(\boldsymbol{\epsilon}) = (V - V_0)/V_0 \tag{2.18}$$

where V_0 is the reference state volume.

The mean normal strain is defined as $\frac{1}{3}\,\epsilon_{ii}$ and we can define the *strain deviator* as the deviatoric part of $\boldsymbol{\epsilon}$, *i.e.*,

$$\boldsymbol{\epsilon}^D = \boldsymbol{\epsilon} - \tfrac{1}{3}\,\text{Tr}\,(\boldsymbol{\epsilon})\,\mathbf{I} \tag{2.19}$$

2.3 Conservation laws . We shall be using Gauss' theorem quite a lot in the following *i.e.*,

$$\int_S \mathbf{v} \cdot \hat{\mathbf{n}}\, dS = \int_V \nabla \cdot \mathbf{v}\, dV \tag{2.20}$$

where $\mathbf{v}$ is a vector. More generally we have that

$$\int_S \hat{\mathbf{n}} * \mathcal{A}\, dS = \int_V \nabla * \mathcal{A}\, dV \tag{2.21}$$

where $\mathcal{A}$ may be a scalar, vector or tensor and $*$ can be an ordinary product, vector dot product or vector cross product depending upon the context. We also note that the *flux* of $\mathcal{A}$ through a surface S is given by

$$\int_S \rho \mathcal{A} \mathbf{v} \cdot \hat{\mathbf{n}}\, dS \tag{2.22}$$

(sometimes $\hat{\mathbf{n}}\, dS$ is denoted by $d\mathbf{A}$).

With these preliminaries out of the way, we now turn to *conservation of mass*. The mass of a volume V is given by

$$M = \int_V \rho\, dV$$

so the rate of increase of M is given by

$$\frac{\partial M}{\partial t} = \int_V \frac{\partial \rho}{\partial t}\, dV$$

provided that the surface of V is fixed in space. As we hypothesize no creation or destruction of mass, it follows that $\partial M/\partial t$ must equal the rate of inflow of mass. From 2.20 and 2.22, the rate of inflow of mass is given by

$$-\int_S \rho \mathbf{v} \cdot \hat{\mathbf{n}}\, dS = -\int_V \nabla \cdot (\rho \mathbf{v})\; dV$$

where the minus sign arises as we are considering an *inward* flux of material. Thus

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla \cdot (\rho \mathbf{v})\; dV$$

therefore

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$$

and, because this is true for an arbitrary volume, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.23}$$

or, equivalently

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{2.24}$$

These last two equations are both statements of the conservation of mass. Note that if $\nabla \cdot \mathbf{v} = 0$, it immediately follows that $D\rho/Dt = 0$ so that the density of a particle does not change with time. This states that the medium is *incompressible* and is a commonly used approximation in fluid mechanics. Obviously, it is not a useful approximation in seismology (P-waves would not exist!).

The most useful conservation law we shall use is the *conservation of linear momentum* but to develop it we shall need the *Reynolds mass transport theorem* which is proved in Malvern (1969, p210). This theorem states that

$$\frac{D}{Dt}\int_V \rho\mathcal{A}\,dV = \int_V \rho\frac{D\mathcal{A}}{Dt}\,dV \tag{2.25}$$

where $\mathcal{A}$ can be a vector, scalar or tensor.

The conservation of linear momentum is a basic postulate of continuum mechanics and can be stated as: the time rate of change of total momentum of a given set of particles equals the vector sum of all external forces acting on the particles. In mathematical form we have

$$\frac{D}{Dt}\int_V \rho\mathbf{v}\,dV = \int_S \mathbf{t}\,dS + \int_V \rho\mathbf{b}\,dV \tag{2.26}$$

or by 2.25

$$\int_V \rho\frac{D\mathbf{v}}{Dt}\,dV = \int_S \mathbf{t}\,dS + \int_V \rho\mathbf{b}\,dV \tag{2.27}$$

Now $\mathbf{t} = \hat{\mathbf{n}}\cdot\mathbf{T}$ so Gauss' theorem can be used to convert the surface integral into a volume integral *i.e.*,

$$\int_S \mathbf{t}\,dS = \int_S \hat{\mathbf{n}}\cdot\mathbf{T}\,dS = \int_V \nabla\cdot\mathbf{T}\,dV$$

