CHAPTER 3

Free oscillations of a spherical Earth model

3.1 Introduction . As a first approximation, we consider a spherical, non-rotating, elastic, transversely isotropic Earth model. (We will relax some of these simplifications later, *e.g.*, we can allow weak anelastic behavior but these departures from our basic model will be treated by perturbation theory.) Such a model is specified by six functions of radius which we take to be ρ_0, A, C, N, L and F which are defined in the previous chapter. An isotropic model is specified by three functions of radius: $\rho_0,\ \lambda$ and μ and can be recovered from the transversely isotropic model by setting A=C=$\lambda + 2\mu$, N=L=μ and F=λ. These moduli can be cast in terms of the seismic velocities, V_p and V_s by using:

$$\begin{aligned} \mu &= \rho_0 V_s^2 \\ K_s &= \rho_0 (V_p^2 - \frac{4}{3} V_s^2) \\ \lambda &= \rho_0 (V_p^2 - 2V_s^2) \end{aligned}$$

Some typical Earth models are plotted in Figure 3.1.

The assumptions we have made allow us to separate variables. We expand scalar fields in surface spherical harmonics, *i.e.*,

$$\phi_1 = \sum_{l,m} \Phi_{1l}^m(r) Y_l^m(\theta, \phi) \tag{3.1}$$

Vector fields cannot be expanded in a similar way and still retain the ability to separate variables. We need a form which is invarient to rigid rotations and do this by invoking the Helmholtz theorem on a sphere. Backus (1986) gives a thorough discussion of this theorem and we quote his results here. If $\mathbf{v}_S$ is a tangent field on a spherical surface, there are unique fields $\mathbf{v}_P$ and $\mathbf{v}_T$ such that

$$\mathbf{v}_S = \mathbf{v}_P + \mathbf{v}_T \quad \text{and} \quad \mathbf{v}_P = \nabla_1 g \quad \text{and} \quad \mathbf{v}_T = -\hat{\mathbf{r}} \times \nabla_1 h$$

where

$$\nabla_1 = \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}} \operatorname{cosec} \theta \frac{\partial}{\partial \phi}$$

g and h are scalar fields which are unique to an additive constant. We make them completely unique by specifying that their average values on the spherical surface be zero.

Now for any vector field, $\mathbf{v}$, there is a unique scalar field, f, such that

$$\begin{aligned} \mathbf{v} &= \hat{\mathbf{r}} f + \mathbf{v}_P + \mathbf{v}_T \\ &= \hat{\mathbf{r}} f + \nabla_1 g - \hat{\mathbf{r}} \times \nabla_1 h \end{aligned}$$

f, g and h may be expanded in spherical harmonics and we have a form which is invariant to rigid rotations. This would not have been true if we had merely expanded the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ components of $\mathbf{v}$ in spherical harmonics.

3.2 Separating the equations . We now consider equations (2.48) when $\mathbf{f}$ is zero and we look for solutions of the form

$$\mathbf{s}(\mathbf{r}, t) = \mathbf{s}_k(\mathbf{r}) e^{i\omega_k t}; \quad \phi_1 = \phi_{1k} e^{i\omega_k t}$$

