CHAPTER 2

Continuum mechanics and the equations of motion

2.1 Introduction . This book is not intended to give a comprehensive view of continuum mechanics, rather we hope to extract that subset which is of relevance to seismology. In particular, we shall be considering infinitesimal deformations and linear constitutive relationships. Finite deformations, *etc*. are treated in detail in any good book on continuum mechanics (*e.g.*, Malvern (1969)) and the reader is encouraged to consult such a work for a complete treatment.

2.2 Forces acting on a body . When dealing with the seismic motion of the Earth, it turns out that both body forces (gravity) and surface forces are important. We consider a body with volume V and surface S and introduce the body force density $\mathbf{b}$ and the *traction* vector $\mathbf{t}$ such that the total body force acting on the body is

$$\int_V \rho \mathbf{b}\, dV \tag{2.1}$$

and the total surface force acting on the body is

$$\int_S \mathbf{t}\, dS \tag{2.2}$$

Note that $\mathbf{b}$ is reckoned per unit mass and $\mathbf{t}$ is reckoned per unit area. $\mathbf{t}$ is most conveniently specified by introducing the stress tensor. If $\hat{\mathbf{n}}$ is the normal to a surface then the traction acting on the plane with that normal is defined by

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \mathbf{T} \tag{2.3}$$

This equation defines the *Cauchy stress tensor*, $\mathbf{T}$, which is the linear vector function which associates with each unit normal $\hat{\mathbf{n}}$ the traction vector $\mathbf{t}$ acting at the point across the surface whose normal is $\hat{\mathbf{n}}$. $\mathbf{T}$ is a tensor and so transforms under a rotation of the coordinate system as

$$\mathbf{T}' = \mathbf{A}^{-1} \mathbf{T}\, \mathbf{A} \tag{2.4}$$

where $\mathbf{T}'$ is the stress tensor in the rotated system and $\mathbf{A}$ is the orthogonal rotation matrix *i.e.*, $\mathbf{A}^{-1} = \mathbf{A}^T$. We shall find out later that $\mathbf{T}$ is symmetric.

We can always find a coordinate system in which $\mathbf{T}$ is diagonal. Such a coordinate system is called the *principal axes* system and the diagonal elements of $\mathbf{T}$ are then called *principal stresses*. One finds the principal stresses by performing an eigenvector-eigenvalue decomposition of $\mathbf{T}$. The eigenvalues of $\mathbf{T}$ do not change under a change of coordinates so the coefficients of the cubic polynomial defining the characteristic equation for the eigenvalues are invariants. In particular, $T_{kk} = \text{Trace}(\mathbf{T})$ is invariant.

The mean normal pressure is defined as

$$p = -\tfrac{1}{3}\,(T_{kk}) \tag{2.5}$$

The minus sign arises because of our sign convention. T_{ij} describes the surface force acting in the j'th direction on the surface with normal in the i'th direction. Thus T_{11} acts in the 1 direction on a plane with normal in the 1 direction and is a tensile stress (fig 2.1). A positive pressure is usually taken to be a compressive stress and so we have the minus sign in equation 2.5.

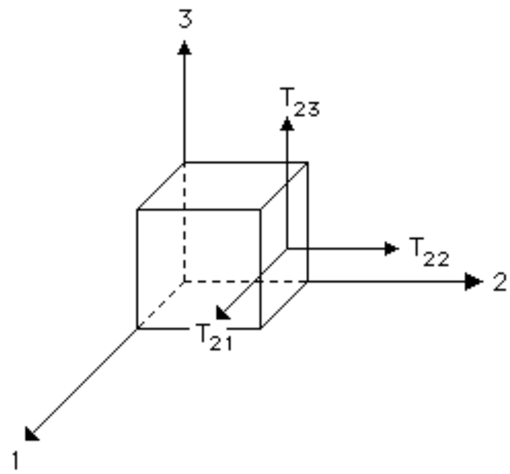


Fig 2.1 Stresses on a face of a cube of material

Sometimes it is convenient to use the *stress deviator* which is the deviatoric part of $\mathbf{T}$, *i.e.*,

$$\mathbf{T^D} = \mathbf{T} + p\mathbf{I} \tag{2.6}$$

where $\mathbf{I}$ is the unit tensor.

2.3 Strain and deformation . Consider an infinitesimal line element $d\mathbf{X}$ joining points P and Q in a material (fig 2.2). After deformation, the particle at point P has moved to p and the particle at point Q has moved to q.

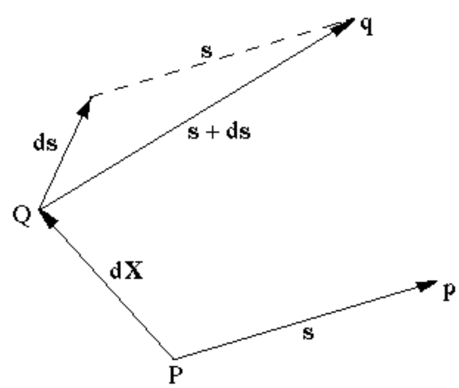


Fig 2.2 Relative displacement of two points in a continuum

The relative displacement is given by

$$d\mathbf{s} = d\mathbf{X} \cdot \nabla\mathbf{s}$$

where, in a Cartesian coordinate system, the elements of ∇s are given by $\partial s_i/\partial X_k$. To make clear what this notation means, think of a line in the 1 direction of initial length ΔX_1 being extended by an amount Δs_1 also in the 1 direction. The ratio $\Delta s_1/\Delta X_1$ is a measure of the deformation which the body is undergoing. We can consider smaller and smaller line segments but the amount of extension will get correspondingly smaller and the ratio has a finite limit as ΔX_1 tends to zero. We write this limit as $\partial s_1/\partial X_1$. The other elements of ∇s are defined similarly.

One way of thinking about deformation of a material is to think of it as a collection of particles which have some natural reference configuration at some time (which we shall take to be $t = 0$). Referring to figure 2.2, we might say that the particle at P is at position $\mathbf{X}$ in the reference configuration and at time t moves to $\mathbf{X} + \mathbf{s}(t)$ where $\mathbf{s}(t)$ is the displacement of the particle at time t. Similarly the particle at Q is at $\mathbf{X} + d\mathbf{X}$ in the reference state and moves to $\mathbf{X} + d\mathbf{X} + \mathbf{s} + d\mathbf{s}$ at time t. Define $\mathbf{x}$ to be the position at time t of the particle originally at $\mathbf{X}$ then we simply have

$$\mathbf{x} = \mathbf{X} + \mathbf{s} \tag{2.7}$$

or, since $\mathbf{s}$ is a function of $\mathbf{X}$ and time, we have

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

(Note that we employ a common short cut in continuum mechanics and use $\mathbf{x}$ twice in this equation to mean different things. On the right, $\mathbf{x}$ is a function and on the left $\mathbf{x}$ is the value of the function.) This equation represents the material (or "Lagrangian") description of motion. It essentially labels particles and is often the most natural description in seismology because the relevant conservation laws apply to particles rather than to some fixed region of space.

An alternative way of looking at things is the spatial (or Eulerian) description, $\mathbf{x}$ and t are taken to be the independent variables and we have

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$$

This description gives the initial position of the particles now (at time t) occupying position $\mathbf{x}$. The velocity field would be written as

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$$

but the velocity is still the velocity of a particle *i.e.*,

$$\mathbf{v} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}} = \frac{D\mathbf{x}}{Dt} \tag{2.8}$$

where we have used a capital D to emphasize that this is a material derivative.
The *particle* acceleration is

$$\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{X}} = \frac{D\mathbf{v}}{Dt} = \frac{D^2\mathbf{x}}{Dt^2} \tag{2.9}$$

This is not the same as the *local* time derivative which is

$$\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}}$$

We can find a relationship connecting these two derivatives using the following formula from calculus:

$$\left(\frac{\partial a}{\partial y}\right)_z = \left(\frac{\partial a}{\partial y}\right)_p + \left(\frac{\partial a}{\partial p}\right)_y \left(\frac{\partial p}{\partial y}\right)_z \tag{2.10}$$

so

$$\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_t \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}}$$

or

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_x \mathbf{v} \tag{2.11}$$

where ∇_x here denotes the gradient with respect to spatial coordinates. This is different from the gradient operator we introduced at the beginning of this section but the difference is unimportant if we are considering small deformations. To see this, consider the gradient of displacement (at fixed time). We have two possibilities: ∇s or ∇_xs. Referring to figure 2.2, $d\mathbf{x} = d\mathbf{s} + d\mathbf{X}$ and goes from p to q. We have

$$\left(\frac{\partial \mathbf{s}}{\partial \mathbf{X}}\right)_t = \left(\frac{\partial \mathbf{s}}{\partial \mathbf{x}}\right)_t \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)_t \tag{2.12}$$

where the tensor with elements $\partial x_i/\partial X_j$ is known as the deformation gradient tensor (usually given the symbol $\mathbf{F}$) and is discussed in detail by Malvern in the context of finite deformations. Taking the gradient of 2.7 (with respect to material coordinates) gives

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{s}$$

and substitution into 2.12 gives

$$\nabla \mathbf{s} = \nabla_x \mathbf{s}(\mathbf{I} + \nabla \mathbf{s})$$

which, given that deformation is small, reduces to

$$\nabla \mathbf{s} \equiv \nabla_x \mathbf{s}$$

so we are able to ignore the distinction between the two gradient definitions. We can use a scalar field in 2.10 such as density, $\rho(\mathbf{x}, t)$ *i.e.*,

$$\left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \rho}{\partial \mathbf{x}}\right)_t \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}}$$

or

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \tag{2.13}$$

and, in general, we have the following relationship between the material and the local derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{2.14}$$

The gradient tensor, ∇s of the displacement vector is a 3×3 tensor whose transpose is sometimes written as s∇. Note that ∇s can be written as the sum of a symmetric tensor and an antisymmetric tensor:

$$\nabla \mathbf{s} = \tfrac{1}{2}\left(\nabla \mathbf{s} + \mathbf{s}\nabla\right) + \tfrac{1}{2}\left(\nabla \mathbf{s} - \mathbf{s}\nabla\right) \tag{2.15}$$

or

$$\nabla \mathbf{s} = \boldsymbol{\epsilon} + \boldsymbol{\Omega} \tag{2.16}$$

$\boldsymbol{\epsilon}$ is the symmetric strain tensor and $\boldsymbol{\Omega}$ is the tensor which describes rigid body rotations when ∇s is small. ($\boldsymbol{\Omega}$ does not describe rigid body rotation during *finite* deformation of a body but we won't have to worry about this here. For infinitesimal deformation, note that both s and ∇s must be small).

