

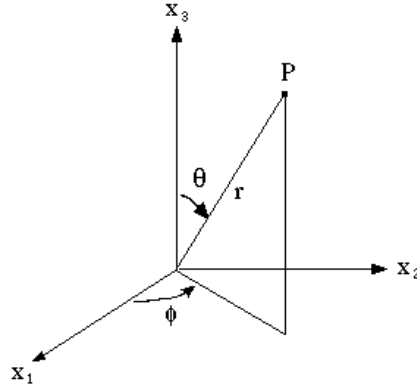
APPENDIX A

Spherical Geometry, Spherical Harmonics and Tensor Calculus

1. Introduction

As the Earth is almost a spherical body, many problems in geophysics require the use of spherical coordinates rather than Cartesian coordinates. Furthermore we often need to be able to specify scalar, vector and tensor fields anywhere within this body and this is most commonly done using expansions in spherical harmonics. Unfortunately, tensor calculus in anything but Cartesian coordinates is algebraically tedious and many techniques have been devised to alleviate the labor. You may be familiar with the use of Christoffel symbols in covariant differentiation of tensors. Another technique, familiar in quantum mechanics, is expansion in generalized spherical harmonics (GSH). This technique was made popular in the geophysical literature in a paper by Phinney and Burridge (1973) and some detail can also be found in the book by Edmonds (1960).

We use standard spherical polar coordinates :



$$x_1 = r \sin \theta \cos \phi \quad \text{and} \quad x_2 = r \sin \theta \sin \phi \quad \text{and} \quad x_3 = r \cos \theta$$

where θ is colatitude and ϕ is east longitude.

The spherical harmonics in most common use in the free oscillation literature are those defined by Edmonds (1960), *i.e.*,

$$Y_l^m(\theta, \phi) = (-1)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{\frac{1}{2}} P_l^m(\cos \theta) e^{im\phi}$$

where the P_l^m are associated Legendre functions which are progressively wigglier functions of θ as l increases. We often need to compute Y_l^m 's and their derivatives with respect to θ . This can easily be done using the recursion relationships given by Edmonds (1960) for the P_l^m .

Some properties of the Y_l^m 's are:

$$Y_l^{-m} = (-1)^m Y_l^{m*} \quad \text{where } * \text{ denotes complex conjugation}$$

$$\nabla_1^2 Y_l^m = -l(l+1)Y_l^m \quad \text{where} \quad \nabla_1^2 = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \operatorname{cosec}^2 \theta \frac{\partial^2}{\partial \phi^2}$$

This relationship allows us to cast higher order θ derivatives in terms of Y_l^m and $\partial Y_l^m / \partial \theta$. Note that ϕ derivatives are trivial, *i.e.*,

$$\frac{\partial}{\partial \phi} Y_l^m = imY_l^m \quad \text{etc.}$$

Finally the Y_l^m 's are fully normalized, *i.e.*,

$$\int_S Y_{l'}^{m'*} Y_l^m dS = \delta_{mm'} \delta_{ll'} \quad \text{where} \quad dS = \sin \theta d\theta d\phi$$

If we wish to specify an appropriately smooth *scalar* function of position inside the Earth, $\rho(r, \theta, \phi)$ say, then we can make an expansion in spherical harmonics, e.g.,

$$\rho(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \rho_l^m(r) Y_l^m(\theta, \phi) \quad (\text{A.1})$$

and the field is completely specified by the coefficients, $\rho_l^m(r)$.

The situation is a little more complicated if we have a vector field and it turns out that it is not useful to expand the r, θ, ϕ components in a way similar to the scalar expansion. The usual way to represent a vector field in spherical geometry is to use the form (Morse and Feshback, 1953):

$$\mathbf{u}(r, \theta, \phi) = \hat{\mathbf{r}}U(r, \theta, \phi) + \nabla_1 V(r, \theta, \phi) - \hat{\mathbf{r}} \times (\nabla_1 W(r, \theta, \phi)) \quad (\text{A.2})$$

where U, V , and W are scalars, ∇_1 is the surface gradient operator defined by

$$\nabla_1 = \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} + \operatorname{cosec} \theta \hat{\boldsymbol{\phi}} \frac{\partial}{\partial \phi}$$

and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are unit vectors in the r, θ, ϕ directions. Note that $\hat{\mathbf{r}} \times \nabla_1$ (\times is vector cross product) is

$$\hat{\mathbf{r}} \times \nabla_1 = -\hat{\boldsymbol{\theta}} \operatorname{cosec} \theta \frac{\partial}{\partial \phi} + \hat{\boldsymbol{\phi}} \frac{\partial}{\partial \theta}$$

The scalars U, V and W can now be expanded as in A1. Equation A2 is the standard form for the expansion of the displacement field in low-frequency seismology.

Canonical components and generalized spherical harmonics

When we come to tensor fields, the algebra gets a little more awkward and it turns out that things simplify if we abandon the r, θ, ϕ coordinates and introduce new ones. We label these new directions $-, 0, +$. If $\mathbf{u}(r, \theta, \phi)$ has components u_r, u_θ, u_ϕ then the new directions are defined as

$$\begin{aligned} u^- &= \frac{1}{\sqrt{2}}(u_\theta + iu_\phi) \\ u^0 &= u_r \\ u^+ &= \frac{1}{\sqrt{2}}(-u_\theta + iu_\phi) \end{aligned}$$

It is convenient to represent this coordinate transformation as a matrix operation. Suppose we let

$$u_1 = u_\theta, \quad u_2 = u_\phi \quad \text{and} \quad u_3 = u_r$$

then

$$u^\alpha = \sum_{i=1}^3 C_{\alpha i}^\dagger u_i \quad \alpha = -, 0, + \quad (\text{A.3})$$

where

$$C_{\alpha i}^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{bmatrix} \quad \text{with Hermitian conjugate} \quad C_{i\alpha} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{A.4})$$

A tensor of any order can be put into the new coordinate system by repetition of the operation A3, *i.e.*,

$$\mathbf{m}^{\alpha\beta\gamma\dots} = C_{\alpha i}^\dagger C_{\beta j}^\dagger C_{\gamma k}^\dagger \dots \mathbf{m}_{ijk\dots} \quad (\text{A.5})$$

where summation over repeated indices is implied. Remember that i, j, k , etc. go from 1 to 3 corresponding to θ, ϕ, r directions respectively. $\mathbf{m}^{\alpha\beta\gamma\dots}$ can be returned to the original coordinate system by the operation

$$\mathbf{m}_{ijk\dots} = C_{i\alpha} C_{j\beta} C_{k\gamma} \dots \mathbf{m}^{\alpha\beta\gamma\dots} \quad (\text{summation implied}) \quad (\text{A.6})$$

Tensor calculus becomes much simpler if we expand the contravariant canonical components in *generalized spherical harmonics* rather than the Y_l^m 's defined earlier, *i.e.*, we set

$$\mathbf{m}^{\alpha\beta\gamma\dots}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l M_l^{\alpha\beta\gamma\dots, m}(r) Y_l^{N, m}(\theta, \phi) \quad (\text{A.7})$$

where $N = \alpha + \beta + \gamma + \dots$. For the contravariant canonical components of the displacement field we have

$$u^\alpha(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l U_l^{\alpha, m} Y_l^{\alpha, m}(\theta, \phi) \quad \text{where} \quad \alpha = -, 0, + \quad (\text{A.8})$$

The generalized spherical harmonics are given by

$$Y_l^{N, m} = P_l^{N, m}(\cos \theta) e^{im\phi}$$

where the $P_l^{N, m}$ are defined by Phinney and Burrige who also give recursion relations for their calculation. We note two of their formulae here:

$$\left. \begin{aligned} \Omega_{N+1}^l Y_l^{N+1, m} + \Omega_N^l Y_l^{N-1, m} &= \sqrt{2} [N \cot \theta - m \operatorname{cosec} \theta] Y_l^{N, m} \\ \Omega_N^l Y_l^{N-1, m} - \Omega_{N+1}^l Y_l^{N+1, m} &= \sqrt{2} \frac{dY_l^{N, m}}{d\theta} \end{aligned} \right\} \quad (\text{A.9})$$

where $\Omega_N^l = [\frac{1}{2}(l+N)(l-N+1)]^{\frac{1}{2}}$. Note that $\Omega_{-N+1}^l = \Omega_N^l$ so that $\Omega_0^l = \Omega_1^l, \Omega_{-1}^l = \Omega_2^l$, etc.

It is convenient to write down some relationships between the GSH and the ordinary Y_l^m 's. Let $\gamma_l = \sqrt{\left(\frac{2l+1}{4\pi}\right)}$. Then

$$\left. \begin{aligned} \gamma_l Y_l^{0,m} &= Y_l^m \\ \frac{1}{\sqrt{2}} \gamma_l \Omega_1^l [Y_l^{-1,m} + Y_l^{+1,m}] &= -m \operatorname{cosec} \theta Y_l^m \\ \frac{1}{\sqrt{2}} \gamma_l \Omega_1^l [Y_l^{-1,m} - Y_l^{+1,m}] &= \frac{\partial Y_l^m}{\partial \theta} \\ \frac{1}{2} \gamma_l \Omega_1^l \Omega_2^l [Y_l^{-2,m} + Y_l^{+2,m}] &= [m^2 \operatorname{cosec}^2 \theta - \frac{l(l+1)}{2}] Y_l^m - \cot \theta \frac{\partial Y_l^m}{\partial \theta} \\ \frac{1}{2} \gamma_l \Omega_1^l \Omega_2^l [Y_l^{-2,m} - Y_l^{+2,m}] &= m \operatorname{cosec} \theta \left[\cot \theta Y_l^m - \frac{\partial Y_l^m}{\partial \theta} \right] \end{aligned} \right\} \quad (\text{A.10})$$

Relationships for higher N can be found from the recursion formulae A9. These relationships are useful if one wants to convert from the canonical components back to expressions in r, θ, ϕ involving ordinary spherical harmonics.