i.e., we look at the *free oscillations* of the model. The equations now become

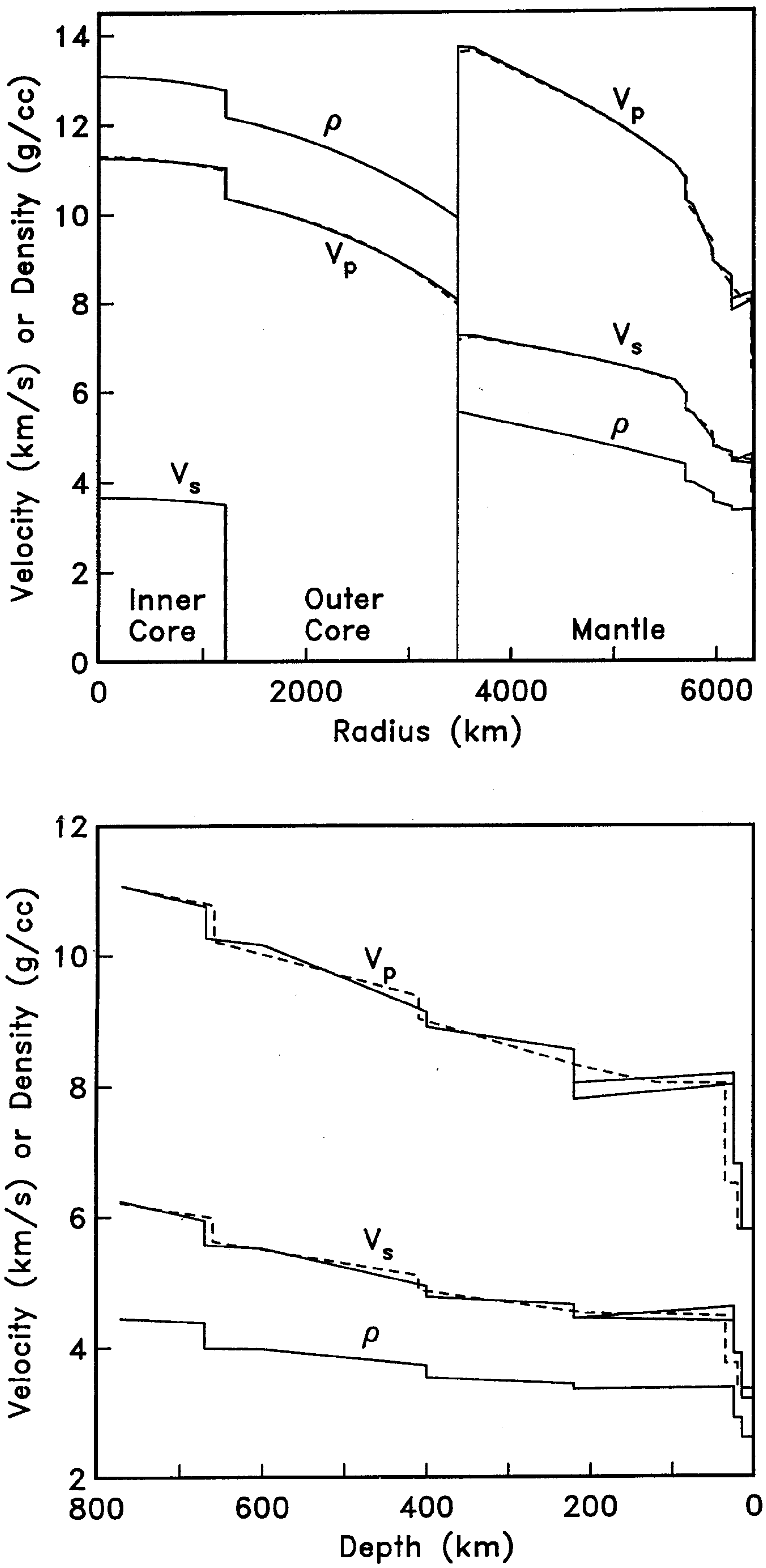

Figure 3.1 Some 1D models of the Earth. The solid lines are (anisotropic) PREM and the dashed line is a variant of IASP91

$$-\rho_0\omega_k^2\mathbf{s}_k = \nabla\cdot\mathbf{T} - \rho_0\nabla\phi_{1k} + \hat{\mathbf{r}}g_0\nabla\cdot(\rho_0\mathbf{s}_k) - \nabla(s_{kr}g_0\rho_0) \tag{3.2}$$

and

$$\nabla^2\phi_{1k} = -4\pi G\nabla\cdot(\rho_0\mathbf{s}_k) \tag{3.3}$$

$\mathbf{s}_k$ and ϕ_{1k} are functions only of position and the equations (3.2) and (3.3) separate if we expand $\mathbf{s}_k$ in vector spherical harmonics and ϕ_{1k} in ordinary spherical harmonics. We write

$$\mathbf{s}_k = \hat{\mathbf{r}}\ {}_kU + \nabla_1\ {}_kV - \hat{\mathbf{r}}\times(\nabla_1\ {}_kW) \tag{3.4}$$

where ${}_kU$, ${}_kV$ and ${}_kW$ are scalars. For reference, we note that

$$\left.\begin{aligned}
\nabla &= \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\nabla_1\\
\nabla_1^2 &= \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \operatorname{cosec}^2\theta\frac{\partial^2}{\partial\phi^2}\\
\text{and}\quad \nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\nabla_1^2
\end{aligned}\right\} \tag{3.5}$$

We now expand ${}_kU$, ${}_kV$ and ${}_kW$ in ordinary spherical harmonics as well as ϕ_{1k}. Each of these functions have an expansion of the form

$${}_kU = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} {}_kU_l^m(r)Y_l^m(\theta,\phi) \tag{3.6}$$

where the $Y_l^m(\theta,\phi)$ are fully normalized spherical harmonics defined in appendix A. Thus $\mathbf{s}_k$ can be written in component form as

$$\left.\begin{aligned}
s_{kr} &= \sum_{l-0}^{\infty}\sum_{m=-l}^{l} {}_kU_l^m(r)Y_l^m\\
s_{k\theta} &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[{}_kV_l^m(r)\frac{\partial Y_l^m}{\partial\theta} + im\operatorname{cosec}\theta\ {}_kW_l^m(r)Y_l^m\right]\\
s_{k\phi} &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[im\operatorname{cosec}\theta\ {}_kV_l^m(r)Y_l^m - {}_kW_l^m(r)\frac{\partial Y_l^m}{\partial\theta}\right]\\
\text{and}\quad \phi_{1k} &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} {}_k\Phi_{1l}^m(r)Y_l^m
\end{aligned}\right\} \tag{3.7}$$

In a similar way, we can expand the traction vector in vector spherical harmonics:

$$\mathbf{t}_k = \hat{\mathbf{r}}\ {}_kR + \nabla_1\ {}_kS - \hat{\mathbf{r}}\times(\nabla_1\ {}_kT) \tag{3.8}$$

where

$${}_kR = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} {}_kR_l^m(r)Y_l^m(\theta,\phi)$$

,

and similarly for ${}_kS$ and ${}_kT$. In what follows, we consider only a single k, l, m component of the expansions so that, without ambiguity, we may write

$$U \quad \text{for} \quad {}_kU_l^m(r), \quad V \quad \text{for} \quad {}_kV_l^m(r) \quad etc.$$

We shall be interested in casting the equations in terms of U, V, W, R, S and T as this will make our boundary conditions on the continuity of displacement and tractions easy to implement.

Now consider equation (3.3). Note that the term $\nabla \cdot (\rho_0 \mathbf{s}_k)$ can be written as

$$\nabla \cdot (\rho_0 \mathbf{s}_k) = \rho_0 \nabla \cdot \mathbf{s}_k + \mathbf{s}_k \cdot \nabla \rho_0$$

and because ρ_0 is only a function of r we have

$$\nabla \cdot (\rho_0 \mathbf{s}_k) = \rho_0 \nabla \cdot \mathbf{s}_k + s_{kr} \frac{\partial \rho_0}{\partial r} \tag{3.9}$$

where s_{kr} is the radial component of $\mathbf{s}_k$. The expression for a single k, l, m component of the divergence, $\nabla \cdot \mathbf{s}_k$, is given by equation A26:

$$\nabla \cdot \mathbf{s}_k = (U' + F) Y_l^m \quad \text{where} \quad F = \frac{1}{r}(2U - l(l+1)V)$$

and prime ($'$) indicates radial derivative. Thus equation (3.9) becomes

$$\nabla \cdot (\rho_0 \mathbf{s}_k) = \left[\rho_0 (U' + F) + \rho_0' U\right] Y_l^m = \left[(\rho_0 U)' + \rho_0 F\right] Y_l^m \tag{3.10}$$

Returning to equation (3.3) and using the expression for a single k, l, m component of the Laplacian (equation A14) *i.e.*,

$$\nabla^2 (\Phi_1 Y_l^m) = \left[\frac{1}{r^2}\left(\frac{d}{dr} r^2 \frac{d\Phi_1}{dr}\right) - \frac{\Phi_1}{r^2} l(l+1)\right] Y_l^m$$

gives

$$\frac{1}{r^2}\left(\frac{d}{dr} r^2 \frac{d\Phi_1}{dr}\right) - l(l+1)\frac{\Phi_1}{r^2} = -4\pi G \left[\frac{d}{dr}(\rho_0 U) + \rho_0 F\right] \tag{3.11}$$

Note that we can use ordinary derivatives because, from (3.7), U, V, W and Φ_1 are functions only of radius.