Because ∇s is a tensor, both $\boldsymbol{\epsilon}$ and $\boldsymbol{\Omega}$ are tensors and so $\boldsymbol{\epsilon}$ changes under a coordinate system rotation as

$$\boldsymbol{\epsilon}' = \mathbf{A}^T \boldsymbol{\epsilon}\, \mathbf{A} \tag{2.17}$$

We can define principal axes of strain and principal strains by analogy with the stress tensor definitions. Similarly, Trace($\boldsymbol{\epsilon}$) is an invariant and is called the *dilatation*, *i.e.*, neglecting second order terms, we have

$$\epsilon_{ii} = \text{Trace}(\boldsymbol{\epsilon}) = (V - V_0)/V_0 \tag{2.18}$$

where V_0 is the reference state volume.

The mean normal strain is defined as $\frac{1}{3}\,\epsilon_{ii}$ and we can define the *strain deviator* as the deviatoric part of $\boldsymbol{\epsilon}$, *i.e.*,

$$\boldsymbol{\epsilon}^D = \boldsymbol{\epsilon} - \tfrac{1}{3}\,\mathrm{Tr}\,(\boldsymbol{\epsilon})\,\mathbf{I} \tag{2.19}$$

2.4 Deformation of boundaries . In following sections, we shall have to consider the conditions at the boundaries of the Earth. There are internal boundaries as well as the boundary at the Earth's surface and we have to deal with the fact that the boundaries are deformed during seismic disturbances. We are dealing with small perturbations to the boundary so we shall neglect squares and higher powers of small quantities in the following. We shall consider a model with boundaries which are spherical in the undeformed state and we must allow for both the radial deformation of this boundary *and* the deflection of the normal. We begin by reviewing some basic ideas. The gradient of a scalar quantity is defined as

$$\frac{dF}{dS} = \nabla F \cdot \hat{\mathbf{u}}$$

where dF is the amount a quantity, F, changes going a distance dS in the direction of the unit vector $\hat{\mathbf{u}}$. Suppose that there is a direction, $\hat{\mathbf{t}}$, in which $dF = 0$ (*i.e.*, $\hat{\mathbf{t}}$ is the tangent direction). Then

$$\nabla F \cdot \hat{\mathbf{t}} = 0$$

Now let $\hat{\mathbf{n}}$ be the unit normal to the plane containing $\hat{\mathbf{t}}$, *i.e.*,

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0$$

and we can identify $\hat{\mathbf{n}}$ as

$$\hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|}$$

Now consider a deformed surface which was initially a sphere of radius r_s

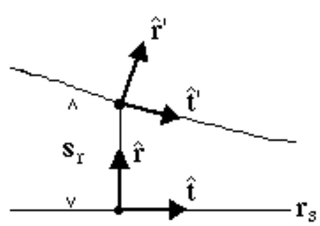


Fig 2.3

The equation of the deformed surface is

$$\psi = 0 = r - (r_s + \hat{\mathbf{r}} \cdot \mathbf{s}) = r - r_s - s_r$$

Now s is evaluated at r_s so $s_r = s_r(r_s, \theta, \phi, t)$. $\nabla\psi$ is proportional to the normal to the surface and so

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \frac{1}{r}\nabla_1\psi = \hat{\mathbf{r}} - \frac{1}{r_s}\nabla_1 s_r$$

and to first-order $|\nabla\psi| = |\hat{\mathbf{r}}| = 1$ so

$$\hat{\mathbf{r}}' = \hat{\mathbf{r}} - \frac{1}{r_s}\nabla_1(\hat{\mathbf{r}} \cdot \mathbf{s}) \tag{2.20}$$

This equation is appropriate when the undeformed surface is a sphere. In general, let ∇_Σ be the derivative in the tangent plane which has normal $\hat{\mathbf{r}}$, then

$$\nabla_\Sigma = \nabla - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \nabla)$$

$$\text{and} \quad \hat{\mathbf{n}} = \hat{\mathbf{r}} - (\nabla_\Sigma \mathbf{s}) \cdot \hat{\mathbf{r}}$$

where $\hat{\mathbf{n}}$ is the normal to the deformed surface.
We can also get an expression for the new tangent vector $\hat{\mathbf{t}}'$. Note that $\hat{\mathbf{t}}' \cdot \hat{\mathbf{r}}' = 0$ and $\hat{\mathbf{t}} \cdot \hat{\mathbf{r}} = 0$ and let

$$\hat{\mathbf{t}}' = \hat{\mathbf{t}} + \boldsymbol{\epsilon}$$

where $\boldsymbol{\epsilon}$ is a small perturbation to $\hat{\mathbf{t}}$. From (2.20), we get

$$\hat{\mathbf{t}}' \cdot \hat{\mathbf{r}}' = (\hat{\mathbf{t}} + \boldsymbol{\epsilon}) \cdot \hat{\mathbf{r}} - (\hat{\mathbf{t}} + \boldsymbol{\epsilon}) \cdot \frac{1}{r_s} \nabla_1 s_r$$

Therefore

$$0 = \boldsymbol{\epsilon} \cdot \hat{\mathbf{r}} - \hat{\mathbf{t}} \cdot \frac{1}{r_s} \nabla_1 s_r$$

to first order, and so

$$\boldsymbol{\epsilon} = \hat{\mathbf{r}} \frac{1}{r_s} (\hat{\mathbf{t}} \cdot \nabla_1 s_r)$$

Thus

$$\hat{\mathbf{t}}' = \hat{\mathbf{t}} + \hat{\mathbf{r}} \frac{1}{r_s} (\hat{\mathbf{t}} \cdot \nabla_1 s_r) \tag{2.21}$$

$|\hat{\mathbf{t}}'|$ is still 1 to first order so (2.21) gives us our perturbed tangent vector and in general we have

$$\hat{\mathbf{t}}' = \hat{\mathbf{t}} + \hat{\mathbf{r}}(\hat{\mathbf{t}} \cdot \nabla_\Sigma \mathbf{s} \cdot \hat{\mathbf{r}})$$

Now consider a patch of the surface which has area dS_0 in the undeformed state and area $dS(t)$ in the deformed state:

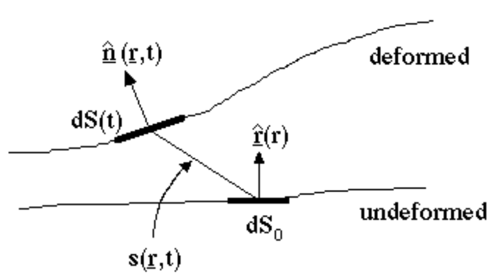


Fig 2.4

Remembering that the deflection of the normal is small, the area of the deformed patch is (to first order)

$$dS(t) = (1 + \nabla_\Sigma \cdot \mathbf{s}) dS_0 \tag{2.22}$$

and the product $\hat{\mathbf{n}} dS$ which we shall need in later sections is given by the product of (2.20) and (2.22):

$$\hat{\mathbf{n}} dS(t) = [\mathbf{I} + (\nabla_\Sigma \cdot \mathbf{s})\mathbf{I} - (\nabla_\Sigma \mathbf{s})] \cdot \hat{\mathbf{r}} dS_0 \tag{2.23}$$

Malvern (page 169) gives a more general analysis of the deformation of a surface element (including the effects of finite deformation). For small deformations, his result reduces to

$$\hat{\mathbf{n}} dS(t) = [\mathbf{I} + (\nabla \cdot \mathbf{s})\mathbf{I} - (\nabla \mathbf{s})] \cdot \hat{\mathbf{r}} dS_0 \tag{2.24}$$

(It is relatively straightforward to show that these two forms are equivalent for a Cartesian coordinate system.) We shall use these results when we consider linearization of boundary conditions.

2.5 Conservation laws . We shall be using Gauss' theorem quite a lot in the following *i.e.*,

$$\int_S \mathbf{v} \cdot \hat{\mathbf{n}} \, dS = \int_V \nabla \cdot \mathbf{v} \, dV \tag{2.25}$$

where $\mathbf{v}$ is a vector. More generally we have that

$$\int_S \hat{\mathbf{n}} * \mathcal{A} \, dS = \int_V \nabla * \mathcal{A} \, dV \tag{2.26}$$

where $\mathcal{A}$ may be a scalar, vector or tensor and $*$ can be an ordinary product, vector dot product or vector cross product depending upon the context. We also note that the *flux* of $\mathcal{A}$ through a surface S is given by

$$\int_S \rho \mathcal{A} \mathbf{v} \cdot \hat{\mathbf{n}} \, dS \tag{2.27}$$

(sometimes $\hat{\mathbf{n}} \, dS$ is denoted by $d\mathbf{A}$).

With these preliminaries out of the way, we now turn to *conservation of mass*. The mass of a volume V is given by

$$M = \int_V \rho \, dV$$

so the rate of increase of M is given by

$$\frac{\partial M}{\partial t} = \int_V \frac{\partial \rho}{\partial t} \, dV$$

provided that the surface of V is fixed in space. As we hypothesize no creation or destruction of mass, it follows that $\partial M / \partial t$ must equal the rate of inflow of mass. From 2.25 and 2.27, the rate of inflow of mass is given by

$$-\int_S \rho \mathbf{v} \cdot \hat{\mathbf{n}} \, dS = -\int_V \nabla \cdot (\rho \mathbf{v}) \, dV$$

where the minus sign arises as we are considering an *inward* flux of material. Thus

$$\int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot (\rho \mathbf{v}) \, dV$$

therefore

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$$

and, because this is true for an arbitrary volume, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.28}$$

or, equivalently

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{2.29}$$

These last two equations are both statements of the conservation of mass. Note that if $\nabla \cdot \mathbf{v} = 0$, it immediately follows that $D\rho/Dt = 0$ so that the density of a particle does not change with time. This states that the medium is *incompressible* and is a commonly used approximation in fluid mechanics. Obviously, it is not a useful approximation in seismology (P-waves would not exist!).

The most useful conservation law we shall use is the *conservation of linear momentum* but to develop it we shall need the *Reynolds mass transport theorem* which is proved in Malvern (1969, p210). This theorem states that

$$\frac{D}{Dt} \int_V \rho \mathcal{A} \, dV = \int_V \rho \frac{D\mathcal{A}}{Dt} \, dV \tag{2.30}$$

where $\mathcal{A}$ can be a vector, scalar or tensor.