Combining this with the previous equation gives

$$\int_V \left(\rho\frac{D\mathbf{v}}{Dt} - \nabla\cdot\mathbf{T} - \rho\mathbf{b}\right)\,dV = 0$$

which must hold for an arbitrary volume so

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla\cdot\mathbf{T} + \rho\mathbf{b} \tag{2.28}$$

These are Cauchy's equations of motion and they apply to the current deformed configuration. We have not yet made any approximation about the constitutive relationship or the size of the deformation. We shall continue to develop these equations in a seismological context but first we shall take a brief look at the *conservation of angular momentum*. This reads

$$\frac{D}{Dt}\int_V (\mathbf{r}\times\rho\mathbf{v})\,dV = \int_S (\mathbf{r}\times\mathbf{t})\,dS + \int_V (\mathbf{r}\times\rho\mathbf{b})\,dV \tag{2.29}$$

Malvern (1969, p215) shows that this can only be satisfied if $\mathbf{T}$ is symmetric *i.e.*,

$$T_{ij} = T_{ji} \tag{2.30}$$

This result is generally true and does not depend upon equilibrium or other conditions.

2.4 Constitutive relationships . We shall consider only perfect elasticity which has a linear relationship between stress and strain of the form

$$T_{ij} = C_{ijkl}\epsilon_{kl} \tag{2.31}$$

The fourth order tensor C_{ijkl} has several symmetries:

$$\begin{aligned} C_{jikl} &= C_{ijkl} \quad \text{because} \quad T_{ij} = T_{ji} \\ C_{ijlk} &= C_{ijkl} \quad \text{because} \quad \epsilon_{ij} = \epsilon_{ji} \\ \text{and} \quad C_{klij} &= C_{ijkl} \end{aligned} \tag{2.32}$$

This last symmetry arises because the elastic work function:

$$W = \tfrac{1}{2}\, C_{ijkl}\epsilon_{ij}\epsilon_{kl} = \tfrac{1}{2}\, T_{ij}\epsilon_{ij} \tag{2.33}$$

must be a positive definite form, *i.e.*, for any strain applied to a material, positive work must be done – a prerequisite for a stable material.

The symmetries of 2.32 leads to a simplified tensor notation where C_{ijkl} is represented as a symmetrix 6×6 matrix. In this notation (called *Voigt* notation), 2.31 becomes

$$T_i = C_{ij}\epsilon_j \tag{2.34}$$

or explicitly

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Similarly

$$W = \tfrac{1}{2}\, T_k\epsilon_k = \tfrac{1}{2}\, \boldsymbol{\epsilon}^T \cdot \mathbf{C} \cdot \boldsymbol{\epsilon}$$

As an example of the equivalence of 2.31 and 2.34, note that $C_{56} \equiv C_{3112} = C_{3121} = C_{1312} = C_{1321}$. The symmetry of C_{ij} in 2.34 also makes it clear that a perfectly elastic material has only 21 independent elastic constants.

The positive definite form of W allows a uniqueness proof for solutions to elastodynamic problems with a perfectly elastic constitutive relationship. The proof is given on pg. 24 of Aki and Richards and will not be reproduced here.

The solution to equation 2.28 conventionally proceeds by separation of variables. For this to be possible, our model of the Earth must satisfy certain symmetry conditions – in particular we shall be concerned about spherical symmetry. An Earth model will only have spherical symmetry for certain special forms of C_{ijkl}. One obvious possibility is that the material is elastically isotropic. This means that wave velocities have no preferred direction so the elements of C_{ijkl} must be invariant to any rotation of the coordinate system. The most general fourth-order isotropic tensor is given by (Chapter 7 of *Cartesian Tensors* by Jeffreys)

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \eta\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} \tag{2.35}$$

Substitution into 2.31 gives

$$T_{ij} = \lambda\delta_{ij}\epsilon_{kk} + \eta\epsilon_{ij} + \nu\epsilon_{ji}$$

Therefore

$$T_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \tag{2.36}$$

where $2\mu = \eta + \nu$.