We now turn to equation (3.2). Equation A43 of appendix A gives expressions for the r, θ, ϕ components of $\nabla \cdot \mathbf{T}$ which we denote P_r, P_θ and P_ϕ. We write the three components of (3.2) out separately:

$$\left.\begin{aligned}
-\rho_0 \omega_k^2 U Y_l^m &= P_r - \rho_0 \Phi_1' Y_l^m + g_0 \left((\rho_0 U)' + \rho_0 F\right) Y_l^m - (\rho_0 g_0 U)' Y_l^m \\
-\rho_0 \omega_k^2 \left(V \frac{\partial Y_l^m}{\partial \theta} + im \operatorname{cosec}\theta W Y_l^m\right) &= P_\theta - \frac{\rho_0 \Phi_1}{r}\frac{\partial Y_l^m}{\partial \theta} - \frac{\rho_0 g_0 U}{r}\frac{\partial Y_l^m}{\partial \theta} \\
-\rho_0 \omega_k^2 \left(im \operatorname{cosec}\theta V Y_l^m - W \frac{\partial Y_l^m}{\partial \theta}\right) &= P_\phi - im \operatorname{cosec}\theta \frac{\rho_0 \Phi_1}{r} Y_l^m - \frac{\rho_0 g_0 U}{r} im \operatorname{cosec}\theta Y_l^m
\end{aligned}\right\} \tag{3.12}$$

Substituting in the expressions for P_r, P_θ, and P_ϕ and rearranging gives

$$\left.\begin{aligned}
-\rho_0 \omega_k^2 U &= \frac{d}{dr}\left(\mathrm{C}U' + \mathrm{F}F\right) - \frac{1}{r}\left[2(\mathrm{F} - \mathrm{C})U' + 2(\mathrm{A} - \mathrm{N} - \mathrm{F})F + l(l+1)\mathrm{L}X\right] \\
&\quad - \rho_0 \Phi_1' + g_0 \left((\rho_0 U)' + \rho_0 F\right) - (\rho_0 g_0 U)' \\
-\rho_0 \omega_k^2 V &= \frac{d}{dr}(\mathrm{L}X) + \frac{1}{r}\left[(\mathrm{A} - \mathrm{N})F + \mathrm{F}U' + 3\mathrm{L}X - \frac{\mathrm{N}V}{r}(l+2)(l-1)\right] \\
&\quad - \frac{\rho_0 \Phi_1}{r} - \frac{\rho_0 g_0 U}{r} \\
-\rho_0 \omega_k^2 W &= \frac{d}{dr}(\mathrm{L}Z) + \frac{1}{r}\left[3\mathrm{L}Z - \frac{\mathrm{N}W}{r}(l+2)(l-1)\right]
\end{aligned}\right\} \tag{3.13}$$

where

$$X = V' + \frac{U - V}{r} \quad \text{and} \quad Z = W' - \frac{W}{r}$$

Equations (3.11) and (3.13) give us four second-order coupled differential equations for U, V, W and Φ_1. If solutions to these equations can be found which satisfy the boundary conditions, we can recover the displacement field and gravitational perturbation for the k'th solution by using equation (3.7).

The first thing to note about (3.11) and (3.13) is that the index m does not appear. This implies that for a particular k, l all the ${}_kU_l^m$ are the same (the same is true for ${}_kV_l^m$, *etc.*) so that the m superscript can be ignored.

The second thing to note is that the last of equation (3.13) is completely decoupled from the others so it can be solved separately. From the analysis of the traction vector in appendix A (equation A46), we have that $T = \mathrm{L}Z$ so that, from the definition of Z we have

$$\frac{dW}{dr} = \frac{T}{\mathrm{L}} + \frac{W}{r}$$

$$\text{and} \quad \frac{dT}{dr} = \frac{\mathrm{N}}{r^2}(l+2)(l-1)W - \rho_0\omega_k^2 W - \frac{3T}{r}$$

which can be written in matrix form as

$$\frac{d}{dr}\begin{bmatrix} W \\ T \end{bmatrix} = \begin{bmatrix} 1/r & 1/L \\ \mathrm{N}(l+2)(l-1)/r^2 - \rho_0\omega_k^2 & -3/r \end{bmatrix}\begin{bmatrix} W \\ T \end{bmatrix} \tag{3.14}$$

This form is convenient for numerical solution.

The other three equations require a little more algebra. First, note that *outside* the Earth, ϕ_1 satisfies Laplaces equation so that (3.11) becomes

$$\frac{1}{r^2}\left(\frac{d}{dr}r^2\frac{d\Phi_1}{dr}\right) - \frac{l(l+1)}{r^2}\Phi_1 = 0$$

and $\Phi_1 \to 0$ as $r \to \infty$. The solution to this equation has the form

$$\Phi_1 = Ar^{-l-1} \quad \text{where } A \text{ is a constant}$$

Thus, *outside* the Earth we have

$$\frac{d\Phi_1}{dr} + \frac{(l+1)}{r}\Phi_1 = 0 \tag{3.15}$$

From the discussion of the boundary conditions we know that

$$\frac{\partial \phi_1}{\partial r} + 4\pi G\rho_0 s_r$$

must be continuous throughout the model so it follows that for a single k, l, m component

$$\frac{d\Phi_1}{dr} + 4\pi G\rho_0 U \quad \text{is continuous}$$

It is therefore convenient to define Ψ_1 as

$$\Psi_1 = \frac{d\Phi_1}{dr} + \frac{(l+1)}{r}\Phi_1 + 4\pi G\rho_0 U \tag{3.16}$$