The conservation of linear momentum is a basic postulate of continuum mechanics and can be stated as: the time rate of change of total momentum of a given set of particles equals the vector sum of all external forces acting on the particles. In mathematical form we have

$$\frac{D}{Dt} \int_V \rho \mathbf{v} \, dV = \int_S \mathbf{t} \, dS + \int_V \rho \mathbf{b} \, dV \tag{2.31}$$

or by 2.30

$$\int_V \rho \frac{D\mathbf{v}}{Dt} \, dV = \int_S \mathbf{t} \, dS + \int_V \rho \mathbf{b} \, dV \tag{2.32}$$

Now $\mathbf{t} = \hat{\mathbf{n}} \cdot \mathbf{T}$ so Gauss' theorem can be used to convert the surface integral into a volume integral *i.e.*,

$$\int_S \mathbf{t} \, dS = \int_S \hat{\mathbf{n}} \cdot \mathbf{T} \, dS = \int_V \nabla \cdot \mathbf{T} \, dV$$

Combining this with the previous equation gives

$$\int_V \left(\rho \frac{D\mathbf{v}}{Dt} - \nabla \cdot \mathbf{T} - \rho \mathbf{b} \right) dV = 0$$

which must hold for an arbitrary volume so

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{b} \tag{2.33}$$

These are Cauchy's equations of motion and they apply to the current deformed configuration. We have not yet made any approximation about the constitutive relationship or the size of the deformation. We shall continue to develop these equations in a seismological context but first we shall take a brief look at the *conservation of angular momentum*. This reads

$$\frac{D}{Dt} \int_V (\mathbf{r} \times \rho \mathbf{v}) \, dV = \int_S (\mathbf{r} \times \mathbf{t}) \, dS + \int_V (\mathbf{r} \times \rho \mathbf{b}) \, dV \tag{2.34}$$

Malvern (1969, p215) shows that this can only be satisfied if $\mathbf{T}$ is symmetric *i.e.*,

$$T_{ij} = T_{ji} \tag{2.35}$$

This result is generally true and does not depend upon equilibrium or other conditions.

2.6 Equilibrium conditions inside the Earth . The initial stress state inside the Earth is mainly due to the pressure of overburden. On long time scales, the Earth behaves like a fluid and we often approximate the initial stress state as one of *hydrostatic pressure*. Such a stress field is isotropic. The deviatoric initial stresses (due to lateral density variations caused dominantly by convective motions) are almost certainly very small and can (perhaps) be treated by perturbation theory. In equilibrium, $\mathbf{v} = 0$ so 2.33 becomes

$$\nabla \cdot \mathbf{T}_0 + \rho_0 \mathbf{b} = 0 \tag{2.36}$$

where the zero subscript refers to the equilibrium state. The body force here is gravity *i.e.*,

$$\mathbf{b} \equiv \mathbf{g} = -\nabla \phi \tag{2.37}$$

where ϕ is the gravitational potential and satisfies Poisson's equation *i.e.*,

$$\nabla^2 \phi = 4\pi G \rho \tag{2.38}$$

Using zero subscripts to denote the equilibrium state gives

$$\nabla^2 \phi_0 = 4\pi G \rho_0 \tag{2.39}$$

and

$$\nabla \cdot \mathbf{T}_0 - \rho_0 \nabla \phi_0 = 0 \tag{2.40}$$

Let $\mathbf{T}_0$ consist of a hydrostatic pressure and a deviatoric part, $\tau_\mathbf{0}$, *i.e.*,

$$\mathbf{T}_0 = -p_0 \mathbf{I} + \tau_\mathbf{0}$$

then

$$-\nabla p_0 + \nabla \cdot \tau_\mathbf{0} - \rho_0 \nabla \phi_0 = 0 \tag{2.41}$$

Equations 2.39 and 2.41 govern the initial stress state but they do not uniquely determine that state. If we assume that $\tau_\mathbf{0} \ll p_0$ and so can be neglected we have

$$\rho_0 \nabla \phi_0 + \nabla p_0 = 0 \tag{2.42}$$

Finally, we often assume that ρ_0 is a function of radius, r alone (in which case it follows that ϕ_0 and p_0 are functions of radius alone – show this using 2.42). If we are given $\rho_0(r)$, it is trivial to compute $\mathbf{g}(r)$ and $p_0(r)$. Note that $\mathbf{g}$ points inwards so it is convenient to write

$$\mathbf{g}(r) = -\hat{\mathbf{r}} g_0(r) \tag{2.43}$$

where

$$g_0(r) = \frac{\partial \phi_0}{\partial r} \tag{2.44}$$

From Poisson's equation with spherical symmetry

$$\nabla^2 \phi_0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi_0}{\partial r} \right) = 4\pi G \rho_0$$

or

$$\left(\frac{\partial}{\partial r} r^2 g_0 \right) = 4\pi G \rho_0 r^2$$

so

$$g_0(r) = \frac{1}{r^2} \int_0^r 4\pi G \rho_0 x^2 \, dx \tag{2.45}$$

Now from 2.42

$$\hat{\mathbf{r}} \frac{\partial p_0}{\partial r} = -\hat{\mathbf{r}} \rho_0 \frac{\partial \phi_0}{\partial r}$$

so

$$\frac{\partial p_0}{\partial r} = -\rho_0 g_0 \tag{2.46}$$

Thus, given $\rho_0(r)$, 2.45 can be evaluated to give $g_0(r)$ and 2.46 can be integrated to give $p_0(r)$ with the boundary condition that the equilibrium pressure at the surface of the Earth is zero (see figure on next page).

2.7 Response to small oscillations . We shall now assume that the Earth is disturbed by a time-dependent external body force density, $\mathbf{f}(\mathbf{r},t)$. (In later chapters, we show how realistic earthquake sources can be represented by an equivalent body force). The disturbance sets up a displacement field $\mathbf{s}(\mathbf{r}, t)$. There will also be a perturbation in the initial density field which we call $\rho_1(\mathbf{r}, t)$ and a perturbation in the gravitational potential, $\phi_1(\mathbf{r}, t)$. Finally, there will be a perturbation in the stress tensor, $\mathbf{T_1}(\mathbf{r}, t)$. Note that all the perturbations are specified at a fixed location in space. Equation 2.33 can now be written as

$$(\rho_0 + \rho_1) \frac{D^2\mathbf{s}}{Dt^2} = \nabla \cdot (\mathbf{T_0} + \mathbf{T_1}) - (\rho_0 + \rho_1) \nabla (\phi_0 + \phi_1) + \mathbf{f} \tag{2.47}$$

In this equation we have separated out the gravitational part of the body force and note that

$$\frac{D^2\mathbf{s}}{Dt^2} = \frac{D\mathbf{v}}{Dt} \text{ where } \mathbf{v} = \frac{D\mathbf{s}}{Dt}$$

Neglecting terms of second order (assuming $\mathbf{s}, \rho_1, \phi_1$ and $\mathbf{T_1}$ are all small) and subtracting 2.36 gives

$$\rho_0 \frac{\partial^2\mathbf{s}}{\partial t^2} = \nabla \cdot \mathbf{T}_1 - \rho_0 \nabla\phi_1 - \rho_1 \nabla\phi_0 + \mathbf{f} \tag{2.48}$$

The interpretation of $\mathbf{T}_1$ requires a little care. The physically relevant quantity that we need is the incremental stress $\mathbf{T}_E$ at the material particle which is at $\mathbf{r}$ at time t but which was at a different position in the reference state. For clarity, we call this initial position of the particle $\mathbf{X}$ so that the incremental stress is given by

$$\mathbf{T}_E(\mathbf{r}, t) = \mathbf{T}(\mathbf{r}, t) - \mathbf{T}_0(\mathbf{X}, 0)$$

Now generally $\mathbf{r} = \mathbf{X} + \mathbf{s}$ so we can rewrite our expression for $\mathbf{T}_E$ as

$$\begin{aligned} \mathbf{T} &= \mathbf{T}_E + \mathbf{T}_0(\mathbf{r} - \mathbf{s}) \\ &= \mathbf{T}_E + \mathbf{T}_0(\mathbf{r}) - \mathbf{s}\nabla\mathbf{T}_0(\mathbf{r}) \end{aligned}$$

correct to first order in s. Now, remembering that

$$\mathbf{T}(\mathbf{r}, t) = \mathbf{T}_0(\mathbf{r}) + \mathbf{T}_1(\mathbf{r}, t)$$

we can obtain an expression for $\mathbf{T}_1$ in terms of $\mathbf{T}_E$ and $\mathbf{T}_0$:

$$\mathbf{T}_1 = \mathbf{T}_E - \mathbf{s} \cdot \nabla\mathbf{T}_0 \tag{2.49}$$

In a perfectly elastic medium, $\mathbf{T}_E$ is related to ∇s by a fourth order tensor and consists of the usual elastic stresses but there is also a contribution caused by an infinitesimal rotation of the initial stress field $\mathbf{T}_0$. If the initial stress field is isotropic, this latter contribution vanishes and $\mathbf{T}_1$ can be regarded as a sum of elastic

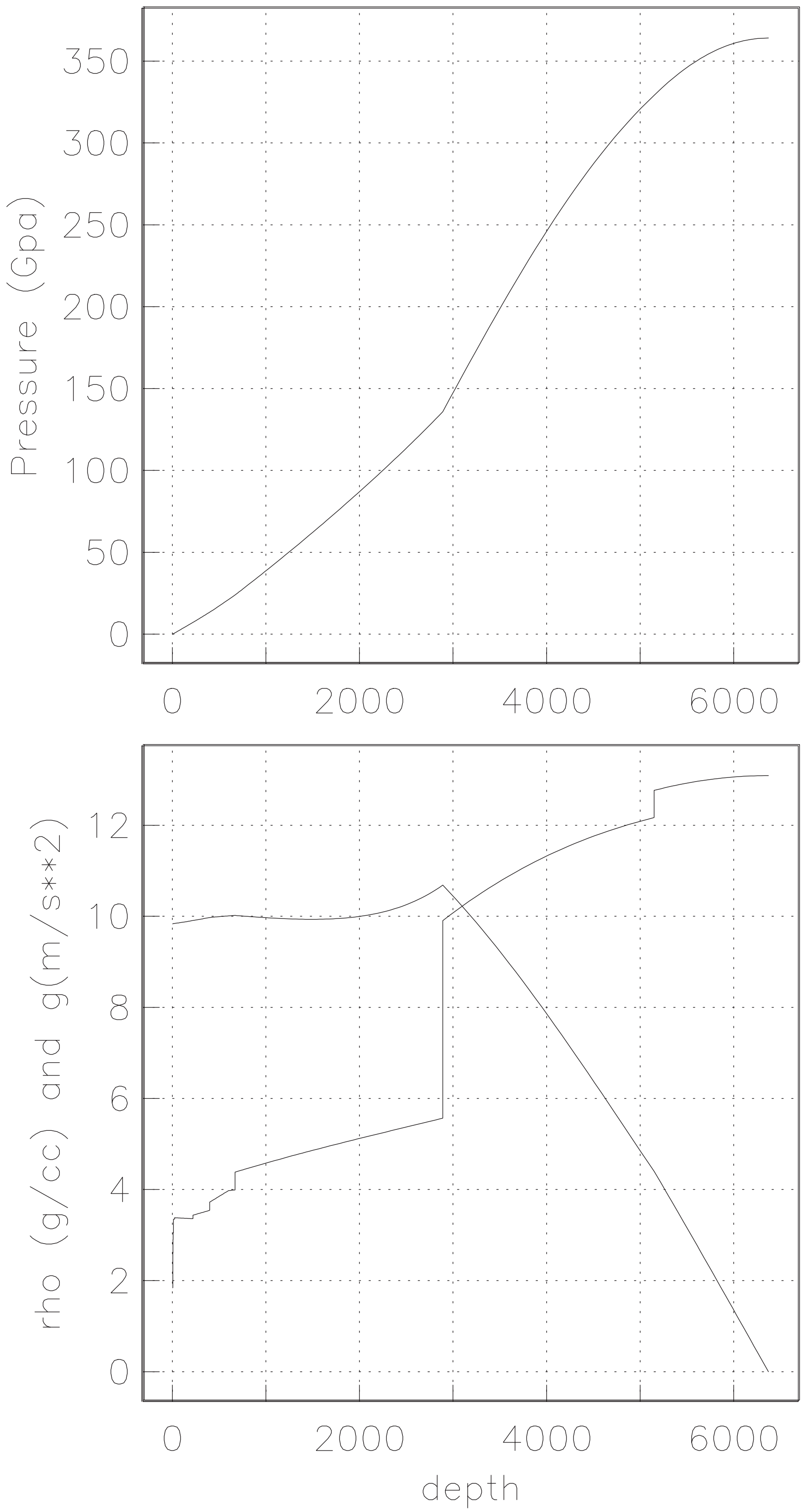


Figure 2.x. Density, acceleration due to gravity and pressure as a function of depth in the equilibrium state – here assumed t be hydrostatic.