Equation A5 defines the contravariant canonical component of $m_{ijk\dots}$ and we can also define covariant components of $m_{ijk\dots}$ (though these are less useful), *i.e.*,

$$m_{\alpha\beta\gamma} = C_{i\alpha} C_{j\beta} C_{k\gamma} \cdots m_{ijk\dots} \quad (\text{summation implied}) \quad (\text{A.11})$$

We will need the covariant components of the tensor δ_{ij} . We define these as

$$\Delta_{\alpha\beta} = C_{i\alpha} C_{j\beta} \delta_{ij} = C_{i\alpha} C_{i\beta} = C_{1\alpha} C_{1\beta} + C_{2\alpha} C_{2\beta} + C_{3\alpha} C_{3\beta}$$

From A4 we find that

$$\Delta_{00} = 1 \quad \text{and} \quad \Delta_{-+} = \Delta_{+-} = -1 \quad \text{and all other components are zero} \quad (\text{A.12})$$

As an example of the use of Δ , we consider the trace of a second-order tensor, *i.e.*,

$$\operatorname{Tr}(m_{ij}) = m_{ij} \delta_{ij} = m_{ii} = m_{11} + m_{22} + m_{33}$$

m_{ii} is a scalar and is equivalent to

$$\begin{aligned} m_{ii} &= \Delta_{\alpha\beta} m^{\alpha\beta} \quad (\text{summation implied}) \\ &= m^{00} - m^{+-} - m^{-+} \quad (\text{as all other components of } \Delta_{\alpha\beta} \text{ are zero}) \end{aligned}$$

In elasticity we have to worry about the derivatives of vectors and tensors. Differentiation is straightforward in the new coordinate system. For definiteness we consider a second-order tensor m_{ij} , differentiation gives a third-order tensor which we write as $m_{ij,k}$.

From equation A7 we have the expansion

$$m^{\alpha\beta}(r, \theta, \phi) = \sum_{l=N}^{\infty} \sum_{m=-l}^l M_l^{\alpha\beta,m}(r) Y_l^{(\alpha+\beta),m}(\theta, \phi)$$

To reduce the number of indices we have to write out, we consider a single l, m component of $m^{\alpha\beta}$ and write this as

$$m^{\alpha\beta} = M^{\alpha\beta} Y^{\alpha+\beta} \quad (l \text{ and } m \text{ understood})$$

Differentiation results in a higher rank tensor which we write as

$$m^{\alpha\beta,\gamma} = M^{\alpha\beta|\gamma} Y^{\alpha+\beta+\gamma}$$

The coefficients $M^{\alpha\beta|\gamma}$ can be found in terms of the $M^{\alpha\beta}$ by using the following recipe which we give for a tensor of any order. Consider $\gamma = -, 0, +$ separately, then we have

$$\left. \begin{aligned} M^{\alpha\beta\dots| -} &= \frac{1}{r} \left[\Omega_N^l M^{\alpha\beta\dots} - \left\{ \begin{array}{l} \text{terms obtained from } M^{\alpha\beta\dots} \text{ by changing} \\ \text{+ into 0, 0 into - one at a time} \end{array} \right\} \right] \\ M^{\alpha\beta\dots| 0} &= \frac{dM^{\alpha\beta}}{dr} \\ M^{\alpha\beta\dots| +} &= \frac{1}{r} \left[\Omega_{N+1}^l M^{\alpha\beta\dots} - \left\{ \begin{array}{l} \text{terms obtained from } M^{\alpha\beta\dots} \text{ by changing} \\ \text{- into 0, 0 into + one at a time} \end{array} \right\} \right] \end{aligned} \right\} \quad (\text{A.13})$$

Here $N = \alpha + \beta + \dots$ and when calculating N , regard $-, 0, +$ as shorthand for $-1, 0$ and $+1$. The use of A13 is best illustrated by example. For a second rank tensor we consider the $+0$ component only so $N = 1$. Thus

$$\begin{aligned} M^{+0| -} &= \frac{1}{r} [\Omega_1^l M^{+0} - M^{00} - M^{+-}] \\ M^{+0| 0} &= \frac{dM^{+0}}{dr} \\ M^{+0| +} &= \frac{1}{r} [\Omega_2^l M^{+0} - M^{++}] \end{aligned}$$

As another example of the use of $\Delta_{\alpha\beta}$, we consider the divergence of a second rank tensor which would be written $m_{ij,j}$. This is equivalent to

$$\Delta_{\beta\gamma} m^{\alpha\beta,\gamma} = m^{\alpha 0,0} - m^{\alpha+,-} - m^{\alpha-,+}$$

Another example of the use of A13 is given by the computation of the Laplacian of a scalar field. A single l, m component of the expansion of a scalar field ϕ can be written

$$\phi = \Phi Y^{0,m}$$

where Φ is an expansion coefficient. Differentiation gives

$$\phi^{,\alpha} = \Phi^{|\alpha} Y^{\alpha,m}$$

which using A13 becomes

$$\begin{aligned} \phi^{,-} &= \frac{1}{r} \Omega_0^l \Phi Y^{-1,m} \\ \phi^{,0} &= \frac{d}{dr} \Phi Y^{0,m} \\ \phi^{,+} &= \frac{1}{r} \Omega_1^l \Phi Y^{+1,m} \end{aligned}$$

The Laplacian is given by

$$\Delta_{\alpha\beta}\phi^{\alpha\beta} = \phi^{,00} - \phi^{,-+} - \phi^{,+}$$

but using A13 again gives

$$\begin{aligned}\phi^{,00} &= \frac{d^2}{dr^2}\Phi Y^{0,m} \\ \phi^{,-+} &= \frac{1}{r} \left[\Omega_0^l \frac{1}{r} \Omega_0^l \Phi - \frac{d}{dr} \Phi \right] Y^{0,m} \\ \phi^{,+} &= \frac{1}{r} \left[\Omega_1^l \frac{1}{r} \Omega_1^l \Phi - \frac{d}{dr} \Phi \right] Y^{0,m}\end{aligned}$$

so the Laplacian of a single l, m component of ϕ is

$$\nabla^2\phi = \nabla^2(\Phi Y^{0,m}) = \left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] \Phi Y^{0,m} \quad (\text{A.14})$$

The canonical components of displacement

These definitions are sufficient for our purposes so now we illustrate how to use these formulae with the free oscillation equations. Suppose we have represented the displacement field by the expression in equation A2, *i.e.*,

$$\begin{aligned}\mathbf{u}(r, \theta, \phi) &= \hat{\mathbf{r}} \sum_{l=0}^{\infty} \sum_{m=-l}^l U_l^m(r) Y_l^m(\theta, \phi) + \hat{\boldsymbol{\theta}} \sum_{l=0}^{\infty} \sum_{m=-l}^l V_l^m(r) \frac{\partial Y_l^m}{\partial \theta}(\theta, \phi) \\ &+ i \operatorname{cosec} \theta \hat{\boldsymbol{\phi}} \sum_{l=0}^{\infty} \sum_{m=-l}^l m V_l^m(r) Y_l^m(\theta, \phi) + i \hat{\boldsymbol{\theta}} \operatorname{cosec} \theta \sum_{l=0}^{\infty} \sum_{m=-l}^l m W_l^m(r) Y_l^m(\theta, \phi) \\ &- \hat{\boldsymbol{\phi}} \sum_{l=0}^{\infty} \sum_{m=-l}^l W_l^m(r) \frac{\partial Y_l^m}{\partial \theta}(\theta, \phi)\end{aligned}$$

For simplicity we consider a single l, m component so that, on collecting terms, we obtain

$$\mathbf{u}(r, \theta, \phi) = \hat{\mathbf{r}} U Y_l^m + \hat{\boldsymbol{\theta}} \left(V \frac{\partial Y_l^m}{\partial \theta} + im \operatorname{cosec} \theta W Y_l^m \right) + \hat{\boldsymbol{\phi}} \left(im \operatorname{cosec} \theta V Y_l^m - W \frac{\partial Y_l^m}{\partial \theta} \right) \quad (\text{A.15})$$

where the indices l and m on U, V , and W are understood. Applying equation A3 gives

$$\begin{aligned}u^- &= \frac{1}{\sqrt{2}} \left[(V - iW) \left(\frac{\partial Y_l^m}{\partial \theta} - m \operatorname{cosec} \theta Y_l^m \right) \right] \\ u^0 &= U Y_l^m \\ u^+ &= \frac{1}{\sqrt{2}} \left[(V + iW) \left(-\frac{\partial Y_l^m}{\partial \theta} - m \operatorname{cosec} \theta Y_l^m \right) \right]\end{aligned} \quad (\text{A.16})$$

Equation A8 for a single l, m component gives

$$\begin{aligned} u^- &= U^- Y_l^{-1,m} \\ u^0 &= U^0 Y_l^{0,m} \\ u^+ &= U^+ Y_l^{+1,m} \end{aligned} \quad (\text{A.17})$$

Comparison of A16 and A17 and use of the relationships between $Y_l^{N,m}$ and Y_l^m gives

$$\begin{aligned} U^- &= (V - iW) \gamma_l \Omega_0^l \\ U^0 &= U \gamma_l \\ U^+ &= (V + iW) \gamma_l \Omega_0^l \end{aligned} \quad (\text{A.18})$$

We now have the relationship between the new and old coordinate system and so can turn to giving examples of tensor differentiation for a single l, m component.