Because of (3.15), Ψ_1 is continuous throughout the model and is zero outside the model. Ψ_1 is convenient from a computational point of view because Ψ_1 is zero at the free surface. We can differentiate equation (3.16) and, using equation (3.11) we find that

$$\frac{d\Psi_1}{dr} = -4\pi G\rho_0 \frac{(l+1)}{r} U + 4\pi G\rho_0 \frac{l(l+1)}{r} V + \frac{(l-1)}{r}\Psi_1 \tag{3.17}$$

Again, from the analysis of the traction vector in appendix A (equation A46) we have

$$R = \mathrm{C}U' + \mathrm{F}F \quad \text{and} \quad S = \mathrm{L}X \tag{3.18}$$

so that using the definitions of F and X gives

$$\left.\begin{aligned} \frac{dU}{dr} &= -\frac{2\mathrm{F}}{\mathrm{C}r}U + \frac{\mathrm{F}}{\mathrm{C}r}l(l+1)V + \frac{1}{\mathrm{C}}R \\ \frac{dV}{dr} &= -\frac{1}{r}U + \frac{1}{r}V + \frac{1}{\mathrm{L}}S \end{aligned}\right\} \tag{3.19}$$

and the first two equations of 3.13 can be written

$$\left.\begin{aligned} \frac{dR}{dr} &= \left[-\rho_0\omega_k^2 + \frac{4\gamma}{r^2} - \frac{4g_0\rho_0}{r}\right]U + \left[g_0\rho_0 - \frac{2\gamma}{r}\right]\frac{l(l+1)}{r}V - \frac{\rho_0(l+1)}{r}\Phi_1 \\ &\quad - \frac{2}{r}\left[1 - \frac{\mathrm{F}}{\mathrm{C}}\right]R + \frac{l(l+1)}{r}S + \rho_0\Psi_1 \\ \frac{dS}{dr} &= \left[\frac{\rho_0 g_0}{r} - \frac{2\gamma}{r^2}\right]U + \left[-\rho_0\omega_k^2 + \frac{l(l+1)}{r^2}(\gamma + \mathrm{N}) - \frac{2\mathrm{N}}{r^2}\right]V + \frac{\rho_0}{r}\Phi_1 \\ &\quad - \frac{\mathrm{F}}{\mathrm{C}r}R - \frac{3}{r}S \end{aligned}\right\} \tag{3.20}$$

where $\gamma = \mathrm{A} - \mathrm{N} - \mathrm{F}^2/\mathrm{C}$.

Equations (3.16), (3.17), (3.19) and (3.20) can now be combined to form a set of six coupled ordinary differential equations which we write in the form

$$\frac{d\mathbf{y}}{dr} = \mathbf{A}\,\mathbf{y} \tag{3.21}$$

The 6×6 coefficient matrix $\mathbf{A}$ takes on a highly symmetric form if we are careful in our definition of $\mathbf{y}$. We choose as our vector

$$\mathbf{y} = \begin{bmatrix} rU \\ rV\mathcal{L} \\ r\Phi_1 \\ rR \\ rS\mathcal{L} \\ r\Psi_1/4\pi G \end{bmatrix}$$

then the matrix, $\mathbf{A}$ is given by

$$\begin{bmatrix} -\frac{2\mathrm{F}}{\mathrm{C}r} + \frac{1}{r} & \mathcal{L}\frac{\mathrm{F}}{\mathrm{C}r} & 0 & \frac{1}{\mathrm{C}} & 0 & 0 \\ -\frac{\mathcal{L}}{r} & \frac{2}{r} & 0 & 0 & \frac{1}{\mathrm{L}} & 0 \\ -4\pi G\rho_0 & 0 & -\frac{l}{r} & 0 & 0 & 4\pi G \\ -\rho_0\omega_k^2 + \frac{4}{r^2}(\gamma - rg_0\rho_0) & \frac{\mathcal{L}}{r^2}(rg_0\rho_0 - 2\gamma) & -\frac{\rho_0(l+1)}{r} & \frac{2\mathrm{F}}{\mathrm{C}r} - \frac{1}{r} & \frac{\mathcal{L}}{r} & 4\pi G\rho_0 \\ \frac{\mathcal{L}}{r^2}(rg_0\rho_0 - 2\gamma) & -\rho_0\omega_k^2 + \frac{\mathcal{L}^2(\gamma+\mathrm{N})-2\mathrm{N}}{r^2} & \frac{\mathcal{L}\rho_0}{r} & -\mathcal{L}\frac{\mathrm{F}}{\mathrm{C}r} & -\frac{2}{r} & 0 \\ -\frac{\rho_0(l+1)}{r} & \frac{\mathcal{L}\rho_0}{r} & 0 & 0 & 0 & \frac{l}{r} \end{bmatrix} \tag{3.22}$$

where $\mathcal{L} = \sqrt{l(l+1)}$ and $\gamma = \mathrm{A} - \mathrm{N} - \mathrm{F}^2/\mathrm{C}$. $\mathbf{A}$ has the symmetry

$$\mathbf{A} = \begin{bmatrix} T & C \\ S & -T^T \end{bmatrix}$$

where S, C and T are 3×3 matrices and both C and S are symmetric. We have only 14 independent coefficients in $\mathbf{A}$ out of a possible 36 so this form is very advantageous from a computational point of view. Equation (3.14) can be written with similar symmetry:

$$\frac{d}{dr}\begin{bmatrix} rW \\ rT \end{bmatrix} = \begin{bmatrix} 2/r & 1/L \\ \mathrm{N}(l+2)(l-1)/r^2 - \rho_0\omega_k^2 & -2/r \end{bmatrix}\begin{bmatrix} rW \\ rT \end{bmatrix} \tag{3.23}$$

The reason that the coefficient matrix takes on this symmetric form is discussed by Chapman and Woodhouse (1981) who show that it arises from the quadratic form of the Lagrangian. We shall discuss this further when we consider variational principles.