stresses and a contribution from the fact that the particle is moving in a stress gradient. This latter term $(\mathbf{s} \cdot \nabla \mathbf{T}_0)$ is sometimes called the *advection of stress* term.
Equation 2.48 now reads

$$\rho_0 \frac{\partial^2 \mathbf{s}}{\partial t^2} = \nabla \cdot \mathbf{T}_E - \nabla \cdot (\mathbf{s} \cdot \nabla \mathbf{T}_0) - \rho_0 \nabla \phi_1 - \rho_1 \nabla \phi_0 + \mathbf{f} \tag{2.50}$$

This equation can be simplified if we approximate $\mathbf{T}_0$ by a hydrostatic pressure, *i.e.*, $\mathbf{T}_0 = -p_0 \mathbf{I}$, then

$$\nabla \mathbf{T}_0 = -\nabla p_0 \mathbf{I}$$

therefore

$$\mathbf{s} \cdot \nabla \mathbf{T}_0 = -\mathbf{s} \cdot \nabla p_0 \mathbf{I} = s_r \rho_0 g_0 \mathbf{I} \qquad (s_r = \hat{\mathbf{r}} \cdot \mathbf{s})$$

(the last step follows because $\nabla p_0 = -\hat{\mathbf{r}} \rho_0 g_0$). Therefore

$$\nabla \cdot (\mathbf{s} \cdot \nabla \mathbf{T}_0) = \nabla (s_r \rho_0 g_0) \tag{2.51}$$

Note also that the term in ρ_1 in 2.50 is also related to s. To see this we use conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

or because $\rho = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t)$ and ρ_1 is small, we get (correct to first order)

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \frac{\partial \mathbf{s}}{\partial t}) = 0$$

Because $\rho_1 = 0$ when $\mathbf{s} = 0$, we can integrate this last equation to give

$$\rho_1 = -\nabla \cdot (\rho_0 \mathbf{s}) \tag{2.52}$$

Finally, we can relate ϕ_1 to s using Poisson's equation, *i.e.*,

$$\nabla^2 \phi = 4\pi G \rho$$

so

$$\nabla^2 (\phi_0 + \phi_1) = 4\pi G (\rho_0 + \rho_1)$$

so

$$\nabla^2 \phi_1 = 4\pi G \rho_1 = -4\pi G \nabla \cdot (\rho_0 \mathbf{s})$$

Substituting 2.51 and 2.52 into 2.50 and collecting terms gives (note $\nabla \phi_0 = \hat{\mathbf{r}} g_0$)

$$\left.\begin{aligned} \rho_0 \frac{\partial^2 \mathbf{s}}{\partial t^2} &= \nabla \cdot \mathbf{T}_E - \nabla (s_r \rho_0 g_0) - \rho_0 \nabla \phi_1 + \hat{\mathbf{r}} g_0 \nabla \cdot (\rho_0 \mathbf{s}) + \mathbf{f} \\ \text{and} \quad \nabla^2 \phi_1 &= -4\pi G \nabla \cdot (\rho_0 \mathbf{s}) \end{aligned}\right\} \tag{2.53}$$

These are our basic equations (for an isotropic initial stress) but before we can solve them we require two things. We require a constitutive relationship relating the Lagrangian incremental stress, $\mathbf{T}_E$, to s and we require a set of boundary conditions. In the following, for notational simplicity, we shall write $\mathbf{T}$ for $\mathbf{T}_E$ providing there is no ambiguity. Before we turn to constitutive relations we take a look at a last conservation law: the conservation of energy.

2.8 Conservation of energy . Work, as you know, is dimensionally a force times a distance. Power is a rate of doing work so dimensionally is a force times a velocity. Seismic motion is accompanied by mechanical work so we shall first look at the input power due to mechanical work. By definition

$$P_{input} = \int_S \mathbf{t} \cdot \mathbf{v} dS + \int_V \rho \mathbf{b} \cdot \mathbf{v} dV \tag{2.54}$$

This can be separated into two contributions; mechanical work performed in deforming the body and work done in changing the kinetic energy of the body. The mathematical development is as follows

$$\begin{aligned}
P_{input} &= \int_S \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \mathbf{v}\, dS + \int_V \rho \mathbf{b} \cdot \mathbf{v}\, dV \\
&= \int_V [\nabla \cdot (\mathbf{T} \cdot \mathbf{v}) + \rho \mathbf{b} \cdot \mathbf{v}]\, dV \qquad \text{(using Gauss' theorem)} \\
&= \int_V [(\nabla \cdot \mathbf{T}) \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v} + \rho \mathbf{b} \cdot \mathbf{v}]\, dV \\
&= \int_V [(\nabla \cdot \mathbf{T} + \rho \mathbf{b}) \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v}]\, dV \\
&= \int_V [\rho \frac{D\mathbf{v}}{Dt} \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v}]\, dV \qquad \text{using 2.33} \\
&= \int_V \tfrac{1}{2}\, \rho \frac{D}{Dt} (\mathbf{v} \cdot \mathbf{v})\, dV + \int_V \mathbf{T} \cdot\cdot \nabla \mathbf{v}\, dV \\
&= \frac{D}{Dt} \int_V \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int_V \mathbf{T} \cdot\cdot \nabla \mathbf{v}\, dV \qquad \text{using 2.30} \\
&= \frac{D}{Dt} \int_V \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int_V \mathbf{T} \cdot\cdot (\dot{\boldsymbol{\epsilon}} + \dot{\Omega})\, dV \qquad \text{using 2.15} \\
&= \frac{D}{Dt} \int_V \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int_V \mathbf{T} \cdot\cdot \dot{\boldsymbol{\epsilon}}\, dV \qquad \text{because } \mathbf{T} \text{ is symmetric and } \Omega \text{ is antisymmetric}
\end{aligned}$$

therefore

$$P_{input} = \frac{D}{Dt} \int_V \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int_V \mathbf{T} : \dot{\boldsymbol{\epsilon}}\, dV \qquad \text{because } \mathbf{T} \text{ is symmetric} \tag{2.55}$$

(Note $A \cdot\cdot B = A_{ij} B_{ji}$ and $A : B = A_{ij} B_{ij}$ in a Cartesian coordinate system.) Equation 2.55 shows the separation into energy of deformation and kinetic energy.

Note that there may be other sources of energy input which we denote Q_{input}. For example, we might have conduction and radioactive heat generation:

$$Q_{input} = - \int_S \mathbf{q} \cdot \hat{\mathbf{n}} dS + \int_V \rho h dV$$

where h is the rate of heat generation per unit mass, $\mathbf{q} = -k \nabla \theta$ is the heat flux and θ is temperature. If E is the total energy of the volume then

$$\dot{E} = P_{input} + Q_{input} \tag{2.56}$$

This is a statement of the first law of thermodynamics. E consists of the change of kinetic energy plus the change of internal energy of the volume so

$$\dot{E} = \frac{D}{Dt}\int\limits_V [\tfrac{1}{2}\,\rho\mathbf{v}\cdot\mathbf{v} + \rho\mathcal{U}]\,dV$$

where $\mathcal{U}$ is the internal energy per unit mass. As we have already separated out the change of kinetic energy in P_{input} we can get an expression for the change of internal energy. Combining 2.55 and 2.56 gives

$$\frac{D}{Dt}\int\limits_V \rho\mathcal{U}dV = \int\limits_V \mathbf{T}:\dot{\boldsymbol{\epsilon}}\,dV - \int\limits_S \mathbf{q}\cdot\hat{\mathbf{n}}\,dS + \int\limits_V \rho h\,dV$$

or using 2.25 and 2.30

$$\int\limits_V \left[\rho\frac{D\mathcal{U}}{Dt} - \mathbf{T}:\dot{\boldsymbol{\epsilon}} + \nabla\cdot\mathbf{q} - \rho h\right] dV = 0$$

or

$$\rho\frac{D\mathcal{U}}{Dt} = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T}:\dot{\boldsymbol{\epsilon}} \tag{2.57}$$

If we recognize that $\rho h - \nabla\cdot\mathbf{q}$ is the rate of change of heat input, DQ/Dt say, then 2.57 takes on a more familiar form

$$\rho D\mathcal{U} = DQ + \mathbf{T}:D\boldsymbol{\epsilon}$$

Also, $\rho D\mathcal{U}/Dt$ is the rate of increase of internal energy per unit volume so let $W = \rho\mathcal{U}$ then

$$DW = DQ + \mathbf{T}:D\boldsymbol{\epsilon} \tag{2.58}$$

For a *reversible* process $DQ = \theta DS$ where S is entropy. In ideal elasticity, seismic motions are considered reversible and are also considered to be sufficiently fast that there is no time for significant heat transfer. Such a process is termed *adiabatic* and $DS = 0$ then

$$DW = \mathbf{T}:D\boldsymbol{\epsilon}$$

or

$$\mathbf{T} = \left(\frac{\partial W}{\partial\boldsymbol{\epsilon}}\right)_S \tag{2.59}$$

This equation has implications for the constitutive relationship and (sometimes) allows us to prove uniqueness of solutions to the elastodynamic equations.