The Strain Tensor

In Cartesian coordinates the strain tensor is given by

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} (u_{i,j} + u_{j,i})$$

In spherical polar coordinates, the answer is not so obvious. In our canonical coordinate system we have

$$\epsilon^{\alpha\beta} = \frac{1}{2} (u^{\alpha,\beta} + u^{\beta,\alpha})$$

where

$$u^{\alpha,\beta} = U^{\alpha|\beta} Y^{\alpha+\beta}$$

Using equation A13 gives

$$\begin{aligned} U^{-|-} &= \frac{1}{r} \Omega_{-1}^l U^- & U^{-|0} &= \frac{dU^-}{dr} & U^{-|+} &= \frac{1}{r} [\Omega_0^l U^- - U^0] \\ U^{0|-} &= \frac{1}{r} [\Omega_0^l U^0 - U^-] & U^{0|0} &= \frac{dU^0}{dr} & U^{0|+} &= \frac{1}{r} [\Omega_1^l U^0 - U^+] \\ U^{+|-} &= \frac{1}{r} [\Omega_1^l U^+ - U^0] & U^{+|0} &= \frac{dU^+}{dr} & U^{+|+} &= \frac{1}{r} \Omega_2^l U^+ \end{aligned}$$

Thus

$$\epsilon^{\alpha\beta} = E^{\alpha\beta} Y^{\alpha+\beta} \quad \text{where} \quad E^{\alpha\beta} = \frac{1}{2} (U^{\alpha|\beta} + U^{\beta|\alpha})$$

and we obtain for the components of $\epsilon^{\alpha\beta}$

$$\begin{aligned} \epsilon^{--} &= \frac{1}{r} \Omega_{-1}^l U^- Y_l^{-2,m} & \epsilon^{-0} &= \epsilon^{0-} = \frac{1}{2} \left[\frac{dU^-}{dr} - \frac{U^-}{r} + \frac{\Omega_0^l U^0}{r} \right] Y_l^{-1,m} \\ \epsilon^{00} &= \frac{dU^0}{dr} Y_l^{0,m} & \epsilon^{+0} &= \epsilon^{0+} = \frac{1}{2} \left[\frac{dU^+}{dr} - \frac{U^+}{r} + \frac{\Omega_1^l U^0}{r} \right] Y_l^{+1,m} \\ \epsilon^{++} &= \frac{1}{r} \Omega_2^l Y_l^{2,m} & \epsilon^{+-} &= \epsilon^{-+} = \frac{1}{2} \frac{1}{r} [\Omega_0^l U^- + \Omega_1^l U^+ - 2U^0] Y_l^{0,m} \end{aligned}$$

If we use the fact that $\Omega_{-1}^l = \Omega_2^l$ and $\Omega_0^l = \Omega_1^l$ (as pointed out earlier) and we substitute for U^- , U^0 and U^+ from equation A18, we finally obtain

$$\left. \begin{aligned} \epsilon^{00} &= U' \gamma_l Y_l^{0,m} \\ \epsilon^{\pm\pm} &= \frac{1}{r} (V \pm iW) \gamma_l \Omega_1^l \Omega_2^l Y_l^{\pm 2,m} \\ \epsilon^{\pm 0} &= \epsilon^{0\pm} = \frac{1}{2} [X \pm iZ] \gamma_l \Omega_1^l Y_l^{\pm 1,m} \\ \epsilon^{\pm\mp} &= -\frac{1}{2} F \gamma_l Y_l^{0,m} \end{aligned} \right\} \quad (\text{A.19})$$

where prime ($'$) denotes differentiation with respect to radius and

$$X = V' + \frac{U - V}{r}, \quad Z = W' - \frac{W}{r} \quad \text{and} \quad F = \frac{1}{r} [2U - l(l+1)V]$$

For reference, we note that inspection of A19 gives the expansion coefficients, $E^{\alpha\beta}$:

$$\left. \begin{aligned} E^{00} &= U' \gamma_l \\ E^{\pm\pm} &= \frac{1}{r} (V \pm iW) \gamma_l \Omega_1^l \Omega_2^l \\ E^{\pm 0} &= E^{0\pm} = \frac{1}{2} [X \pm iZ] \gamma_l \Omega_1^l \\ E^{\pm\mp} &= -\frac{1}{2} F \gamma_l \end{aligned} \right\} \quad (\text{A.20})$$

Often we need go no further as the canonical components can be used throughout the calculations. If we wish to obtain expressions for ϵ in the original r, θ, ϕ coordinate system we must use equation A6, *i.e.*,

$$\begin{aligned} \epsilon_{ij} &= C_{i\alpha} C_{j\beta} \epsilon^{\alpha\beta} \\ &= C_{i-} C_{j-} \epsilon^{--} + C_{i-} C_{j0} \epsilon^{-0} + C_{i-} C_{j+} \epsilon^{-+} \\ &\quad + C_{i0} C_{j-} \epsilon^{0-} + C_{i0} C_{j0} \epsilon^{00} + C_{i0} C_{j+} \epsilon^{0+} \\ &\quad + C_{i+} C_{j-} \epsilon^{+-} + C_{i+} C_{j0} \epsilon^{+0} + C_{i+} C_{j+} \epsilon^{++} \end{aligned}$$

Remember that i and j go from 1 to 3 and $1 \equiv \theta$, $2 \equiv \phi$, $3 \equiv r$ directions respectively. As an example, consider the $\epsilon_{11} = \epsilon_{\theta\theta}$ component. We have from A4 that

$$C_{1-} = \frac{1}{\sqrt{2}}, \quad C_{10} = 0, \quad \text{and} \quad C_{1+} = -\frac{1}{\sqrt{2}}$$

Substitution gives

$$\begin{aligned} \epsilon_{\theta\theta} &= \frac{1}{2} \epsilon^{--} - \frac{1}{2} \epsilon^{-+} - \frac{1}{2} \epsilon^{+-} + \frac{1}{2} \epsilon^{++} \\ &= \frac{V}{2r} \gamma_l \Omega_0^l \Omega_2^l \left[Y_l^{+2,m} + Y_l^{-2,m} \right] + \frac{iW}{2r} \gamma_l \Omega_0^l \Omega_2^l \left[Y_l^{+2,m} - Y_l^{-2,m} \right] + \frac{1}{2} F \gamma_l Y_l^{0,m} \end{aligned}$$

where we have made use of equation A19. A complete list of strain components is

$$\left. \begin{aligned}
\epsilon_{rr} &= U' K_0 \\
\epsilon_{\theta\theta} &= \frac{V}{r} K_2^+ - \frac{iW}{r} K_2^- + \frac{F}{2} K_0 \\
\epsilon_{\phi\phi} &= -\frac{V}{r} K_2^+ + \frac{iW}{r} K_2^- + \frac{F}{2} K_0 \\
2\epsilon_{r\theta} &= X K_1^- - iZ K_1^+ \\
2\epsilon_{r\phi} &= -iX K_1^+ - Z K_1^- \\
2\epsilon_{\theta\phi} &= -\frac{i2V}{r} K_2^- - \frac{2W}{r} K_2^+
\end{aligned} \right\} \quad (\text{A.21})$$

where

$$\begin{aligned}
K_0 &= \gamma_l Y_l^{0,m} \\
K_1^\pm &= \frac{1}{\sqrt{2}} \gamma_l \Omega_1^l (Y_l^{-1,m} \pm Y_l^{+1,m}) \\
K_2^\pm &= \frac{1}{2} \gamma_l \Omega_1^l \Omega_2^l (Y_l^{-2,m} \pm Y_l^{+2,m}) \\
K_3^\pm &= \frac{1}{2\sqrt{2}} \gamma_l \Omega_1^l \Omega_2^l \Omega_3^l (Y_l^{-3,m} \pm Y_l^{+3,m})
\end{aligned}$$

Equation A20 can be cast in terms of ordinary spherical harmonics using the relationships in A10. The result is:

$$\left. \begin{aligned}
\epsilon_{rr} &= U' Y_l^m \\
\epsilon_{\theta\theta} &= \frac{1}{r} \left[U Y_l^m + V \left(m^2 \operatorname{cosec}^2 \theta Y_l^m - l(l+1) Y_l^m - \cot \theta \frac{\partial Y_l^m}{\partial \theta} \right) \right. \\
&\quad \left. + i \frac{W}{r} m \operatorname{cosec} \theta \left(\frac{\partial Y_l^m}{\partial \theta} - \cot \theta Y_l^m \right) \right] \\
\epsilon_{\phi\phi} &= \frac{1}{r} \left[U Y_l^m + V \left(\cot \theta \frac{\partial Y_l^m}{\partial \theta} - m^2 \operatorname{cosec}^2 \theta Y_l^m \right) \right. \\
&\quad \left. - i \frac{W}{r} m \operatorname{cosec} \theta \left(\frac{\partial Y_l^m}{\partial \theta} - \cot \theta Y_l^m \right) \right] \\
2\epsilon_{r\theta} &= X \frac{\partial Y_l^m}{\partial \theta} + im \operatorname{cosec} \theta Z Y_l^m \\
2\epsilon_{r\phi} &= im \operatorname{cosec} \theta X Y_l^m - Z \frac{\partial Y_l^m}{\partial \theta} \\
2\epsilon_{\theta\phi} &= i \frac{2mV}{r} \operatorname{cosec} \theta \left(\frac{\partial Y_l^m}{\partial \theta} - \cot \theta Y_l^m \right) \\
&\quad + \frac{W}{r} \left(2 \cot \theta \frac{\partial Y_l^m}{\partial \theta} - 2m^2 \operatorname{cosec}^2 \theta Y_l^m + l(l+1) Y_l^m \right)
\end{aligned} \right\} \quad (\text{A.22})$$

where X and Z are defined in equation A19.