Because the equation for W, T separates from the others we can regard the free oscillations of the Earth as being separated into two groups. The first group consists of *spheroidal* oscillations with displacements of the form

$$\hat{\mathbf{r}}UY_l^m + \hat{\boldsymbol{\theta}}V\frac{\partial Y_l^m}{\partial\theta} + \hat{\boldsymbol{\phi}}\operatorname{cosec}\theta V imY_l^m \tag{3.24}$$

and the second group consists of *toroidal* oscillations with displacements of the form

$$\hat{\boldsymbol{\theta}}\operatorname{cosec}\theta W imY_l^m - \hat{\boldsymbol{\phi}}W\frac{\partial Y_l^m}{\partial\theta} \tag{3.25}$$

Note that toroidal free oscillations have no vertical component of motion and no dilatation thus they do not perturb the gravitational field and their equation of motion is very simple. The motion (3.25) can be thought of as a twisting motion on concentric shells.

Another reason for our choice of vector in equation 3.21 is that the boundary conditions are easy to implement. Note that the scalars U, V and W are proportional to the components of the displacement field and so satisfy all the boundary conditions appropriate for $\mathbf{s}$. Similarly, R, S and T are proportional to the components of the traction vector and so must satisfy the boundary conditions on $\mathbf{t}$. Φ_1 is proportional to the perturbation in gravitational potential, ϕ_1 and so is continuous throughout the model and Ψ_1 has been carefully chosen to be continuous throughout the model and zero at the free surface.

Inspection of equation 3.22 shows that this form is not useful in a fluid region where L and N are zero. Of course, the shear traction is also zero (y_5), so in an isotropic fluid, 3.21 becomes

$$\frac{dy_1}{dr} = -\frac{1}{r}y_1 + \frac{\mathcal{L}}{r}y_2 + \frac{y_4}{\lambda} \tag{a}$$

$$\frac{dy_3}{dr} = -4\pi G\rho_0 y_1 - \frac{ly_3}{r} + 4\pi G y_6 \tag{b}$$

$$\frac{dy_4}{dr} = (-\rho_0\omega_k^2 - \frac{4g_0\rho_0}{r})y_1 + \frac{\mathcal{L}}{r}\rho_0 g_0 y_2 - \rho_0\frac{(l+1)}{r}y_3 + \frac{y_4}{r} + 4\pi G\rho_0 y_6 \tag{c}$$

$$0 = \frac{\mathcal{L}}{r}\rho_0 g_0 y_1 - \rho_0\omega_k^2 y_2 + \frac{\rho_0\mathcal{L}}{r}y_3 - \frac{\mathcal{L}}{r}y_4 \tag{d}$$

$$\frac{dy_6}{dr} = -\rho_0\frac{(l+1)}{r}y_1 + \frac{\rho_0\mathcal{L}}{r}y_2 + \frac{l}{r}y_6 \tag{e}$$

where $\mathcal{L} = \sqrt{l(l+1)}$.

Equation (d) allows y_2 to be cast in terms of y_1, y_3, and y_4. y_2 can then be eliminated from (a), (b), (c), and (e). We end up with a system of four equations to solve:

$$\frac{d}{dr}\begin{bmatrix} y_1 \\ y_3 \\ y_4 \\ y_6 \end{bmatrix} = \begin{bmatrix} g_0\beta - \frac{1}{r} & \beta & \frac{1}{\lambda} - \frac{\beta}{\rho_0} & 0 \\ -4\pi G\rho_0 & -\frac{l}{r} & 0 & 4\pi G \\ -\rho_0\left[\omega_k^2 + \frac{4g_0}{r} - g_0^2\beta\right] & \rho_0\left[g_0\beta - \frac{(l+1)}{r}\right] & \frac{1}{r} - g_0\beta & 4\pi G\rho_0 \\ \rho_0\left[g_0\beta - \frac{(l+1)}{r}\right] & \rho_0\beta & -\beta & \frac{l}{r} \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \\ y_4 \\ y_6 \end{bmatrix} \tag{3.26}$$

where $\beta = \mathcal{L}^2/(r^2\omega_k^2)$. Equation 3.26 has the same symmetry as 3.21 and there are only 9 independent elements of the 4×4 coefficient matrix. There are two independent solutions which are regular at the origin and the general solution is a linear combination of them.

Another special case of the spheroidal mode equations is when $l = 0$. First, we note that $Y_0^0(\theta, \phi)$ is a constant and so

$$\frac{\partial Y_0}{\partial \theta} = \frac{\partial Y_0}{\partial \phi} = 0$$

Thus, From equation 3.7 it follows that when $l = 0$

$$\mathbf{s}_k = \hat{\mathbf{r}}\ {}_k U_0(r) Y_0$$

i.e., these oscillations only have a radial component of motion and are therefore called *radial* oscillations. Thus V and S are zero and equation 3.21 reduces to

$$\frac{dy_1}{dr} = -\left(\frac{2\text{F}}{\text{C}r} - \frac{1}{r}\right) y_1 + \frac{y_4}{\text{C}} \tag{a}$$

$$\frac{dy_3}{dr} = -4\pi G\rho_0 y_1 + 4\pi G y_6 \tag{b}$$

$$\frac{dy_4}{dr} = \left[-\rho_0\omega_k^2 + \frac{4}{r^2}(\gamma - r g_0\rho_0)\right] y_1 - \frac{\rho_0}{r} y_3 + \left(\frac{2\text{F}}{\text{C}r} - \frac{1}{r}\right) y_4 + 4\pi G\rho_0 y_6 \tag{c}$$

$$\frac{dy_6}{dr} = -\frac{\rho_0}{r} y_1 \tag{d}$$

Combining (b) and (d) gives

$$\frac{dy_3}{dr} = 4\pi G r \frac{dy_6}{dr} + 4\pi G y_6 = 4\pi G \frac{d}{dr}(r y_6)$$