2.9 Constitutive relationships . The first case we shall consider is that of perfect elasticity which has a linear relationship between stress and strain of the form

$$T_{ij} = C_{ijkl}\epsilon_{kl} \tag{2.60}$$

The fourth order tensor C_{ijkl} has several symmetries:

$$\begin{aligned} C_{jikl} &= C_{ijkl} \quad \text{because} \quad T_{ij} = T_{ji} \\ C_{ijlk} &= C_{ijkl} \quad \text{because} \quad \epsilon_{ij} = \epsilon_{ji} \\ \text{and} \quad C_{klij} &= C_{ijkl} \end{aligned} \tag{2.61}$$

This last symmetry arises if 2.59 is valid, *i.e.*,

$$\frac{\partial W}{\partial \epsilon_{ij}} = T_{ij} = C_{ijkl}\epsilon_{kl} \tag{2.62}$$

but

$$\frac{\partial^2 W}{\partial \epsilon_{ij}\partial \epsilon_{kl}} = \frac{\partial^2 W}{\partial \epsilon_{kl}\partial \epsilon_{ij}}$$

which implies $C_{ijkl} = C_{klij}$. Inspection of 2.62 shows that

$$W = \tfrac{1}{2}\,C_{ijkl}\epsilon_{ij}\epsilon_{kl} = \tfrac{1}{2}\,T_{ij}\epsilon_{ij} \tag{2.63}$$

Given the symmetries of C_{ijkl}, equation 2.63 is a positive definite form which is a prerequisite for a stable material.

The symmetry of 2.61 leads to a simplified tensor notation where C_{ijkl} is represented as a symmetrix 6×6 matrix. In this notation (called *Voigt* notation), 2.60 becomes

$$T_i = C_{ij}\epsilon_j \tag{2.64}$$

or explicitly

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Similarly

$$W = \tfrac{1}{2}\,T_k\epsilon_k = \tfrac{1}{2}\,\boldsymbol{\epsilon}^T \cdot \mathbf{C} \cdot \boldsymbol{\epsilon}$$

As an example of the equivalence of 2.60 and 2.64, note that $C_{56} \equiv C_{3112} = C_{3121} = C_{1312} = C_{1321}$. The symmetry of C_{ij} in 2.64 also makes it clear that a perfectly elastic material has only 21 independent elastic constants.

The positive definite form of W allows a uniqueness proof for solutions to elastodynamic problems with a perfectly elastic constitutive relationship. The proof is given on pg. 24 of Aki and Richards and will not be reproduced here.

The solution to equation 2.53 conventionally proceeds by separation of variables. For this to be possible, our model of the Earth must satisfy certain symmetry conditions – in particular we shall be concerned about spherical symmetry. An Earth model will only have spherical symmetry for certain special forms of C_{ijkl}. One obvious possibility is that the material is elastically isotropic. This means that wave velocities have no preferred direction so the elements of C_{ijkl} must be invariant to any rotation of the coordinate system. The most general fourth-order isotropic tensor is given by (Chapter 7 of *Cartesian Tensors* by Jeffreys)

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \eta\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} \tag{2.65}$$

Substitution into 2.60 gives

$$T_{ij} = \lambda\delta_{ij}\epsilon_{kk} + \eta\epsilon_{ij} + \nu\epsilon_{ji}$$

Therefore

$$T_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \tag{2.66}$$

where $2\mu = \eta + \nu$.

λ and μ are the well-known Lamé constants and μ is also called the rigidity or shear modulus. Other combinations of elastic moduli that are commonly referred to for isotropic materials are

$$\left.\begin{aligned} K_s &= \lambda + \frac{2}{3}\mu \\ E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \\ \sigma &= \frac{\lambda}{2(\lambda + \mu)} \end{aligned}\right\} \tag{2.67}$$

where K_s is the adiabatic bulk modulus, E is Young's modulus, and σ is Poisson's ratio. The Earth's mantle has a Poisson's ratio of close to 0.25 which is achieved when $\lambda \simeq \mu$. ($\lambda = \mu$ defines a *Poisson* solid). In a fluid $\mu = 0$ so $\sigma = 0.5$. The fluid outer core is often modeled as a perfectly elastic (inviscid) fluid which is obtained from 2.66 by setting $\mu = 0$, *i.e.*,

$$T_{ij} = \lambda \delta_{ij} \epsilon_{kk} \tag{2.68}$$

Thus there are no off-diagonal (shear) stresses.
In Voigt notation, an elastically isotropic material is given by

$$\mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & 0 & & & \mu & \\ & & & & & \mu \end{bmatrix} \tag{2.69}$$

which is invariant to any rotation of the coordinate system. Equation 2.66 is not the most general form of $\mathbf{C}$ which still allows us to construct a model with spherical symmetry. A little thought will show that elastic velocities in the radial direction can be different from elastic velocities perpendicular to the radial direction. $\mathbf{C}$ then has rotational symmetry about the $\hat{\mathbf{r}}$ direction. Such a material is called *transversely isotropic*. If the 3-direction is chosen as the $\hat{\mathbf{r}}$-direction, a transversely isotropic material in the Voigt notation looks like:

$$\mathbf{C} = \begin{bmatrix} \lambda' + 2\mu' & \lambda' & \lambda & & & \\ \lambda' & \lambda' + 2\mu' & \lambda & & 0 & \\ \lambda & \lambda & \beta & & & \\ & & & \mu & & \\ & 0 & & & \mu & \\ & & & & & \mu' \end{bmatrix} \tag{2.70}$$

Note that there are five independent elastic coefficients (as opposed to two for the isotropic case). A commonly used alternative notation in 2.70 is A $= \lambda' + 2\mu'$, C $= \beta$, F $= \lambda$, L $= \mu$, and N $= \mu'$.

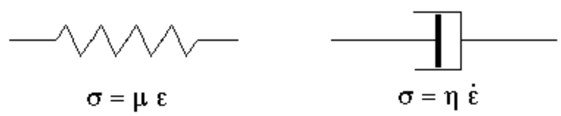
Problem 2.1 Prove that 2.70 is the most general form of $\mathbf{C}$ invariant under rotations about the 3-direction.

The perfectly elastic constitutive relationship is not a bad approximation to the real Earth – in fact the real Earth is quite close to being isotropic though it is now commonly modeled as being transversely isotropic. General anisotropy is quite weak and hopefully can be modeled by perturbation theory.

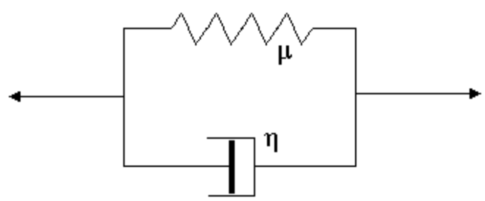
2.10 Visco-elastic constitutive relations . One obvious failure of the perfectly elastic medium is that there is no attenuation. Attenuation of seismic energy is quite weak and (perhaps) can also be modeled by perturbation theory. From a philosophical point of view, the difference between dissipative and non-dissipative materials is quite profound and so we now consider some constitutive relationships which admit attenuation. Of course, we can't get too fancy or we can't solve our equations. In particular we would like equation 2.53 to stay linear, *i.e.*, no powers of $\mathbf{s}$ should appear. Thus our constitutive relationship should remain linear. One class of materials which satisfies this is that of linear visoelastic materials but even within this class, a wide range of constitutive relationships is possible. To keep the discussion simple we consider one-dimensional constitutive relationships so that perfect elasticity takes the form

$$\sigma(t) = M\epsilon(t) \tag{2.71}$$

Note that M is independent of time. When we apply a stress we get an immediate deformation but the material returns to the original configuration when the stress is released. Real materials don't behave like this. When we stress a rock, we do get an immediate deformation but this is followed by some creep (relaxation) behavior. We can model this by adding some perfectly viscous linear fluid behavior to our perfectly linear elastic model. A convenient way of investigating possible constitutive relationships is to introduce the two basic elements: a spring for perfect elastic behavior and a dashpot for viscous fluid behavior.



One obvious thing to do is to put these two elements in parallel, *i.e.*,



This is called a Kelvin-Voigt solid and has the constitutive relationship

$$\sigma = \mu\epsilon + \eta\dot{\epsilon} \tag{2.72}$$

Suppose we apply a stress of the form

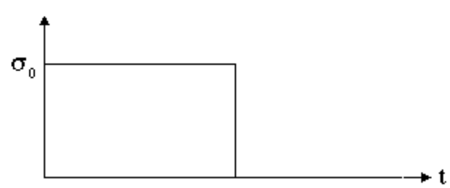


Fig 2.7

Then we get a response like

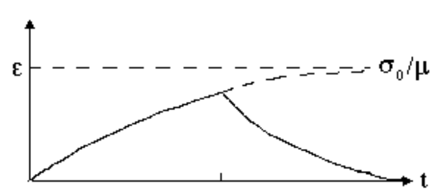


Fig 2.8

As we apply the steady stress, the dashpot controls the extension until equilibrium with the spring is achieved. When the stress is released the spring acts against the dashpot to compress the system.

Suppose we apply a stress $\sigma = \sigma_0 H(t)$ to the Kelvin-Voigt model. Then it is easy to show that

$$\epsilon(t) = \frac{\sigma_0}{\mu}(1 - e^{-t/\tau_0})H(t) \quad \text{where} \quad \tau_0 = \frac{\eta}{\mu} \tag{2.73}$$

τ_0 is called the Maxwell relaxation time. The response of the material to a stress of the form $H(t)$ is called a creep function while the response of the material to a unit step function in strain is called a relaxation function.

The response of a Kelvin-Voigt (K-V) solid is not very realistic in that there is no instantaneous deformation. We can patch this up by putting a spring in series with a K-V element, *i.e.*,

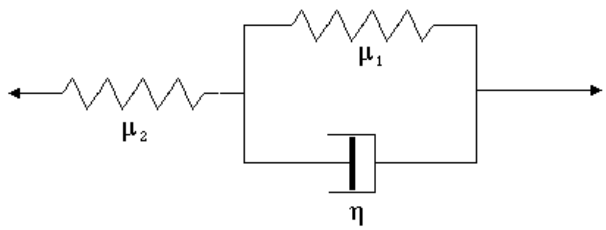


Fig 2.9

This body is called a standard linear solid. Let ϵ_2 be the deformation of the spring μ_2 and let ϵ_1 be the deformation of the K-V element. Then, since the force is the same in the K-V element and in the second spring, we can write

$$\sigma = \mu_1 \epsilon_1 + \eta \dot{\epsilon}_1 = \mu_2 \epsilon_2$$

and $\epsilon = \epsilon_1 + \epsilon_2$ is the total deformation. Rearranging gives

$$\sigma + \tau_\sigma \dot{\sigma} = M_R[\epsilon + \tau_\epsilon \dot{\epsilon}] \tag{2.74}$$

where

$$\tau_\sigma = \frac{\eta}{\mu_1 + \mu_2}, \quad \tau_\epsilon = \frac{\eta}{\mu_1} \quad \text{and} \quad M_R = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}$$

Note that 2.74 is the most general stress-strain law involving only zero'th and first derivatives.