It is sometimes convenient to work in “epicentral coordinates,” *i.e.*, with the pole of the coordinate system at the earthquake epicenter. The strain tensor must then be evaluated at $\theta, \phi \rightarrow 0$. Equation A20 makes this evaluation particularly simple as generalized spherical harmonics have the property

$$Y_l^{N,m}(0,0) = \delta_{Nm} \quad (\text{A.23})$$

At $(\theta, \phi) = (0, 0)$, it therefore follows that

$$\left. \begin{aligned} K_0 &= d_l^0 \quad \text{for } m = 0 \text{ only} \\ K_k^+ &= d_l^k \quad \text{for } m = \pm k \text{ only} \\ K_k^- &= \mp d_l^k \quad \text{for } m = \pm k \text{ only} \end{aligned} \right\} \quad (\text{A.24})$$

where

$$d_l^k = \frac{1}{2^k} \left[\frac{2l+1}{4\pi} \frac{(l+k)!}{(l-k)!} \right]^{\frac{1}{2}} \quad (\text{A.25})$$

and the nonzero values of the strain tensor can be easily evaluated (Table A1).

The Divergence of the Displacement Vector $\nabla \cdot \mathbf{u}$

In Cartesian coordinates $\nabla \cdot \mathbf{u} = \partial u_i / \partial x_i = \epsilon_{ii}$ (summation implied). In canonical components we have $\epsilon_{ii} = \epsilon_{ij} \delta_{ij} \equiv \Delta_{\alpha\beta} \epsilon^{\alpha\beta}$ and from equation A12 this becomes

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \epsilon^{00} - \epsilon^{+-} - \epsilon^{-+} \\ &= (U' + F) \gamma_l Y_l^{0,m} = (U' + F) Y_l^m \end{aligned} \quad (\text{A.26})$$

where we have used equation A19. This is easily verified by summing $\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi}$ in equation A21.

The gradient of the strain tensor

When computing the excitation of modes by extended sources, we need to know the strain tensor of a mode throughout the source volume. One way of doing this is to expand the strain tensor in a Taylor series about some fiducial point. This requires (at least) the first derivative of the strain tensor. Fortunately, the calculation is not too onerous in canonical components. Our starting point is (for a single l, m component)

$$\epsilon^{\alpha\beta,\gamma} = E^{\alpha\beta|\gamma} Y^{\alpha+\beta+\gamma}$$

Using equation A13 gives

$$\begin{aligned} E^{00|-} &= \frac{1}{r} [\Omega_0^l E^{00} - 2E^{-0}] & E^{00|0} &= E^{00} & E^{00|+} &= \frac{1}{r} [\Omega_0^l E^{00} - 2E^{+0}] \\ E^{++|-} &= \frac{1}{r} [\Omega_2^l E^{++} - 2E^{+0}] & E^{++|0} &= E^{++} & E^{++|+} &= \frac{1}{r} \Omega_3^l E^{++} \\ E^{--|-} &= \frac{1}{r} \Omega_3^l E^{--} & E^{--|0} &= \frac{d}{dr} E^{--} & E^{--|+} &= \frac{1}{r} [\Omega_0^l E^{--} - 2E^{-0}] \\ E^{+0|-} &= \frac{1}{r} [\Omega_0^l E^{+0} - E^{00} - E^{+-}] & E^{+0|0} &= \frac{d}{dr} E^{+0} & E^{+0|+} &= \frac{1}{r} [\Omega_2^l E^{+0} - E^{++}] \\ E^{-0|-} &= \frac{1}{r} [\Omega_2^l E^{-0} - E^{--}] & E^{-0|0} &= \frac{d}{dr} E^{-0} & E^{-0|+} &= \frac{1}{r} [\Omega_0^l E^{-0} - E^{00} - E^{-+}] \\ E^{+-|-} &= \frac{1}{r} [\Omega_0^l E^{+-} - E^{0-}] & E^{+-|0} &= \frac{d}{dr} E^{+-} & E^{+-|+} &= \frac{1}{r} [\Omega_0^l E^{+-} - E^{+0}] \end{aligned}$$

so we obtain

$$\left. \begin{aligned} \epsilon^{00,0} &= U'' \gamma_l Y^{0,m} \\ \epsilon^{00,\pm} &= \frac{1}{r} [U' - X \mp iZ] \gamma_l \Omega_0^l Y^{\pm 1,m} \\ \epsilon^{\pm\pm,0} &= \frac{1}{r} \left[(V' \pm iW') - \frac{(V \pm iW)}{r} \right] \gamma_l \Omega_0^l \Omega_2^l Y^{\pm 2,m} \\ \epsilon^{\pm\pm,\pm} &= \frac{1}{r^2} (V \pm iW) \gamma_l \Omega_0^l \Omega_2^l \Omega_3^l Y^{\pm 3,m} \\ \epsilon^{\pm\pm,\mp} &= \frac{1}{r} \left[\frac{\Omega_2^l \Omega_2^l}{r} (V \pm iW) - (X \pm iZ) \right] \gamma_l \Omega_0^l Y^{\pm 1,m} \\ \epsilon^{\pm 0,0} &= \frac{1}{2} (X' \pm iZ') \gamma_l \Omega_0^l Y^{\pm 1,m} \\ \epsilon^{\pm 0,\pm} &= \frac{1}{r} \left[\frac{1}{2} (X \pm iZ) - \frac{1}{r} (V \pm iW) \right] \gamma_l \Omega_0^l \Omega_2^l Y^{\pm 2,m} \\ \epsilon^{\pm 0,\mp} &= \frac{1}{r} \left[\frac{\Omega_0^l \Omega_0^l}{2} (X \pm iZ) - U' + \frac{1}{2} F \right] \gamma_l Y^{0,m} \\ \epsilon^{+-,0} &= -\frac{1}{2} F' \gamma_l Y^{0,m} \\ \epsilon^{+-,\pm} &= -\frac{1}{2r} [F + X \pm iZ] \gamma_l \Omega_0^l Y^{\pm 1,m} \end{aligned} \right\} \quad (\text{A.27})$$

Equation A6 can now be used to retrieve the r, θ, ϕ components giving

$$\begin{aligned}
\epsilon_{rr,r} &= U'' K_0 \\
\epsilon_{rr,\theta} &= \frac{1}{r}(U' - X)K_1^- + \frac{iZ}{r}K_1^+ \\
\epsilon_{rr,\phi} &= -\frac{i}{r}(U' - X)K_1^+ + \frac{Z}{r}K_1^- \\
\epsilon_{\theta\theta,r} &= \frac{1}{r}(V' - \frac{V}{r})K_2^+ - \frac{iZ}{r}K_2^- + \frac{1}{2}F'K_0 \\
\epsilon_{\theta\theta,\theta} &= \frac{V}{r^2}K_3^- - \frac{iW}{r^2}K_3^+ - \frac{1}{2r}\left(\frac{(\Omega_2)^2}{r}V - 2X - F\right)K_1^- + \frac{i}{2r}\left(\frac{(\Omega_2)^2}{r}W - 2Z\right)K_1^+ \\
\epsilon_{\theta\theta,\phi} &= -\frac{iV}{r^2}K_3^+ - \frac{W}{r^2}K_3^- - \frac{i}{2r}\left(\frac{(\Omega_2)^2}{r}V + F\right)K_1^+ - \frac{1}{2r}\left(\frac{(\Omega_2)^2}{r}W\right)K_1^- \\
\epsilon_{\phi\phi,r} &= -\frac{1}{r}(V' - \frac{V}{r})K_2^+ + \frac{iZ}{r}K_2^- + \frac{1}{2}F'K_0 \\
\epsilon_{\phi\phi,\theta} &= -\frac{V}{r^2}K_3^- + \frac{iW}{r^2}K_3^+ + \frac{1}{2r}\left(\frac{(\Omega_2)^2}{r}V + F\right)K_1^- - \frac{i}{2r}\left(\frac{(\Omega_2)^2}{r}W\right)K_1^+ \\
\epsilon_{\phi\phi,\phi} &= \frac{iV}{r^2}K_3^+ + \frac{W}{r^2}K_3^- + \frac{i}{2r}\left(\frac{(\Omega_2)^2}{r}V - 2X - F\right)K_1^+ + \frac{1}{2r}\left(\frac{(\Omega_2)^2}{r}W - 2Z\right)K_1^- \\
2\epsilon_{r\theta,r} &= X'K_1^- - iZ'K_1^+ \\
2\epsilon_{r\theta,\theta} &= \frac{1}{r}(X - \frac{2V}{r})K_2^+ - \frac{i}{r}(Z - \frac{2W}{r})K_2^- - \frac{1}{r}((\Omega_0)^2X - 2U' + F)K_0 \\
2\epsilon_{r\theta,\phi} &= -\frac{i}{r}(X - \frac{2V}{r})K_2^- - \frac{1}{r}(Z - \frac{2W}{r})K_2^+ - \frac{(\Omega_0)^2}{r}ZK_0 \\
2\epsilon_{r\phi,r} &= -iX'K_1^+ - Z'K_1^- \\
2\epsilon_{r\phi,\theta} &= -\frac{i}{r}(X - \frac{2V}{r})K_2^- - \frac{1}{r}(Z - \frac{2W}{r})K_2^+ + \frac{(\Omega_0)^2}{r}ZK_0 \\
2\epsilon_{r\phi,\phi} &= -\frac{1}{r}(X - \frac{2V}{r})K_2^+ + \frac{i}{r}(Z - \frac{2W}{r})K_2^- - \frac{1}{r}((\Omega_0)^2X - 2U' + F)K_0 \\
2\epsilon_{\theta\phi,r} &= -\frac{2i}{r}(V' - \frac{V}{r})K_2^- - \frac{2Z}{r}K_2^+ \\
2\epsilon_{\theta\phi,\theta} &= -\frac{2iV}{r^2}K_3^+ - \frac{2W}{r^2}K_3^- + \frac{i}{r}\left(\frac{(\Omega_2)^2}{r}V - X\right)K_1^+ + \frac{1}{r}\left(\frac{(\Omega_2)^2}{r}W - Z\right)K_1^- \\
2\epsilon_{\theta\phi,\phi} &= -\frac{2V}{r^2}K_3^- + \frac{2iW}{r^2}K_3^+ - \frac{1}{r}\left(\frac{(\Omega_2)^2}{r}V - X\right)K_1^- + \frac{i}{r}\left(\frac{(\Omega_2)^2}{r}W - Z\right)K_1^+
\end{aligned} \tag{A.28}$$