λ and μ are the well-known Lamé constants and μ is also called the rigidity or shear modulus. Other combinations of elastic moduli that are commonly referred to for isotropic materials are

$$\left.\begin{aligned} K_s &= \lambda + \frac{2}{3}\mu \\ E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \\ \sigma &= \frac{\lambda}{2(\lambda + \mu)} \end{aligned}\right\} \tag{2.37}$$

where K_s is the adiabatic bulk modulus, E is Young's modulus, and σ is Poisson's ratio. The Earth's mantle has a Poisson's ratio of close to 0.25 which is achieved when $\lambda \simeq \mu$. ($\lambda = \mu$ defines a *Poisson* solid). In a fluid $\mu = 0$ so $\sigma = 0.5$. The fluid outer core is often modeled as a perfectly elastic (inviscid) fluid which is obtained from 2.36 by setting $\mu = 0$, *i.e.*,

$$T_{ij} = \lambda \delta_{ij} \epsilon_{kk} \tag{2.38}$$

Thus there are no off-diagonal (shear) stresses.
In Voigt notation, an elastically isotropic material is given by

$$\mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & 0 & & & \mu & \\ & & & & & \mu \end{bmatrix} \tag{2.39}$$

which is invariant to any rotation of the coordinate system. Equation 2.35 is not the most general form of $\mathbf{C}$ which still allows us to construct a model with spherical symmetry. A little thought will show that elastic velocities in the radial direction can be different from elastic velocities perpendicular to the radial direction. $\mathbf{C}$ then has rotational symmetry about the $\hat{\mathbf{r}}$ direction. Such a material is called *transversely isotropic*. If the 3-direction is chosen as the $\hat{\mathbf{r}}$-direction, a transversely isotropic material in the Voigt notation looks like:

$$\mathbf{C} = \begin{bmatrix} \lambda' + 2\mu' & \lambda' & \lambda & & & \\ \lambda' & \lambda' + 2\mu' & \lambda & & 0 & \\ \lambda & \lambda & \beta & & & \\ & & & \mu & & \\ & 0 & & & \mu & \\ & & & & & \mu' \end{bmatrix} \tag{2.40}$$

Note that there are five independent elastic coefficients (as opposed to two for the isotropic case). A commonly used alternative notation in 2.70 is A $= \lambda' + 2\mu'$, C $= \beta$, F $= \lambda$, L $= \mu$, and N $= \mu'$.

The perfectly elastic constitutive relationship is not a bad approximation to the real Earth – in fact the real Earth is quite close to being isotropic though it is now commonly modeled as being transversely isotropic. General anisotropy is quite weak and hopefully can be modeled by perturbation theory.

2.5 Boundary conditions . There are several kinds of boundaries we must deal with in the real Earth. The surface of the Earth is usually modeled as being "free" which means that tractions must vanish here as we assume there are no external forces acting on the Earth. (We also neglect coupling of seismic motion into the atmosphere.) Another type of boundary is one between fluid and solid (*e.g.*, inner core-outer core, mantle-outer core, crust-ocean) and a final type is a welded type (*e.g.*, the 660 km discontinuity). Because, to a first approximation, we consider fluids to be inviscid we can have slip at a fluid-solid boundary but not at a welded boundary.

We first consider the boundary conditions that must be satisfied on the deformed boundaries and then later consider the equivalent conditions on the undeformed surface.
The conditions are:

1) $\hat{\mathbf{n}} \cdot \mathbf{s}$ is continuous at all boundaries so that we allow no gapping or interpenetration. Note that $\hat{\mathbf{n}}$ is the normal to the *deformed* surface.
2) $\mathbf{s}$ is continuous at all *welded* boundaries, *i.e.*, we allow no slip at such boundaries.
3) $\mathbf{t}$ (the traction vector) is continuous at all deformed boundaries and is zero at the free surface.
4) ϕ is continuous at all boundaries.
5) $\hat{\mathbf{n}} \cdot \nabla\phi$ is continuous at all (deformed) boundaries.
6) The displacements and tractions must be regular at the origin and the gravitational potential must be zero at infinity.

These boundary conditions are valid on the deformed boundaries and we must linearize to get the equivalent boundary conditions on the undeformed boundaries. A summary of the boundary conditions on the undeformed boundary with an isotropic prestress is

$\hat{\mathbf{r}} \cdot \mathbf{s}$ is continuous at all boundaries
$\mathbf{s}$ is regular at the origin
$\mathbf{s}$ is continuous at welded boundaries
ϕ_1 is continuous at all boundaries
ϕ_1 is zero at infinity
$\partial \phi_1 / \partial r + 4\pi G \rho_0 s_r$ is continuous at all boundaries
$\mathbf{t}$ is continuous at all boundaries
$\mathbf{t}$ is zero at the free surface
$\mathbf{t}$ is regular at the origin.