Problem 2.2 Suppose we apply a stress $\sigma = \sigma_0 H(t)$ to the standard linear solid. Show that

$$\epsilon(t) = \frac{\sigma_0}{M_R}\left[1 - (1 - \frac{\tau_\sigma}{\tau_\epsilon})e^{-t/\tau_\epsilon}\right] H(t) \tag{2.75}$$

Note that $\epsilon(\infty) = \sigma_0 / M_R$ so M_R is often called the *relaxed* modulus. Also note that $\epsilon(0) = \sigma_0(\tau_\sigma / \tau_\epsilon M_R)$ so that $\tau_\epsilon M_R / \tau_\sigma$ is often called the *unrelaxed modulus* and is given the symbol M_U

If a material has an instantaneous elastic response, it is conventional to separate this out from the subsequent creep response. For example, 2.75 can be rewritten as

$$\epsilon(t) = \frac{\sigma_0}{M_U}\left[1 + (\frac{\tau_\epsilon}{\tau_\sigma} - 1)(1 - e^{-t/\tau_\epsilon})\right] H(t) \tag{2.76}$$

or, more generally, the response to a step load can be written as

$$\epsilon(t) = \frac{\sigma_0}{M_U}\left[1 + \phi(t)\right] H(t) \tag{2.77}$$

where $\phi(t)$ is a function characteristic of the material and is called the *creep function* for that material. A material which follows 2.77 has a memory of its stress history and the total response to an arbitrary load will be a combination of the instantaneous elastic response and a part which integrates over the loading history of the material. Consider the part of the response which depends upon the loading history. A set of discrete stress increments of the form $\Delta\sigma(t_i)$ will produce an increment in strain at time t of the form

$$M_U \Delta\epsilon_i(t) = \Delta\sigma(t_i')\phi(t - t_i')$$

We suppose that the stress increments are applied at a time interval Δt and we assume that the total strain at time, t, is just the sum of the result of all the increments in stress that have happened up to time t (this is the Boltzmann Superposition principle). Then

$$\begin{aligned} M_U \epsilon(t) &= \sum_i \Delta\sigma(t_i')\phi(t - t_i') \\ &= \sum_i \frac{\Delta\sigma}{\Delta t}(t_i')\phi(t - t_i')\Delta t \end{aligned}$$

or, as $\Delta t \to 0$

$$M_U \epsilon(t) = \int_{-\infty}^{t} \dot{\sigma}(t')\phi(t - t')dt' \qquad (2.78)$$

Adding in the instantaneous elastic response gives the total response of the material:

$$M_U \epsilon(t) = \sigma(t) + \int_{-\infty}^{t} \dot{\sigma}(t')\phi(t - t')dt' \qquad (2.79)$$

Because a material has no knowledge of its future we note that $\phi(t)$ for $t < 0$ must be zero so we can integrate the above equation by parts to give

$$M_U \epsilon(t) = \sigma(t) + \int_{-\infty}^{t} \sigma(t')\dot{\phi}(t - t')dt' \qquad (2.80)$$

Now suppose we have a sinusoidal disturbance, $\sigma(t) = \sigma e^{i\omega t}$, then

$$\begin{aligned} M_U \epsilon(t) &= \sigma(t) + \int_{-\infty}^{t} \sigma e^{i\omega t'}\dot{\phi}(t - t')dt' \\ &= \sigma(t) + \int_{0}^{\infty} \sigma e^{i\omega t} e^{-i\omega t''}\dot{\phi}(t'')dt'' \qquad (t'' = t - t') \\ &= \left[1 + \int_{0}^{\infty} e^{-i\omega t''}\dot{\phi}(t'')dt''\right]\sigma(t) \end{aligned}$$

Thus for a sinusoidal disturbance at frequency ω, a linear viscoelastic material obeys

$$\epsilon(t) = M^{-1}(\omega)\sigma(t) \qquad (2.81)$$

where

$$M^{-1}(\omega) = M_U^{-1}(1 + d(\omega) - iq(\omega))$$

and

$$\left.\begin{aligned} d(\omega) &= \int_0^\infty \cos\omega t\ \dot{\phi}(t)dt \\ q(\omega) &= \int_0^\infty \sin\omega t\ \dot{\phi}(t)dt \end{aligned}\right\} \tag{2.82}$$

Note that if d and q are small

$$M(\omega) = M_U[1 - d(\omega) + iq(\omega)] \tag{2.83}$$

If this is compared with the perfectly elastic case, we find that the real part of M_U is slightly frequency-dependent (a phenomenon called physical dispersion) and there is a small imaginary part. The elastic moduli are therefore complex functions. For example, we can write the shear modulus as

$$\mu(\omega) = \mu_0 \left[1 - d_\mu(\omega) + iq_\mu(\omega)\right] \tag{2.84}$$

Conventionally, $1/Q_\mu(\omega)$ is defined as $Im(\mu(\omega)/Re(\mu(\omega)$ or

$$\frac{1}{Q_\mu(\omega)} = \frac{q_\mu(\omega)}{1 - d_\mu(\omega)} \tag{2.85}$$

A similar equation can be defined for the bulk modulus and the quality of attenuation in compression, $Q_K(\omega)$. Thus Q_μ and Q_K provide a phenomenological description of attenuation. Note that, from 2.82, $d(\omega)$ and $q(\omega)$ are allied functions so, in principle, if we are given one we can compute the other. We shall discuss this further below. The fact that the elastic moduli are complex leads to complex frequencies of oscillation which simply means that the oscillations decay. Observation of the decay rates of the oscillations can, in principle, be used to constrain Q_μ and Q_K as a function of depth and frequency. The existing measurements are sufficiently imprecise that no frequency dependence of Q_μ or Q_K has been detected. This is sometimes interpreted as both Q_μ and Q_K being independent of frequency. If this is true, it follows that a simple standard linear solid is inconsistent with the observations. Comparison of 2.76 with 2.77 shows that the creep function for a standard linear solid is given by

$$\phi(t) = (\frac{\tau_\epsilon}{\tau_\sigma} - 1)(1 - e^{-t/\tau_\epsilon})$$

Substitution into 2.82 gives

$$\left.\begin{aligned} q(\omega) &= \frac{\omega\tau_\epsilon}{1 + \omega^2\tau_\epsilon^2}\left(\frac{\tau_\epsilon}{\tau_\sigma} - 1\right) \\ d(\omega) &= \frac{1}{1 + \omega^2\tau_\epsilon^2}\left(\frac{\tau_\epsilon}{\tau_\sigma} - 1\right) \end{aligned}\right\} \tag{2.86}$$

It is convenient to introduce dimensionless frequency $y = \omega\tau_\epsilon$ and let $C = \tau_\epsilon/\tau_\sigma - 1 = M_U/M_R - 1$ then

$$q(y) = \frac{Cy}{1 + y^2} \quad \text{and} \quad d(y) = \frac{C}{1 + y^2}$$

$$\frac{1}{Q} = \frac{q}{1 - d} = \frac{Cy}{1 - C + y^2}$$

Note that C must be small for d and q to be small so that $q \simeq 1/Q$ and that q has a peak at $y = 1$ so here $\omega = 1./\tau_\epsilon$. The real part, $1 - d$ has a value of $1 - C$ at small values of y and tends to 1 at large values. We plot q and $1 - d$ below:

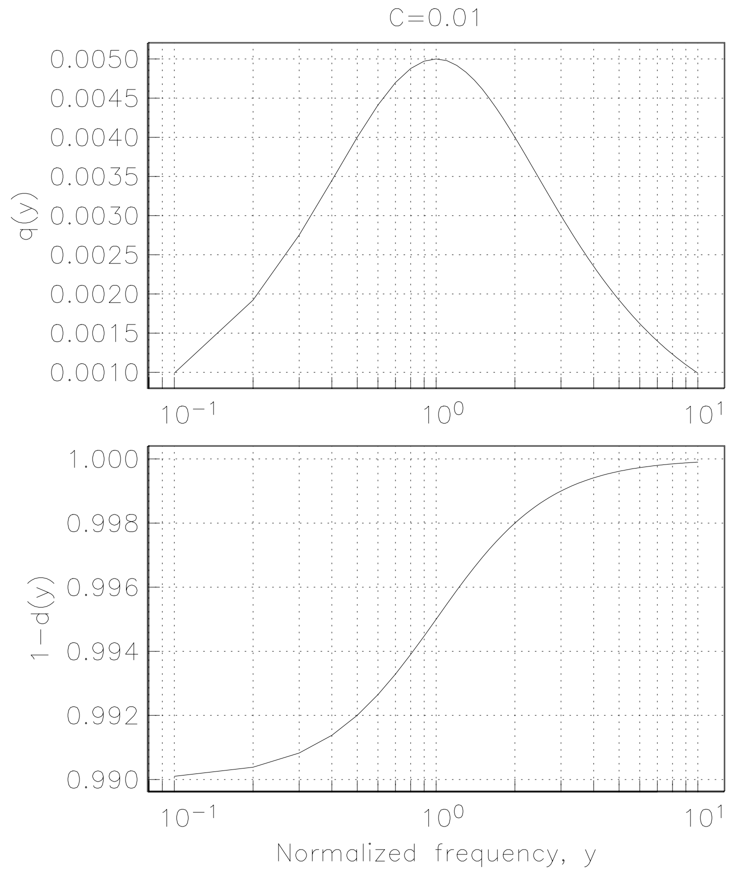


Fig 2.10

This response looks like the response when a single attenuation (or *relaxation*) mechanism is acting. In rocks, a whole spectrum of relaxation mechanisms is operating. This can be modeled by putting more K-V elements in series in the standard linear solid. By judicious choice of time constants an “absorption band” can be constructed, *i.e.*,

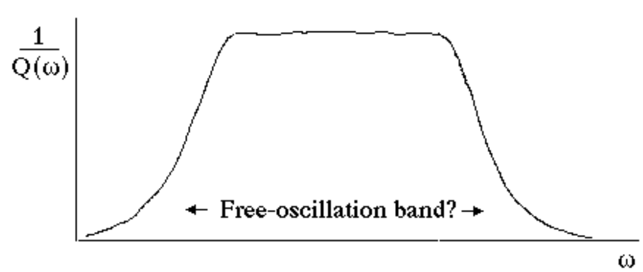


Fig 2.11

and an apparently frequency-independent Q constructed. If we have a discrete set of relaxation mechanisms we can write

$$q(\omega) = \sum_k \frac{\omega \tau_{\epsilon_k}}{1 + \omega^2 \tau_{\epsilon_k}^2} \left(\frac{\tau_{\epsilon_k}}{\tau_{\sigma_k}} - 1 \right) \tag{2.87}$$

and similarly for $d(\omega)$.