Finally, we can evaluate this at the pole using equation A24. The non-zero values are given in Table A1.

The Divergence of the Elastic Stress Tensor

In the equations of motion we require $\nabla \cdot \tau$. As pointed out before, in canonical components this can be written

$$\nabla \cdot \tau \equiv \Delta_{\beta\gamma} \tau^{\alpha\beta,\gamma} = \tau^{\alpha 0,0} - \tau^{\alpha+,-} - \tau^{\alpha-,+} \equiv P^\alpha \quad \text{say} \tag{A.29}$$

Now suppose that the stress tensor has been expanded in generalised spherical harmonics as in A7 with a single l, m component being

TABLE A1. The strain tensor and its gradient at the origin

	$m = 0$	$m = \pm 1$	$m = \pm 2$	$m = \pm 3$
ϵ_{rr}	U'	-	-	-
$\epsilon_{\theta\theta}$	$\frac{1}{2}F$	-	$\frac{V \pm iW}{r}$	-
$\epsilon_{\phi\phi}$	$\frac{1}{2}F$	-	$-\frac{V \mp iW}{r}$	-
$2\epsilon_{r\theta}$	-	$\mp X - iZ$	-	-
$2\epsilon_{r\phi}$	-	$-iX \pm Z$	-	-
$2\epsilon_{\theta\phi}$	-	-	$\pm \frac{2iV}{r} - \frac{2iW}{r}$	-
$\epsilon_{rr,r}$	U''	-	-	-
$\epsilon_{rr,\theta}$	-	$\mp \frac{1}{r}(U' - X) + \frac{iZ}{r}$	-	-
$\epsilon_{rr,\phi}$	-	$-\frac{i}{r}(U' - X) \mp \frac{iZ}{r}$	-	-
$\epsilon_{\theta\theta,r}$	$\frac{1}{2}F'$	-	$\frac{1}{r}(V' - \frac{V}{r}) \pm \frac{iZ}{r}$	-
$\epsilon_{\theta\theta,\theta}$	-	$\pm \frac{1}{2r} \left(\frac{(\Omega_2)^2}{r} V - 2X - F \right) \parallel + \frac{i}{2r} \left(\frac{(\Omega_2)^2}{r} W - 2Z \right) \parallel$	-	$\pm \frac{V}{r^2} - \frac{iW}{r^2}$
$\epsilon_{\theta\theta,\phi}$	-	$-\frac{i}{2r} \left(\frac{(\Omega_2)^2}{r} V + F \right) \parallel \pm \frac{1}{2r} \left(\frac{(\Omega_2)^2}{r} W \right) \parallel$	-	$-\frac{iV}{r^2} \pm \frac{W}{r^2}$
$\epsilon_{\phi\phi,r}$	$\frac{1}{2}F'$	-	$-\frac{1}{r}(V' - \frac{V}{r}) \mp \frac{iZ}{r}$	-
$\epsilon_{\phi\phi,\theta}$	-	$\mp \frac{1}{2r} \left(\frac{(\Omega_2)^2}{r} V + F \right) \parallel - \frac{i}{2r} \left(\frac{(\Omega_2)^2}{r} W \right) \parallel$	-	$\pm \frac{V}{r^2} + \frac{iW}{r^2}$
$\epsilon_{\phi\phi,\phi}$	-	$\frac{i}{2r} \left(\frac{(\Omega_2)^2}{r} V - 2X - F \right) \parallel \mp \frac{1}{2r} \left(\frac{(\Omega_2)^2}{r} W - 2Z \right) \parallel$	-	$\frac{iV}{r^2} \mp \frac{W}{r^2}$
$2\epsilon_{r\theta,r}$	-	$\mp X' - iZ'$	-	-
$2\epsilon_{r\theta,\theta}$	$-\frac{1}{r}((\Omega_0)^2 X - 2U' + F) \parallel$	-	$\frac{1}{r}(X - \frac{2V}{r}) \pm \frac{i}{r}(Z - \frac{2iW}{r})$	-
$2\epsilon_{r\theta,\phi}$	$-\frac{1}{r}(\Omega_0)^2 Z$	-	$\pm \frac{i}{r}(X - \frac{2V}{r}) - \frac{1}{r}(Z - \frac{2iW}{r})$	-
$2\epsilon_{r\phi,r}$	-	$-iX' \pm Z'$	-	-
$2\epsilon_{r\phi,\theta}$	$\frac{1}{r}(\Omega_0)^2 Z$	-	$\pm \frac{i}{r}(X - \frac{2V}{r}) - \frac{1}{r}(Z - \frac{2iW}{r})$	-
$2\epsilon_{r\phi,\phi}$	$-\frac{1}{r}((\Omega_0)^2 X - 2U' + F) \parallel$	-	$-\frac{1}{r}(X - \frac{2V}{r}) \mp \frac{i}{r}(Z - \frac{2iW}{r})$	-
$2\epsilon_{\theta\phi,r}$	-	-	$\pm \frac{2i}{r}(V' - \frac{V}{r}) - \frac{2Z}{r}$	-
$2\epsilon_{\theta\phi,\theta}$	-	$\frac{i}{r} \left(\frac{(\Omega_2)^2}{r} V - X \right) \parallel \mp \frac{1}{r} \left(\frac{(\Omega_2)^2}{r} W - Z \right) \parallel$	-	$-\frac{2iV}{r^2} \pm \frac{2iW}{r^2}$
$2\epsilon_{\theta\phi,\phi}$	-	$\pm \frac{1}{r} \left(\frac{(\Omega_2)^2}{r} V - X \right) \parallel + \frac{i}{r} \left(\frac{(\Omega_2)^2}{r} W - Z \right) \parallel$	-	$\pm \frac{2V}{r^2} + \frac{2iW}{r^2}$

Note that each column should be multiplied by $d_l^{(m)}$.

$$\tau^{\alpha\beta} = T^{\alpha\beta} Y_l^{\alpha+\beta} \quad (\text{A.30})$$

The covariant derivative is therefore

$$\tau^{\alpha\beta,\gamma} = T^{\alpha\beta|\gamma} Y_l^{\alpha+\beta+\gamma}$$

where the coefficients $T^{\alpha\beta|\gamma}$ can be determined from the $T^{\alpha\beta}$ using the recipe of equation A13. Substituting this last equation into A29 gives

$$\begin{aligned} P^- &= (T^{-0|0} - T^{-+|-} - T^{-|-|+}) Y_l^{-1,m} \\ P^0 &= (T^{00|0} - T^{0+|-} - T^{0-|+}) Y_l^{0,m} \\ P^+ &= (T^{+0|0} - T^{++|-} - T^{+-|+}) Y_l^{+1,m} \end{aligned}$$

and using equation A13 gives

$$\begin{aligned} T^{-+|-} &= \frac{1}{r} [\Omega_0^l T^{-+} - T^{-0}] & T^{-0|0} &= \frac{d}{dr} T^{-0} & T^{-|-|+} &= \frac{1}{r} [\Omega_{-1}^l T^{-} - T^{-0} - T^{0-}] \\ T^{0+|-} &= \frac{1}{r} [\Omega_1^l T^{0+} - T^{00} - T^{-+}] & T^{00|0} &= \frac{d}{dr} T^{00} & T^{0-|+} &= \frac{1}{r} [\Omega_0^l T^{0-} - T^{00} - T^{+-}] \\ T^{++|-} &= \frac{1}{r} [\Omega_2^l T^{++} - T^{+0} - T^{+0}] & T^{+0|0} &= \frac{d}{dr} T^{+0} & T^{+-|+} &= \frac{1}{r} [\Omega_1^l T^{+-} - T^{+0}] \end{aligned}$$