and because both y_3 and y_6 are zero at infinity, we can integrate to give $y_3 = 4\pi G r y_6$. This result allows us to decouple (a) and (c) from (b) and (d) and we obtain

$$\left.\begin{aligned} \frac{dy_1}{dr} &= -\left(\frac{2\text{F}}{\text{C}r} - \frac{1}{r}\right) y_1 + \frac{y_4}{\text{C}} \\ \frac{dy_4}{dr} &= \left[-\rho_0\omega_k^2 + \frac{4}{r^2}(\gamma - r g_0\rho_0)\right] y_1 + \left(\frac{2\text{F}}{\text{C}r} - \frac{1}{r}\right) y_4 \end{aligned}\right\} \tag{3.27}$$

$$\text{with} \quad \frac{dy_6}{dr} = -\frac{\rho_0}{r} y_1 \quad \text{and} \quad y_3 = 4\pi G r y_6$$

where $y_1 = rU, y_4 = rR, y_3 = r\Phi_1,$ and $y_6 = r\Psi_1/4\pi G$.

3.3 The Cowling approximation . Solving the full spheroidal mode problem with self-gravitation is computationally tricky and we expect that gravitational effects should become less important at high frequencies where the elastic forces will dominate. A common approximation, therefore, is to set ϕ_1 equal to zero in the equations of motion. We do however keep the gravitational buoyancy and advection of stress terms (the last two terms on the right hand side of 3.2). In this approximation, Φ_1 and $d\Phi_1/dr$ are both set to zero and $\Psi_1 = 4\pi G\rho_0 U$. Equation 3.21 now reduces to a system of four equations:

$$\mathbf{y} = \begin{bmatrix} rU \\ rV\mathcal{L} \\ rR \\ rS\mathcal{L} \end{bmatrix}$$

then the matrix, $\mathbf{A}$ is given by

$$\begin{bmatrix} -\frac{2\mathrm{F}}{\mathrm{C}r} + \frac{1}{r} & \mathcal{L}\frac{\mathrm{F}}{\mathrm{C}r} & \frac{1}{\mathrm{C}} & 0 \\ -\frac{\mathcal{L}}{r} & \frac{2}{r} & 0 & \frac{1}{\mathrm{L}} \\ -\rho_0\omega_k^2 + \frac{4}{r^2}(\gamma - rg_0\rho_0) + 4\pi G\rho_0 & \frac{\mathcal{L}}{r^2}(rg_0\rho_0 - 2\gamma) & \frac{2\mathrm{F}}{\mathrm{C}r} - \frac{1}{r} & \frac{\mathcal{L}}{r} \\ \frac{\mathcal{L}}{r^2}(rg_0\rho_0 - 2\gamma) & -\rho_0\omega_k^2 + \frac{\mathcal{L}^2(\gamma+\mathrm{N})-2\mathrm{N}}{r^2} & -\mathcal{L}\frac{\mathrm{F}}{\mathrm{C}r} & -\frac{2}{r} \end{bmatrix} \tag{3.28}$$

which has the same symmetry as before. In a fluid, 3.26 reduces to a system of two coupled equations:

$$\frac{d}{dr}\begin{bmatrix} y_1 \\ y_3 \end{bmatrix} = \begin{bmatrix} g_0\beta - \frac{1}{r} & \frac{1}{\lambda} - \frac{\beta}{\rho_0} \\ -\rho_0\left[\omega_k^2 + \frac{4g_0}{r} - g_0^2\beta\right] & \frac{1}{r} - g_0\beta \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \end{bmatrix} \tag{3.29}$$

where y_1 and y_3 are defined as in 3.28 and β is as defined in 3.26.

In summary, for toroidal modes, we solve the system of equations 3.23; for radial modes, we solve 3.27; for spheroidal modes with self gravitation, we solve 3.21 (solid) and 3.26 (fluid); and for spheroidal modes without self-gravitation, we solve 3.28 (solid) and 3.29 (fluid). In general, the parameters that describe the properties of the spherical Earth will not be simple analytic functions and so we must solve the systems numerically. In the following, we outline the simplest numerical algorithms for the different cases though we shall return to spheroidal modes later on to see how to handle some numerical problems.

3.4 Numerical solution for toroidal modes . Consider toroidal oscillations in the mantle (3.22). The traction vector is continuous at all solid/solid interfaces and is zero at the free surface or in a fluid. This is accomplished by making T continuous at all solid/solid interfaces and vanish in a fluid or at the free surface. Similarly, the displacement field must be continuous at all solid/solid interfaces so W must be continuous.

We can now design a solution method for equation 3.22. We choose a particular value of l and ω_k and construct the coefficient matrix in equation 3.22. We integrate from the core-mantle boundary to the ocean floor (or the free surface if the model has no ocean). This integration over depth can be accomplished by using a Runge-Kutta scheme for example and a good starting solution for the vector $[rW, rT]$ would be $[1, 0]$ at the core mantle boundary. rW and rT are both continuous at all solid/solid interfaces within the mantle. At the ocean floor (or free surface) rT should be zero. If rT is not zero at the ocean floor (or free surface) we must repeat the integration with a different value of ω_k until we find a value at which rT is zero at the surface. There are an infinite number of such ω_k's for every value of l.