We note that, because $q(\omega)$ is always positive we have that $\tau_\epsilon > \tau_\sigma$ so $M_U > M_R$. In the limit of many relaxation mechanisms we can generalize the sum to an integral over a distribution (Liu *et al.*, 1976), *i.e.*,

$$q(\omega) = \int_0^\infty d\tau_\sigma \int_0^\infty d\tau_\epsilon D(\tau_\sigma, \tau_\epsilon) \frac{\omega\tau_\epsilon}{1+\omega^2\tau_\epsilon^2}\left(\frac{\tau_\epsilon}{\tau_\sigma} - 1\right) \tag{2.88}$$

and similarly for $d(\omega)$. Liu *et al.*, (1976) consider some simple models for the absorption band. For example, a good representation of the seismic case can be had if we set

$$\tau_\sigma = (1 - C)\tau_\epsilon$$

where $C > 0$ because $\tau_\sigma < \tau_\epsilon$ and we let

$$\begin{aligned} D &= \frac{1}{\tau_\epsilon} \quad \text{for} \quad \tau_1 < \tau_\epsilon < \tau_2 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{Q(\omega)} &= C \tan^{-1} \frac{\omega(\tau_1 - \tau_2)}{1 + \omega^2\tau_1\tau_2} \\ d(\omega) &= -\frac{C}{2} \ln \left[\frac{(1+\omega^2\tau_1^2)\tau_2^2}{(1+\omega^2\tau_2^2)\tau_1^2}\right] \end{aligned}$$

Now if τ_1 is very large and τ_2 is very small:

$$\frac{1}{Q(\omega)} \simeq \frac{C\pi}{2} \quad \text{(a constant)}$$

and if we consider the real part of the elastic modulus:

$$M^{-1}(\omega) = M_U^{-1}(1 + d(\omega)) = \mu^{-1}(\omega) \quad \text{say}$$

we find that

$$\frac{\mu(\omega_1)}{\mu(\omega_2)} = 1 + \frac{2}{\pi Q_\mu} \ln\left(\frac{\omega_1}{\omega_2}\right) \tag{2.89}$$

This means that seismic velocities change by about 1% for a period change of $1 \to 1000$ secs if $Q \simeq 200$. The number is increased to 4% if $Q \simeq 60$.

The construction of this absorption band may seem a little *ad hoc* but several other lines of reasoning lead to the relationship 2.89. The fact that $d(\omega)$ and $q(\omega)$ are the real and imaginary parts of the Fourier transform of a one-sided function means that they are Hilbert transforms of one another. In fact

$$d(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{q(\omega')d\omega'}{\omega' - \omega} \tag{2.90}$$

Note that this stems from the fact that $\phi = 0$ for $t < 0$ and so is a direct consequence of causality. In principle, we can compute $d(\omega)$ if we know $q(\omega)$ but in practice, we only know $q(\omega)$ over a small frequency band. It is possible to construct $q(\omega)$'s which give radically different $d(\omega)$'s but which are still nearly constant over the seismic band. This is discussed by Aki and Richards (l980) where they show that empirically observed creep functions also lead to 2.89, so that if attenuation is weak and roughly independent of ω, 2.89 seems to give a reasonable description of the physical dispersion of elastic moduli. The fact that the elastic moduli are frequency-dependent means that we must specify the model at a certain frequency and provide a Q structure so that equations like 2.89 can be used to calculate the model at any desired frequency (*e.g.*, the frequency of a mode of oscillation). For an isotropic model we would also need an equivalent expression to 2.89 for the bulk modulus, *i.e.*,

$$\frac{K(\omega_1)}{K(\omega_2)} = 1 + \frac{2}{\pi Q_K} \ln \left(\frac{\omega_1}{\omega_2} \right) \tag{2.91}$$

where Q_K measures the attenuation in compression and dilatation. Attenuation of seismic energy is much greater during shearing than during compression so $Q_K \gg Q_\mu$. Sometimes Q_K is chosen to be infinite but this is inconsistent with the observations of attenuation of radial modes of the Earth. Reasonable agreement is achieved if $Q_K \simeq 50 Q_\mu$.

In following chapters, we shall be using an isotropic or transversely isotropic elastic material. We shall be solving the equation of motion at specific frequencies (the frequencies of the modes of oscillation) and we use the correspondence principle to substitute complex elastic moduli into the constitutive relation to account for attenuation. The frequencies and eigenfunctions of modes of oscillation are then complex. If attenuation is strong, we must know the creep function of the material to properly calculate the frequency dependence of the moduli (c.f., eqn. 2.84). If attenuation is weak and only weakly dependent upon frequency we can use 2.89 to calculate the physical dispersion. We may then treat the model as being effectively perfectly elastic and solve as if the frequencies were purely real. The imaginary part of the frequency can then be computed by perturbation theory assuming that the imaginary part of the elastic moduli can be treated as a perturbation to the real part. This latter route is the one we shall take because: (1) attenuation of long-period oscillations *is* weak and (2) the full complex problem requires a squaring of the numerical effort to solve it!

2.11 Boundary conditions . There are several kinds of boundaries we must deal with in the real Earth. The surface of the Earth is usually modeled as being "free" which means that tractions must vanish here as we assume there are no external forces acting on the Earth. (We also neglect coupling of seismic motion into the atmosphere.) Another type of boundary is one between fluid and solid (*e.g.*, inner core-outer core, mantle-outer core, crust-ocean) and a final type is a welded type (*e.g.*, the 660 km discontinuity). Because, to a first approximation, we consider fluids to be inviscid we can have slip at a fluid-solid boundary but not at a welded boundary.

We first consider the boundary conditions that must be satisfied on the deformed boundaries and then later consider the equivalent conditions on the undeformed surface.

The conditions are:

1) $\hat{\mathbf{n}} \cdot \mathbf{s}$ is continuous at all boundaries so that we allow no gapping or interpenetration. Note that $\hat{\mathbf{n}}$ is the normal to the *deformed* surface.
2) $\mathbf{s}$ is continuous at all *welded* boundaries, *i.e.*, we allow no slip at such boundaries.
3) $\mathbf{t}$ (the traction vector) is continuous at all deformed boundaries and is zero at the free surface.
4) ϕ is continuous at all boundaries.
5) $\hat{\mathbf{n}} \cdot \nabla\phi$ is continuous at all (deformed) boundaries.
6) The displacements and tractions must be regular at the origin and the gravitational potential must be zero at infinity.

Most of these conditions are self-evident. For example, if ϕ were not continuous we would have singular behavior of $\mathbf{g}$ which would be interesting to say the least. The continuity of $\mathbf{t}$ is shown by using the equations of motion. We consider a volume intersected by a boundary, *i.e.*,

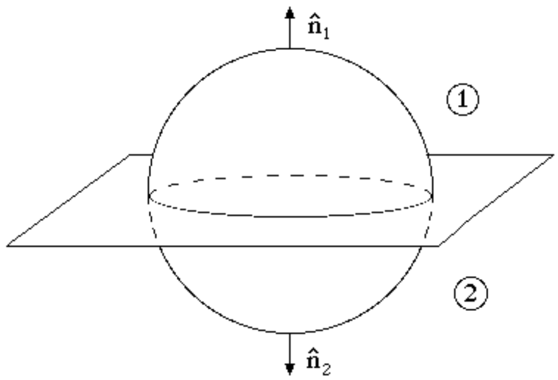


Fig 2.12

From 2.33 we have

$$\int_V \left[\rho \frac{D\mathbf{v}}{Dt} - \rho\mathbf{b}\right] dV = \int_V \nabla \cdot \mathbf{T} dV = \int_S \hat{\mathbf{n}} \cdot \mathbf{T} dS$$

Now the integrand of the volume integral is bounded so if we let $V \to 0$ to give a patch of area S, the volume integral goes to zero. Thus

$$\int_S [\hat{\mathbf{n}}_1 \cdot \mathbf{T}^{(1)} + \hat{\mathbf{n}}_2 \cdot \mathbf{T}^{(2)}] dS = 0$$

where $\mathbf{T}^{(1)}$ is the Cauchy stress tensor on the (1) side of the boundary and $\mathbf{T}^{(2)}$ is the Cauchy stress tensor on the (2) side. As $V \to 0$ we have that

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2$$

So

$$\int_S \hat{\mathbf{n}}[\mathbf{T}^{(1)} - \mathbf{T}^{(2)}] dS = 0$$

which must be true for any patch of boundary so

$$\hat{\mathbf{n}} \cdot \mathbf{T}^{(1)} = \hat{\mathbf{n}} \cdot \mathbf{T}^{(2)} \equiv \mathbf{t}$$

which proves the continuity of $\mathbf{t}$. Note that only $\mathbf{t}$ need be continuous and the stress tensor, $\mathbf{T}$, need not be.

A similar argument can be used to show that $\hat{\mathbf{n}} \cdot \nabla\phi$ must be continuous. Consider Poisson's equation

$$\nabla^2\phi = \nabla \cdot \nabla\phi = 4\pi G\rho$$

Integrate this over a volume intersecting the boundary and use Gauss' theorem to give

$$\int_S \hat{\mathbf{n}} \cdot \nabla\phi dS = \int_V 4\pi G\rho dV$$

Because the integrand of the right-hand side is bounded it follows that, as $V \to 0$,

$$\int_S \hat{\mathbf{n}} \cdot \nabla\phi dS = 0 \qquad \text{for any patch}$$

so $\hat{\mathbf{n}} \cdot \nabla\phi$ is continuous.

The boundary conditions are valid on the *deformed* boundary and we would like to find the equivalent conditions on the undeformed boundary. We are dealing with small perturbations to the boundary so we shall neglect squares and higher powers of small quantities in the following. We shall consider a model with boundaries which are spherical in the undeformed state and we must allow for both the radial deformation of this boundary *and* the deflection of the normal (see section 2.4 for some basic results). In the following, we shall use the notation that the current position at time t is given by $\mathbf{r} = \mathbf{x} + \mathbf{s}(t)$. Continuity of $\mathbf{s}$ at a deformed welded boundary becomes continuity of

$$\mathbf{s}(\mathbf{x}) + s_r\partial_r\mathbf{s}$$

where s_r is the radial deflection of the boundary and $\mathbf{x}$ is the location of the boundary in the undeformed state. The second term is of second-order so that it is immaterial whether $\mathbf{s}$ is evaluated at $\mathbf{x}$ or $\mathbf{r}$ and we may match the boundary conditions at the undeformed surface. We can treat the boundary condition on $\mathbf{s} \cdot \hat{\mathbf{n}}$ in a similar fashion. Using (2.20) gives

$$\mathbf{s} \cdot \hat{\mathbf{n}} = \mathbf{s} \cdot \hat{\mathbf{r}} - \frac{1}{r_s}\mathbf{s} \cdot \nabla_1(\hat{\mathbf{r}} \cdot \mathbf{s}) \tag{2.92}$$

but $\mathbf{s}$ and its gradient are small so, to first order,

$$\mathbf{s} \cdot \hat{\mathbf{n}} \equiv \mathbf{s} \cdot \hat{\mathbf{r}} \tag{2.93}$$

and it is still immaterial where we evaluate $\mathbf{s}$ so we can match boundary conditions at the undeformed boundary (with the undeformed normal).