Combining these and remembering that the stress tensor is symmetric (so $T^{\alpha\beta} = T^{\beta\alpha}$) gives

$$\left. \begin{aligned} P^- &= \left[\frac{d}{dr} T^{-0} - \frac{1}{r} (\Omega_0^l T^{-+} - 3T^{-0} + \Omega_2^l T^{--}) \right] Y_l^{-1,m} \\ P^0 &= \left[\frac{d}{dr} T^{00} - \frac{1}{r} (\Omega_0^l T^{0+} - 2T^{00} - 2T^{-+} + \Omega_0^l T^{0-}) \right] Y_l^{0,m} \\ P^+ &= \left[\frac{d}{dr} T^{+0} - \frac{1}{r} (\Omega_0^l T^{-+} - 3T^{+0} + \Omega_2^l T^{++}) \right] Y_l^{+1,m} \end{aligned} \right\} \quad (\text{A.31})$$

Finally, the r, θ, ϕ components can be recovered from the $-, 0, +$ components by applying equation A6. Thus

$$\begin{aligned} P_i &= C_{i\alpha} P^\alpha = C_{i-} P^- + C_{i0} P^0 + C_{i+} P^+ \\ \text{so } P_\theta &= P_1 = \frac{1}{\sqrt{2}} (P^- - P^+) \\ P_\phi &= P_2 = -\frac{i}{\sqrt{2}} (P^- + P^+) \\ P_r &= P_3 = P^0 \end{aligned}$$

To proceed further we must specify the form of the stress-strain relation. The case of an isotropic elastic solid is particularly simple as the canonical components of stress are simply proportional to the canonical components of strain except for the terms involving the dilatation which require a little care. We have

$$\tau = 2\mu\epsilon + \lambda(\nabla \cdot \mathbf{u}) \mathbf{I}$$

where \mathbf{I} is the identity tensor. In canonical components the identity tensor is given by

$$\Delta^{\alpha\beta} = C_{\alpha i}^\dagger C_{\beta j}^\dagger \delta_{ij} = C_{\alpha i}^\dagger C_{\beta i}^\dagger$$

so that using equation A3 gives

$$\Delta^{00} = 1, \quad \Delta^{-+} = \Delta^{+-} = -1 \quad \text{and all other elements are zero}$$

(Note that $\Delta^{\alpha\beta} = \Delta_{\alpha\beta}$.) We can now write out the canonical components of τ , *i.e.*,

$$\tau^{\alpha\beta} = 2\mu\epsilon^{\alpha\beta} + \lambda(\nabla \cdot \mathbf{u}) \Delta^{\alpha\beta}$$

Thus

$$\left. \begin{aligned} \tau^{00} &= [2\mu U' + \lambda(U' + F)] \gamma_l Y_l^{0,m} \\ \tau^{\pm\pm} &= \frac{2\mu}{r} (V \pm iW) \gamma_l \Omega_0^l \Omega_2^l Y_l^{\pm 2,m} \\ \tau^{\pm 0} &= \mu[X \pm iZ] \gamma_l \Omega_0^l Y_l^{\pm 1,m} \\ \tau^{\pm\mp} &= [-\mu F - \lambda(U' + F)] \gamma_l Y_l^{0,m} \end{aligned} \right\} \quad (\text{A.32})$$

Comparison with equation A30 gives the expansion coefficients, $T^{\alpha\beta}$ *i.e.*,

$$\left. \begin{aligned} T^{00} &= [2\mu U' + \lambda(U' + F)] \gamma_l \\ T^{\pm\pm} &= \frac{2\mu}{r} (V \pm iW) \gamma_l \Omega_0^l \Omega_2^l \\ T^{\pm 0} &= T^{0\pm} = \mu[X \pm iZ] \gamma_l \Omega_0^l \\ T^{\pm\mp} &= -[\mu F + \lambda(U' + F)] \gamma_l \end{aligned} \right\} \quad (\text{A.33})$$

Substituting these into equation A31 gives

$$\left. \begin{aligned} P^- &= \left\{ \frac{d}{dr}(\mu X) - i \frac{d}{dr}(\mu Z) \right. \\ &\quad \left. + \frac{1}{r} \left[\mu F + \lambda(U' + F) + 3\mu(X - iZ) - \frac{2\mu}{r} (V - iW) \Omega_2^l \Omega_2^l \right] \right\} \gamma_l \Omega_0^l Y_l^{-1,m} \\ P^0 &= \left\{ \frac{d}{dr} (2\mu U' + \lambda(U' + F)) \right. \\ &\quad \left. - \frac{1}{r} [2\mu X \Omega_0^l \Omega_0^l - 4\mu U' + 2\mu F] \right\} \gamma_l Y_l^{0,m} \\ P^+ &= \left\{ \frac{d}{dr}(\mu X) + i \frac{d}{dr}(\mu Z) \right. \\ &\quad \left. + \frac{1}{r} \left[\mu F + \lambda(U' + F) + 3\mu(X + iZ) - \frac{2\mu}{r} (V + iW) \Omega_2^l \Omega_2^l \right] \right\} \gamma_l \Omega_0^l Y_l^{+1,m} \end{aligned} \right\} \quad (\text{A.34})$$

so that the (r, θ, ϕ) components are

$$\left. \begin{aligned}
P_r &= \left\{ \frac{d}{dr} (2\mu U' + \lambda(U' + F)) + \frac{\mu}{r} [4U' - 2F - l(l+1)X] \right\} Y_l^m \\
P_\theta &= \left\{ \frac{d}{dr} (\mu X) + \frac{1}{r} \left[\mu F + \lambda(U' + F) + 3\mu X - \frac{\mu V}{r} (l+2)(l-1) \right] \right\} \frac{\partial Y_l^m}{\partial \theta} \\
&\quad + \left\{ \frac{d}{dr} (\mu Z) + \frac{\mu}{r} \left[3Z - \frac{W}{r} (l+2)(l-1) \right] \right\} im \operatorname{cosec} \theta Y_l^m \\
P_\phi &= \left\{ \frac{d}{dr} (\mu X) + \frac{1}{r} \left[\mu F + \lambda(U' + F) + 3\mu X - \frac{\mu V}{r} (l+2)(l-1) \right] \right\} im \operatorname{cosec} \theta Y_l^m \\
&\quad - \left\{ \frac{d}{dr} (\mu Z) + \frac{\mu}{r} \left[3Z - \frac{W}{r} (l+2)(l-1) \right] \right\} \frac{\partial Y_l^m}{\partial \theta}
\end{aligned} \right\} \quad (\text{A.35})$$

The case of a more general elastic tensor is much more complicated. The most general relationship is

$$\tau_{ij} = C_{ijkl} \epsilon_{kl}$$

where the fourth-order tensor C has the symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$

and so has only 21 independent components. Applying A5 gives the tensor in canonical components with the same symmetries *i.e.*,

$$C^{\alpha\beta\gamma\delta} = C^{\beta\alpha\gamma\delta} = C^{\alpha\beta\delta\gamma} = C^{\gamma\delta\alpha\beta}$$

The 21 independent components are

$$\left. \begin{aligned}
C^{0000} &= C_{rrrr} \\
C^{++--} &= \frac{1}{4} C_{\theta\theta\theta\theta} + \frac{1}{4} C_{\phi\phi\phi\phi} - \frac{1}{2} C_{\theta\theta\phi\phi} + C_{\theta\phi\theta\phi} \\
C^{+-+-} &= \frac{1}{4} C_{\theta\theta\theta\theta} + \frac{1}{4} C_{\phi\phi\phi\phi} + \frac{1}{2} C_{\theta\theta\phi\phi} \\
C^{+-00} &= -\frac{1}{2} C_{\theta\theta rr} - \frac{1}{2} C_{\phi\phi rr} \\
C^{+0-0} &= -\frac{1}{2} C_{\theta r\theta r} - \frac{1}{2} C_{\phi r\phi r} \\
C^{\pm 000} &= \mp \frac{1}{\sqrt{2}} C_{\theta rrr} + \frac{i}{\sqrt{2}} C_{\phi rrr} \\
C^{\pm\pm\mp 0} &= \pm \frac{1}{2\sqrt{2}} (C_{\theta\theta\theta r} + 2C_{\theta\phi\phi r} - C_{\phi\phi\theta r}) + \frac{i}{2\sqrt{2}} (C_{\theta\theta\phi r} - 2C_{\theta\phi\theta r} - C_{\phi\phi\phi r}) \\
C^{+-\pm 0} &= \pm \frac{1}{2\sqrt{2}} (C_{\theta\theta\theta r} + C_{\phi\phi\theta r}) - \frac{i}{2\sqrt{2}} (C_{\theta\theta\phi r} + C_{\phi\phi\phi r}) \\
C^{\pm\pm 00} &= \frac{1}{2} C_{\theta\theta rr} - \frac{1}{2} C_{\phi\phi rr} \mp i C_{\theta\phi rr} \\
C^{\pm 0\pm 0} &= \frac{1}{2} C_{\theta r\theta r} - \frac{1}{2} C_{\phi r\phi r} \mp i C_{\theta r\phi r} \\
C^{\pm\pm\pm-} &= -\frac{1}{4} C_{\theta\theta\theta\theta} + \frac{1}{4} C_{\phi\phi\phi\phi} \pm \frac{i}{2} (C_{\theta\theta\theta\phi} + C_{\theta\phi\phi\phi}) \\
C^{\pm\pm\pm 0} &= \mp \frac{1}{2\sqrt{2}} (C_{\theta\theta\theta r} - 2C_{\theta\phi\phi r} - C_{\phi\phi\theta r}) + \frac{i}{2\sqrt{2}} (C_{\theta\theta\phi r} + 2C_{\theta\phi\theta r} - C_{\phi\phi\phi r}) \\
C^{\pm\pm\pm\pm} &= \frac{1}{4} C_{\theta\theta\theta\theta} + \frac{1}{4} C_{\phi\phi\phi\phi} - \frac{1}{2} C_{\theta\theta\phi\phi} - C_{\theta\phi\theta\phi} \mp i (C_{\theta\theta\theta\phi} - C_{\theta\phi\phi\phi})
\end{aligned} \right\} \quad (\text{A.36})$$