A plot of acceptable ω_k's (*i.e.*, ones for which the boundary conditions are satisfied) versus l for a typical Earth model is shown in Figure 3.2. The lowest frequency ω_k for each l is given the label $n = 0$ (the fundamental mode), the next highest frequency ω_k for a particular l is labelled $n = 1$ and so on. Each acceptable mode of oscillation is therefore given the label

$${}_nT_l$$

where n is called the overtone index and l is the harmonic degree of the mode. As noted previously, m does not appear in equation 3.22 thus for each n, l value there are $2l+1$ $(-l \leq m \leq +l)$ oscillations with exactly the same frequency. The displacement field of each of these oscillations has the form (equation 3.7)

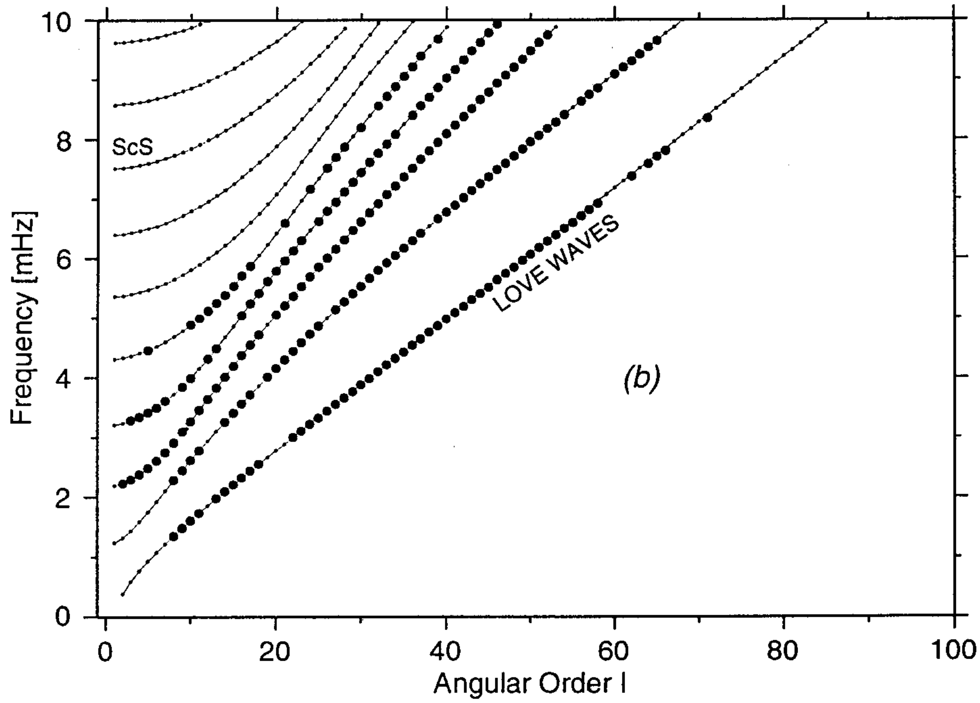

Figure 3.2 ω/ℓ plot for toroidal modes. Large symbols indicate modes which have been observed

$$ {}_n\mathbf{s}_l^m = \left[\hat{\boldsymbol{\theta}} \operatorname{cosec}\theta \; {}_nW_l(r) \frac{\partial Y_l^m}{\partial \phi} - \hat{\boldsymbol{\phi}} \; {}_nW_l(r) \frac{\partial Y_l^m}{\partial \theta} \right] e^{i \; {}_n\omega_l t} \quad (-l \leq m \leq l) $$

The fact that there are $2l+1$ oscillations with exactly the same frequency is the familiar phenomenon of degeneracy and is the result of the spherical symmetry of the model. The group of $2l+1$ oscillations with the same degenerate frequency is called a *multiplet* while a single oscillation is called a *singlet*. In the case of toroidal modes, n is equivalent to the number of nodes in radius of the function ${}_nW_l(r)$. Thus ${}_0T_l$ modes are called fundamental modes and have no nodes in radius. As l increases, ${}_0W_l(r)$ becomes increasingly concentrated near the surface so these modes in the typical frequency band of interest ($\omega > 2$ mHz) sample only the upper mantle. As n increases for a particular l, ${}_nW_l(r)$ is progressively wigglier and, as a rule, samples deeper into the mantle. Thus overtone toroidal modes can be used to infer the density and shear modulus of the whole mantle.

Toroidal modes also exist in the inner core but are neither excited nor observed by sources or receivers at or near the surface. The algorithm described above can be used to compute the toroidal modes of the inner core except that we must use a different starting solution which is valid near the center of the Earth. To do this, we approximate a region near the center of the Earth as a homogeneous isotropic sphere. In this case, the third equation of 3.12 can be written as

$$ \frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} + W\left(\frac{\omega_k^2}{V_s^2} - \frac{l(l+1)}{r^2}\right) = 0 \tag{3.30} $$

This is the equation for spherical Bessel functions with solutions: $j_l(kr)$ and $y_l(kr)$ where $k = \omega/V_s$ (Appendix B). In general, W is a linear combination of j_l and y_l, i.e.,

$$W(r) = Aj_l(kr) + By_l(kr)$$

For our homogeneous sphere, $B = 0$ because the solutions must be regular at the origin. Recursion relations for the spherical Bessel functions can be used to compute the radial derivative of W so it is straightforward to compute T and so get a complete starting solution.

3.5 Numerical solution for radial modes . Equation 3.26 can be integrated from the center of the Earth to the surface. Because y_1 is directly related to the radial component of displacement, it is continuous at all interfaces. Similarly y_4 is directly related to τ_{rr} (equation A45) and so must be continuous at all interfaces and zero at the free surface. The procedure for solving (3.26) is thus similar to that for solving the toroidal mode equations. A trial value of ω_k is chosen and a starting solution constructed at the center of the Earth. The solution is integrated to the surface making y_1 and y_4 continuous at all interfaces and the value of y_4 is checked at the surface. ω_k is varied until y_4 is zero at the surface (all boundary conditions are then satisfied). There are an infinite number of ω_k's at which y_4 is zero at the free surface. The solution with the lowest value of frequency is labeled ${}_0S_0$ and higher frequency solutions are labeled ${}_nS_0$ with $n = 1, \ldots, \infty$. The displacement field of the n'th overtone has the form

$${}_n\mathbf{s}_0 = [\hat{\mathbf{r}}\ {}_nU_0(r)Y_0]e^{i\ {}_n\omega_0 t}$$

Because $l = 0$, there is only one oscillation for each value of n.