The boundary conditions in the equilibrium state are:

$$\left.\begin{aligned} \phi_0 &\quad \text{continuous} \\ \hat{\mathbf{r}} \cdot \nabla\phi_0 &\quad \text{continuous} \\ \hat{\mathbf{r}} \cdot \mathbf{T}_0 &\quad \text{continuous} \end{aligned}\right\} \tag{2.94}$$

Because ϕ_0 and ϕ are continuous, it follows that ϕ_1 must be continuous and continuity of ϕ_1 at the deformed boundary implies continuity of $\phi_1(\mathbf{x}) + s_r\partial_r\phi_1$ which, to first-order, implies continuity of ϕ_1 at $\mathbf{x}$. Now consider $\nabla\phi$ at the deformed boundary which is given by

$$\begin{aligned} \nabla\phi &= \nabla\phi_0 + \nabla\phi_1 \\ &= g_0(\mathbf{x}) + s_r\frac{\partial g_0}{\partial r} + \nabla\phi_1(\mathbf{x}) + s_r\frac{\partial}{\partial r}\nabla\phi_1(\mathbf{x}) \\ &= 4\pi G\rho_0 s_r - \frac{2g_0 s_r}{r} + g_0(\mathbf{x}) + \nabla\phi_1(\mathbf{x}) \end{aligned}$$

to first order. Now g_0 is continuous so it follows that $\hat{\mathbf{n}} \cdot (\nabla\phi_1 + 4\pi G\rho_0 s_r)$ is continuous. Because ϕ_1 and s_r are both small it follows that continuity of $\hat{\mathbf{n}} \cdot \nabla\phi$ reduces to continuity of

$$\hat{\mathbf{r}} \cdot (\nabla\phi_1 + 4\pi G\rho_0 s_r) \tag{2.95}$$

The other boundary conditions are that the solution should be regular at the origin and that ϕ (or equivalently, ϕ_1) must vanish at infinity.

Finally, we consider the equivalent boundary conditions for the traction vector $\mathbf{t}$. The difficulty is that we are concerned with the force balance on a patch of boundary whose shape is a function of time. The problem is simplified if we introduce the incremental Piola-Kirchoff stress tensor $\tilde{\mathbf{T}}$. Consider Fig 2.4, $\mathbf{T}_0 + \tilde{\mathbf{T}}$ gives the actual force on the deformed dS but reckoned per unit area of the deformed dS_0 and is defined by

$$\hat{\mathbf{r}} \cdot (\mathbf{T}_0(\mathbf{x}) + \tilde{\mathbf{T}})dS_0 = \hat{\mathbf{n}} \cdot (\mathbf{T})dS$$

The traction vector on the deformed boundary is given by

$$\hat{\mathbf{n}}(\mathbf{r}, t) \cdot \mathbf{T}(\mathbf{r}, t) = \hat{\mathbf{n}}(\mathbf{r}, t) \cdot [\mathbf{T}_0(\mathbf{x}) + \mathbf{T}_E(\mathbf{r}, t)] = \mathbf{t}$$

where $\mathbf{T}_E$ is the incremental Lagrangian stress. The force exerted on a patch of the surface $dS(t)$ is

$$\mathbf{F}(t) = \int_{dS(t)} \mathbf{t}dS$$

Using equation (2.24), we can write

$$\mathbf{T}_0 + \tilde{\mathbf{T}} = (\mathbf{I} + (\nabla \cdot \mathbf{s})\mathbf{I} - (\nabla\mathbf{s})^T) \cdot (\mathbf{T}_E + \mathbf{T}_0) \tag{2.96}$$

and to first order

$$\tilde{\mathbf{T}} = \mathbf{T}_E + [(\nabla \cdot \mathbf{s})\mathbf{I} - (\nabla\mathbf{s})^T] \cdot \mathbf{T}_0$$

where it is immaterial to first order whether $\mathbf{T}_0$ is evaluated at $\mathbf{r}$ or $\mathbf{x}$. Thus $\mathbf{F}(t)$, the force exerted on a patch of surface $dS(t)$, is

$$\mathbf{F}(t) = \int_{dS_0} dS_0 \hat{\mathbf{r}} \cdot [\mathbf{T}_0 + \tilde{\mathbf{T}}] \tag{2.97}$$

Now $\hat{\mathbf{r}} \cdot \mathbf{T}_0$ is already continuous so that continuity of $\mathbf{F}(t)$ reduces to the requirement that $\hat{\mathbf{r}} \cdot \tilde{\mathbf{T}}$ be continuous at the undeformed boundary. (This is only true if dS_0 is continuous which may not be true if we have slip). For a general prestress, $\tilde{\mathbf{T}}$ can be written in a variety of ways. Dahlen (1972) gives

$$\tilde{\mathbf{T}} = \mathbf{\Lambda} : \nabla\mathbf{s} \tag{2.98}$$

where

$$\Lambda_{ijkl} = C_{ijkl} + \tfrac{1}{2}\left(T^0_{ij}\delta_{kl} + T^0_{kl}\delta_{ij} + T^0_{ik}\delta_{jl} - T^0_{jk}\delta_{il} - T^0_{il}\delta_{jk} - T^0_{jl}\delta_{ik}\right) \tag{2.99}$$

which, for an isotropic prestress reduces to

$$\Lambda_{ijkl} = C_{ijkl} - p_0(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}) \tag{2.100}$$

Even in this case $\tilde{\mathbf{T}}$ is not symmetric, *i.e.*,

$$\tilde{T}_{ij} = C_{ijkl}\epsilon_{kl} - p_0(\partial_k s_k \delta_{ij} - \partial_j s_i) \tag{2.101}$$

Because p_0 is continuous across any boundary, it follows that the elastic stress, $\mathbf{T}_E$ given by

$$C_{ijkl}\epsilon_{kl} \tag{2.102}$$

satisfies the continuity condition $\hat{\mathbf{r}} \cdot \mathbf{T}_E$ at a welded boundary. This is the condition that we usually impose but you should remember that it is only valid for an isotropic prestress.

The boundary condition at a fluid/solid interface is more complicated and is discussed in considerable detail by Woodhouse and Dahlen (1978) (Geophys. J. R astr. Soc, 1978, v53, see pages 335–339). Inspection of their equation 23 shows that the continuity of $\hat{\mathbf{r}} \cdot \mathbf{T}_E$ also holds at a frictionless boundary if the pre-stress is isotropic.

A summary of the boundary conditions on the undeformed boundary with an isotropic prestress is
$\hat{\mathbf{r}} \cdot \mathbf{s}$ is continuous at all boundaries
s is regular at the origin

$\mathbf{s}$ is continuous at welded boundaries
ϕ_1 is continuous at all boundaries
ϕ_1 is zero at infinity
$\partial\phi_1/\partial r + 4\pi G\rho_0 s_r$ is continuous at all boundaries
$\mathbf{t}$ is continuous at all boundaries
$\mathbf{t}$ is zero at the free surface
$\mathbf{t}$ is regular at the origin.

2.12 Effect of rotation . The equations are easily modified if we wish to include the Earth's rotation. The acceleration in a rotating frame is related to the acceleration in the intertial frame by

$$\rho_0 \frac{D^2\mathbf{s}}{Dt^2} \rightarrow \rho_0 \frac{D^2\mathbf{s}}{Dt^2} + 2\rho_0 \mathbf{\Omega} \times \frac{D\mathbf{s}}{Dt} + \rho_0 \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \tag{2.103}$$

where $\mathbf{\Omega}$ is the rotation vector and is taken to be

$$\mathbf{\Omega} = \Omega_0 \hat{\mathbf{z}} \tag{2.104}$$

The second term in 2.103 is the Coriolis force while the last term is the centripetal force. The full equation of motion, 2.47, becomes

$$(\rho_0 + \rho_1)\left[\frac{D^2\mathbf{s}}{Dt^2} + 2\mathbf{\Omega} \times \frac{D\mathbf{s}}{Dt}\right] = \nabla \cdot \mathbf{T} - (\rho_0 + \rho_1)\nabla(\phi_0 + \psi + \phi_1) + \mathbf{f} \tag{2.105}$$

where we have included the centripetal acceleration as the gradient of a *rotation potential*, ψ where

$$\psi(\mathbf{r}) = -\tfrac{1}{2}\left[\Omega_0^2 r^2 - (\mathbf{\Omega}\cdot\mathbf{r})^2\right] \equiv -\tfrac{1}{2}\,\Omega_0^2 r^2 \sin^2\theta \qquad \theta \text{ is colatitude} \tag{2.106}$$

In the following diagram, note that the centripetal acceleration is $\mathbf{g}_\omega = -\Omega_0^2 x = -\Omega_0^2 r \sin\theta$ and points directly away from the rotation axis and so in a spherical polar coordinate system is

$$-\hat{\mathbf{r}}\Omega_0^2 r \sin^2\theta - \hat{\boldsymbol{\theta}}\Omega_0^2 r \sin\theta\cos\theta$$

It is then easy to verify that $\mathbf{g}_\omega = \nabla\psi$ with ψ as given above.

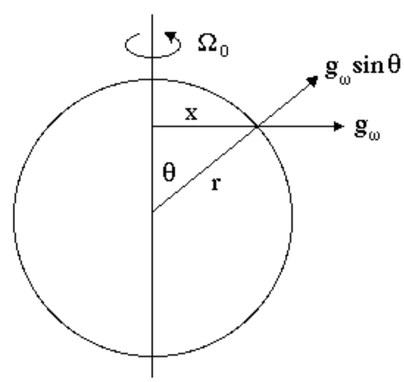


Fig 2.13

When we linearize (2.105) we must remember that the initial stress is also subject to the centripetal force, *i.e.*,

$$\nabla p_0 = -\rho_0 \nabla(\phi_0 + \psi) \tag{2.107}$$

Thus 2.53 becomes

$$\rho_0 \frac{\partial^2 \mathbf{s}}{\partial t^2} + 2\rho_0 \mathbf{\Omega}_0 \times \frac{\partial \mathbf{s}}{\partial t} = \nabla \cdot \mathbf{T} - \rho_0 \nabla \phi_1 - \rho_1 \nabla(\phi_0 + \psi) - \nabla \cdot (\mathbf{s} \cdot \nabla \mathbf{T}_0) + \mathbf{f} \tag{2.108}$$

$|\mathbf{\Omega}|$ is the sidereal rotation frequency (about 11.6 μHz) so $\psi(r)$, the rotation potential, can often be neglected. If we consider an oscillation with $\mathbf{s} = \mathbf{S}e^{i\omega t}$ then the ratio of the second (Coriolis) to the first term on the left-hand side of 2.108 is about $2\Omega_0/\omega$. Thus, except for very low frequencies and some special coupling cases, the coriolis force can also be neglected.