We are particularly interested in the case of transverse isotropy when only the five components with $N = \alpha + \beta + \gamma + \delta = 0$ are non-zero. In the usual notation, we have

$$\left. \begin{aligned} C^{0000} &= C \\ C^{++--} &= 2N \\ C^{+-+-} &= A - N \\ C^{+-00} &= -F \\ C^{+0-0} &= -L \end{aligned} \right\} \quad (\text{A.37})$$

The double contraction of the elastic tensor with the strain tensor to give the stress tensor is written in canonical components as

$$\tau^{\alpha\beta} = C^{\alpha\beta\gamma\delta} \epsilon^{\eta\xi} \Delta_{\gamma\eta} \Delta_{\delta\xi}$$

Performing this contraction gives

$$\begin{aligned} \tau^{\alpha\beta} &= C^{\alpha\beta--} \epsilon^{++} + C^{\alpha\beta 00} \epsilon^{00} + C^{\alpha\beta ++} \epsilon^{--} \\ &\quad - 2C^{\alpha\beta-0} \epsilon^{+0} - 2C^{\alpha\beta+0} \epsilon^{-0} + 2C^{\alpha\beta+-} \epsilon^{+-} \end{aligned} \quad (\text{A.38})$$

and specialising to the case of a transversely isotropic solid gives

$$\begin{aligned} \tau^{00} &= C^{0000} \epsilon^{00} + 2C^{00+-} \epsilon^{+-} \\ \tau^{\pm\pm} &= C^{++--} \epsilon^{\pm\pm} \\ \tau^{\pm 0} &= \tau^{0\pm} = -2C^{+0-0} \epsilon^{\pm 0} \\ \tau^{\pm\mp} &= C^{+-00} \epsilon^{00} + 2C^{+-+-} \epsilon^{+-} \end{aligned} \quad (\text{A.39})$$

It is interesting to compare this with the isotropic case and, using the notation of equation A37, we get

$$\begin{aligned} \tau^{00} &= C \epsilon^{00} - 2F \epsilon^{+-} & \tau^{00} &= (\lambda + 2\mu) \epsilon^{00} - 2\lambda \epsilon^{+-} \\ \tau^{\pm\pm} &= 2N \epsilon^{\pm\pm} & \tau^{\pm\pm} &= 2\mu \epsilon^{\pm\pm} \\ \tau^{\pm 0} &= \tau^{0\pm} = 2L \epsilon^{\pm 0} & \tau^{\pm 0} &= \tau^{0\pm} = 2\mu \epsilon^{\pm 0} \\ \tau^{\pm\mp} &= -F \epsilon^{00} + 2(A - N) \epsilon^{+-} & \tau^{\pm\mp} &= -\lambda \epsilon^{00} + 2(\lambda + \mu) \epsilon^{+-} \end{aligned} \quad \text{versus} \quad (\text{A.40})$$

from which we see that the isotropic case can be recovered by setting $A = C = \lambda + 2\mu$, $L = N = \mu$ and $F = \lambda$. The expansion coefficients are (by comparison with A30)

$$\left. \begin{aligned} T^{00} &= [CU' + FF] \gamma_l \\ T^{\pm\pm} &= \frac{2N}{r} (V \pm iW) \gamma_l \Omega_0^l \Omega_2^l \\ T^{\pm 0} &= T^{0\pm} = L[X \pm iZ] \gamma_l \Omega_0^l \\ T^{\pm\mp} &= -[(A - N)F + FU'] \gamma_l \end{aligned} \right\} \quad (\text{A.41})$$

Substituting these into equation A31 gives

$$\left. \begin{aligned}
P^- &= \left\{ \frac{d}{dr}(\mathbf{L}X) - i \frac{d}{dr}(\mathbf{L}Z) \right. \\
&\quad \left. + \frac{1}{r} \left[(\mathbf{A} - \mathbf{N})F + \mathbf{F}U' + 3\mathbf{L}(X - iZ) - \frac{2\mathbf{N}}{r}(V - iW)\Omega_2^l \Omega_2^l \right] \right\} \gamma_l \Omega_0^l Y_l^{-1,m} \\
P^0 &= \left\{ \frac{d}{dr}(\mathbf{C}U' + \mathbf{F}F) \right. \\
&\quad \left. - \frac{1}{r} [2\mathbf{L}X\Omega_0^l \Omega_0^l + 2(\mathbf{F} - \mathbf{C})U' + 2(\mathbf{A} - \mathbf{N} - \mathbf{F})F] \right\} \gamma_l Y_l^{0,m} \\
P^+ &= \left\{ \frac{d}{dr}(\mathbf{L}X) + i \frac{d}{dr}(\mathbf{L}Z) \right. \\
&\quad \left. + \frac{1}{r} \left[(\mathbf{A} - \mathbf{N})F + \mathbf{F}U' + 3\mathbf{L}(X + iZ) - \frac{2\mathbf{N}}{r}(V + iW)\Omega_2^l \Omega_2^l \right] \right\} \gamma_l \Omega_0^l Y_l^{+1,m}
\end{aligned} \right\} \quad (\text{A.42})$$

so that the (r, θ, ϕ) components are

$$\left. \begin{aligned}
P_r &= \left\{ \frac{d}{dr}(\mathbf{C}U' + \mathbf{F}F) - \frac{1}{r} [2(\mathbf{F} - \mathbf{C})U' + 2(\mathbf{A} - \mathbf{N} - \mathbf{F})F + l(l+1)\mathbf{L}X] \right\} Y_l^m \\
P_\theta &= \left\{ \frac{d}{dr}(\mathbf{L}X) + \frac{1}{r} \left[(\mathbf{A} - \mathbf{N})F + \mathbf{F}U' + 3\mathbf{L}X - \frac{\mathbf{N}V}{r}(l+2)(l-1) \right] \right\} \frac{\partial Y_l^m}{\partial \theta} \\
&\quad + \left\{ \frac{d}{dr}(\mathbf{L}Z) + \frac{1}{r} \left[3\mathbf{L}Z - \frac{\mathbf{N}W}{r}(l+2)(l-1) \right] \right\} im \operatorname{cosec} \theta Y_l^m \\
P_\phi &= \left\{ \frac{d}{dr}(\mathbf{L}X) + \frac{1}{r} \left[(\mathbf{A} - \mathbf{N})F + \mathbf{F}U' + 3\mathbf{L}X - \frac{\mathbf{N}V}{r}(l+2)(l-1) \right] \right\} im \operatorname{cosec} \theta Y_l^m \\
&\quad - \left\{ \frac{d}{dr}(\mathbf{L}Z) + \frac{1}{r} \left[3\mathbf{L}Z - \frac{\mathbf{N}W}{r}(l+2)(l-1) \right] \right\} \frac{\partial Y_l^m}{\partial \theta}
\end{aligned} \right\} \quad (\text{A.43})$$

The Traction Vector

The traction vector $\mathbf{t} = \hat{\mathbf{r}}\tau_{rr} + \hat{\boldsymbol{\theta}}\tau_{r\theta} + \hat{\boldsymbol{\phi}}\tau_{r\phi}$ can also be expanded in vector spherical harmonics (as we have implicitly showed in the last section). We can recover the usual expression for a single l, m component of the elements of \mathbf{t} from equation A30 along with equations A3 and A5, *i.e.*,

$$\tau_{ij} = C_{i\alpha} C_{j\beta} \tau^{\alpha\beta}$$

Thus

$$\begin{aligned}
\tau_{rr} &= \tau_{33} = C_{3\alpha} C_{3\beta} \tau^{\alpha\beta} = \tau^{00} \\
\tau_{r\theta} &= \tau_{31} = C_{3\alpha} C_{1\beta} \tau^{\alpha\beta} = \frac{1}{\sqrt{2}}(\tau^{0-} - \tau^{0+}) \\
\tau_{r\phi} &= \tau_{32} = C_{3\alpha} C_{2\beta} \tau^{\alpha\beta} = -\frac{i}{\sqrt{2}}(\tau^{0-} + \tau^{0+})
\end{aligned}$$

For an isotropic solid we get

$$\begin{aligned}
\tau_{rr} &= [(\lambda + 2\mu)U' + \lambda F] K_0 \\
\tau_{r\theta} &= \mu X K_1^- - i\mu Z K_1^+ \\
\tau_{r\phi} &= -i\mu X K_1^+ - \mu Z K_1^-
\end{aligned}$$

so

$$\left. \begin{aligned}
\tau_{rr} &= [(\lambda + 2\mu)U' + \lambda F] Y_l^m \\
\tau_{r\theta} &= \mu X \frac{\partial Y_l^m}{\partial \theta} + im \operatorname{cosec} \theta \mu Z Y_l^m \\
\tau_{r\phi} &= im \operatorname{cosec} \theta \mu X Y_l^m - \mu Z \frac{\partial Y_l^m}{\partial \theta}
\end{aligned} \right\} \quad (\text{A.44})$$

and for a transversely isotropic solid we have

$$\left. \begin{aligned}
\tau_{rr} &= \left[\mathbf{C} \frac{dU}{dr} + \mathbf{F}F \right] Y_l^m \\
\tau_{r\theta} &= \mathbf{L}X \frac{\partial Y_l^m}{\partial \theta} + im \operatorname{cosec} \theta \mathbf{L}Z Y_l^m \\
\tau_{r\phi} &= im \operatorname{cosec} \theta \mathbf{L}X Y_l^m - \mathbf{L}Z \frac{\partial Y_l^m}{\partial \theta}
\end{aligned} \right\} \quad (\text{A.45})$$

Inspection of A.44 and A.45 shows that defining

$$R = \mathbf{C}U' + \mathbf{F}F \quad \text{and} \quad S = \mathbf{L}X \quad \text{and} \quad T = \mathbf{L}Z \quad (\text{A.46})$$

gives the usual vector spherical harmonic expansion of \mathbf{t} :

$$\mathbf{t} = \hat{\mathbf{r}}R(r)Y_l^m + S(r)\nabla_1 Y_l^m - T(r)\hat{\mathbf{r}} \times \nabla_1 Y_l^m \quad (\text{A.47})$$

Dot products and contraction

We often have to evaluate integrals of functions which involve dot products and contractions, e.g.

$$\int \rho_0 \mathbf{u}^* \cdot \mathbf{u} dV$$

Transformation to the canonical basis can make such integrals easy but we have to be careful about the complex conjugation since the new basis is complex and not real. For a real basis $(u_i)^* = (u^*)_i$ and since

$$u_i = C_{i\alpha} u^\alpha$$

we have

$$(u_i)^* = C_{i\alpha}^* (u^\alpha)^* \quad \text{and} \quad (u^*)_i = C_{i\alpha} (u^*)^\alpha$$

Thus

$$C_{\beta i}^\dagger C_{i\alpha} (u^*)^\alpha = C_{\beta i}^\dagger C_{i\alpha}^* (u^\alpha)^*$$

which evaluates to

$$\delta_{\alpha\beta}(u^*)^\alpha = \Delta_{\alpha\beta}(u^\alpha)^*$$

or

$$(u^*)^\beta = \Delta_{\alpha\beta}(u^\alpha)^*$$

Now consider the dot product $(\mathbf{u} \cdot \mathbf{v}^*)$ which in canonical components is

$$(\mathbf{u} \cdot \mathbf{v}^*) = u^\alpha (v^*)^\beta \Delta_{\alpha\beta} = u^\alpha (v^\gamma)^* \Delta_{\gamma\beta} \Delta_{\alpha\beta} = u^\alpha (v^\gamma)^* \delta_{\alpha\gamma} \quad (\text{A.48})$$

which is the conventional form for the inner product. We shall also need the contraction of $\boldsymbol{\tau}$ with the complex conjugate of the strain tensor which can be written:

$$\boldsymbol{\tau} : \boldsymbol{\epsilon}^* = \tau^{\alpha\beta} (\epsilon^{\gamma\delta})^* \delta_{\alpha\gamma} \delta_{\beta\delta} \quad (\text{A.49})$$

Integrals of the products of spherical harmonics

The major result for the integral of the products of two generalized spherical harmonics is

$$\gamma_l^2 \int_{\Omega} Y_l^{Nm} Y_l^{Nm'}^* d\Omega = \delta_{ll'} \delta_{mm'} \quad (\text{A.50})$$

where $\gamma_l = \sqrt{(2l+1)/4\pi}$ and $d\Omega = \sin\theta d\theta d\phi$

Now consider

$$\int_V \rho_0 \mathbf{u}^* \cdot \mathbf{u} dV$$

In canonical components, we have using A48

$$\begin{aligned} \mathbf{u}^* \cdot \mathbf{u} &= u^- u^{-*} + u^0 u^{0*} + u^+ u^{+*} \\ &= (V - iW)(V + iW) \gamma_l^2 \frac{1}{2} l(l+1) \left[Y_l^{-1,m} Y_l^{-1,m*} + Y_l^{+1,m} Y_l^{+1,m*} \right] \\ &\quad + U^2 \gamma_l^2 Y_l^{0,m} Y_l^{0,m*} \end{aligned} \quad (\text{A.51})$$

Substituting A51 into the integral and using A50 gives

$$\begin{aligned} \int_V \rho_0 \mathbf{u}^* \cdot \mathbf{u} dV &= \int \rho_0 (U^2 + l(l+1)V^2) r^2 dr \quad (\text{spheroidals}) \\ &= \int \rho_0 l(l+1)W^2 r^2 dr \quad (\text{toroidals}) \end{aligned} \quad (\text{A.52})$$

As a second example, consider the elastic energy density integral:

$$\int_V \boldsymbol{\epsilon}^* \cdot \cdot \mathbf{C} : \boldsymbol{\epsilon} dV \quad (\text{A.53})$$

Now

$$\begin{aligned}\boldsymbol{\epsilon}^* \cdot \cdot \mathbf{C} : \boldsymbol{\epsilon} &= \boldsymbol{\epsilon}^* \cdot \cdot \boldsymbol{\tau} = \boldsymbol{\epsilon}^* : \boldsymbol{\tau} = \tau^{\alpha\beta} (\epsilon^{\gamma\delta})^* \delta_{\alpha\gamma} \delta_{\beta\delta} \\ &= \tau^{--} (\epsilon^{--})^* + 2\tau^{-0} (\epsilon^{-0})^* + 2\tau^{-+} (\epsilon^{-+})^* \\ &\quad + \tau^{00} (\epsilon^{00})^* + 2\tau^{0+} (\epsilon^{0+})^* + \tau^{++} (\epsilon^{++})^*\end{aligned}$$

Substituting the form for $\tau^{\alpha\beta}$ for an isotropic solid gives

$$\begin{aligned}\boldsymbol{\epsilon}^* : \boldsymbol{\tau} &= 2\mu\epsilon^{--} (\epsilon^{--})^* + 4\mu\epsilon^{-0} (\epsilon^{-0})^* + 2 \left(-\lambda\epsilon^{00} (\epsilon^{-+})^* + 2(\lambda + \mu)\epsilon^{+-} (\epsilon^{+-})^* \right) \\ &\quad + (\lambda + 2\mu)\epsilon^{00} (\epsilon^{00})^* - 2\lambda\epsilon^{+-} (\epsilon^{00})^* + 4\mu\epsilon^{0+} (\epsilon^{0+})^* + 2\mu\epsilon^{++} (\epsilon^{++})^* \\ &= \frac{2\mu}{r^2} (V - iW)(V - iW)^* \gamma_l^2 \Omega_1^{l^2} \Omega_2^{l^2} Y_l^{-2,m} Y_l^{-2,m*} \\ &\quad + \frac{2\mu}{r^2} (V + iW)(V + iW)^* \gamma_l^2 \Omega_1^{l^2} \Omega_2^{l^2} Y_l^{+2,m} Y_l^{+2,m*} \\ &\quad + \mu(X - iZ)(X - iZ)^* \gamma_l^2 \Omega_1^{l^2} Y_l^{-1,m} Y_l^{-1,m*} \\ &\quad + \mu(X + iZ)(X + iZ)^* \gamma_l^2 \Omega_1^{l^2} Y_l^{+1,m} Y_l^{+1,m*} \\ &\quad + [2\lambda U'F + (\lambda + \mu)F^2 + (\lambda + 2\mu)U'^2] \gamma_l^2 Y_l^{0,m} Y_l^{0,m*}\end{aligned}$$

Substituting into A53 and using A50 gives

$$\int \left[\frac{\mu}{r^2} l(l+1)(l-1)(l+2)(V^2 + W^2) + \mu l(l+1)(X^2 + Z^2) + 2\lambda U'F + (\lambda + \mu)F^2 + (\lambda + 2\mu)U'^2 \right] r^2 dr$$

This can be rearranged in terms of the compressional and shear energy densities by substituting the bulk modulus K instead of λ using the relationship $K = \lambda + 2/3\mu$ yielding

$$\begin{aligned}\int_V \boldsymbol{\epsilon}^* \cdot \cdot \mathbf{C} : \boldsymbol{\epsilon} dV &= \int \left[\frac{\mu}{r^2} l(l+1)(l-1)(l+2)W^2 + \mu l(l+1)Z^2 \right] r^2 dr \quad (\text{toroidals}) \\ &= \int \left[\frac{\mu}{r^2} l(l+1)(l-1)(l+2)V^2 + \mu l(l+1)X^2 + \frac{\mu}{3}(2U' - F)^2 + K(U' + F)^2 \right] r^2 dr\end{aligned} \quad (\text{A.54})$$

for spheroidals. We can also evaluate the integral A53 for a transversely isotropic solid by using A40. The result is

$$\begin{aligned}\int_V \boldsymbol{\epsilon}^* \cdot \cdot \mathbf{C} : \boldsymbol{\epsilon} dV &= \int \left[l(l+1)(l-1)(l+2) \frac{N}{r^2} W^2 + l(l+1)LZ^2 \right] r^2 dr \quad (\text{toroidals}) \\ &= \int \left[l(l+1)(l-1)(l+2) \frac{N}{r^2} V^2 + l(l+1)LX^2 + 2FU'F + (A - N)F^2 + CU'^2 \right] r^2 dr\end{aligned} \quad (\text{A.55})$$

for spheroidals.