As in the inner core toroidal mode case, we construct a staring solution by considering a very small region about the center of the Earth and approximating it as a homogeneous sphere. Consider the first of the equations in (3.12) (or we can combine equations 3.26 into a single second-order differential equation). For simplicity, we consider an elastically isotropic sphere, then in the *homogeneous* case, we have

$$(\lambda + 2\mu)\left[\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} - \frac{2}{r^2}U\right] + \rho_0\left[\frac{2Ug_0}{r} - U\frac{dg_0}{dr} - \frac{d\Phi_1}{dr} + \omega_k^2 U\right] = 0 \tag{3.31}$$

(from equation (3.12)) and

$$\frac{1}{r^2}\left(\frac{d}{dr}r^2\frac{d\Phi_1}{dr}\right) = -4\pi G\rho_0\left[\frac{dU}{dr} + \frac{2U}{r}\right] = \frac{-4\pi G\rho_0}{r^2}\left(\frac{d}{dr}r^2 U\right)$$

(from equation (3.10)). The second of these can be integrated directly to give

$$\frac{d\Phi_1}{dr} = -4\pi G\rho_0 U$$

In a homogeneous sphere we also have that

$$g_0(r) = \frac{4}{3}\pi G\rho_0 r \quad \text{so} \quad \frac{dg_0}{dr} = \frac{4}{3}\pi G\rho_0$$

Substitution into 3.30 gives

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} + \left[k^2 - \frac{2}{r^2}\right]U = 0 \tag{3.32}$$

where

$$k^2 = \rho_0\frac{\frac{16}{3}\pi G\rho_0 + \omega_k^2}{\lambda + 2\mu} \tag{3.33}$$

Equation 3.31 is the differential equation satisfied by spherical Bessel functions of the first order. Because the solution is regular at the origin we may take

$$U = A\left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right) = Aj_1(kr) \tag{3.34}$$

where A is an arbitrary constant. This may be readily differentiated to give R and so we can easily construct the starting solution y_1, y_4 which is equivalent to $r_0U(r_0)$, $r_0R(r_0)$ where r_0 is some small distance from the origin.

Finally, we note that the equation for k (3.32) has a gravitational term, $16/3\pi G\rho_0$, which has units of a frequency squared and corresponds to a frequency of about 0.4mHz. Clearly, when ω_k is much greater than this value, the effect of gravity will be unimportant.

3.6 Numerical solution for spheroidal modes . The case of a general spheroidal oscillation requires the use of 3.20 and 3.25. Note that there are three solutions of these equations which are regular at the origin and the general solution is a linear combination of these three. The most obvious method of solving the spheroidal mode equations is as follows. Choose an l and a value for ω_k.

1) Find the three independent solutions of 3.20 near $r = 0$ assuming that the Earth can be modelled as a homogeneous sphere here. The details of this calculation were first given by Pekeris and Jarosch (1958) and a surfeit of detail is given in Lapwood and Usami (1981).
2) Integrate all three solutions to the inner core boundary. Equation A45 shows that, for a spheroidal mode

$$\begin{aligned} \tau_{r\theta} =& \mathrm{L}X\frac{\partial Y_l^m}{\partial\theta} = S\frac{\partial Y_l^m}{\partial\theta} \\ \text{and}\quad \tau_{r\phi} =& im\,\mathrm{cosec}\,\theta \mathrm{L}XY_l^m = \mathrm{cosec}\,\theta S\frac{\partial Y_l^m}{\partial\phi} \\ \text{and}\quad \tau_{rr} =& RY_l^m \quad \text{(as for radial modes)} \end{aligned}$$

Therefore, at the inner core boundary (a fluid/solid interface), R is continuous, S is zero, and U is continuous – Φ_1 and Ψ_1 are continuous at every interface. It follows from the definition of $\mathbf{y}$ that y_1, y_3, y_4, and y_6 are continuous and y_5 is zero at a fluid/solid interface.
There are two independent linear combinations of the three solutions at the top of the inner core which satisfy $y_5 = 0$. We compute these two solutions and use equation 3.25 to integrate the solution through the fluid outer core boundary.
3) At the mantle core boundary we now have two four vectors $(y_1^1, y_3^1, y_4^1, y_6^1)$ and $(y_1^2, y_3^2, y_4^2, y_6^2)$. We construct three independent vectors at the base of the solid mantle.

$$\begin{bmatrix} y_2^1 \\ 0 \\ y_3^1 \\ y_4^1 \\ 0 \\ y_6^1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_1^2 \\ 0 \\ y_3^2 \\ y_4^2 \\ 0 \\ y_6^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Any combination of these three vectors satisfies the boundary condition that y_5 be zero and the last vector has been added because we do not yet know what the value of y_2 is at the base of the mantle. These three solutions can be integrated to the ocean floor or to the free surface (if there is no ocean). All the y's are continuous at solid/solid interfaces. If there is an ocean, we must proceed as we did at the inner core/outer core boundary. If there is no ocean we consider a linear combination of our three vectors at the free surface. If we have chosen ω_k correctly then R, S, and Ψ_1 should all be zero at the free surface or, equivalently, y_4, y_5, and y_6 will all be zero. Thus we require

$$\mathbf{y} = a_1\mathbf{y}^1 + a_2\mathbf{y}^2 + a_3\mathbf{y}^3$$

where the a's are arbitrary constants such that

$$a_1\begin{bmatrix} y_4^1 \\ y_5^1 \\ y_6^1 \end{bmatrix} + a_2\begin{bmatrix} y_4^2 \\ y_5^2 \\ y_6^2 \end{bmatrix} + a_3\begin{bmatrix} y_4^3 \\ y_5^3 \\ y_6^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or equivalently

$$\det \begin{bmatrix} y_4^1 & y_4^2 & y_4^3 \\ y_5^1 & y_5^2 & y_5^3 \\ y_6^1 & y_6^2 & y_6^3 \end{bmatrix} = 0 \quad (3.35)$$

If this is not true we must start again with a new value of ω_k.

There are many numerical diffculties associated with this calculation and, as we shall see later, there are better ways to compute the solution.

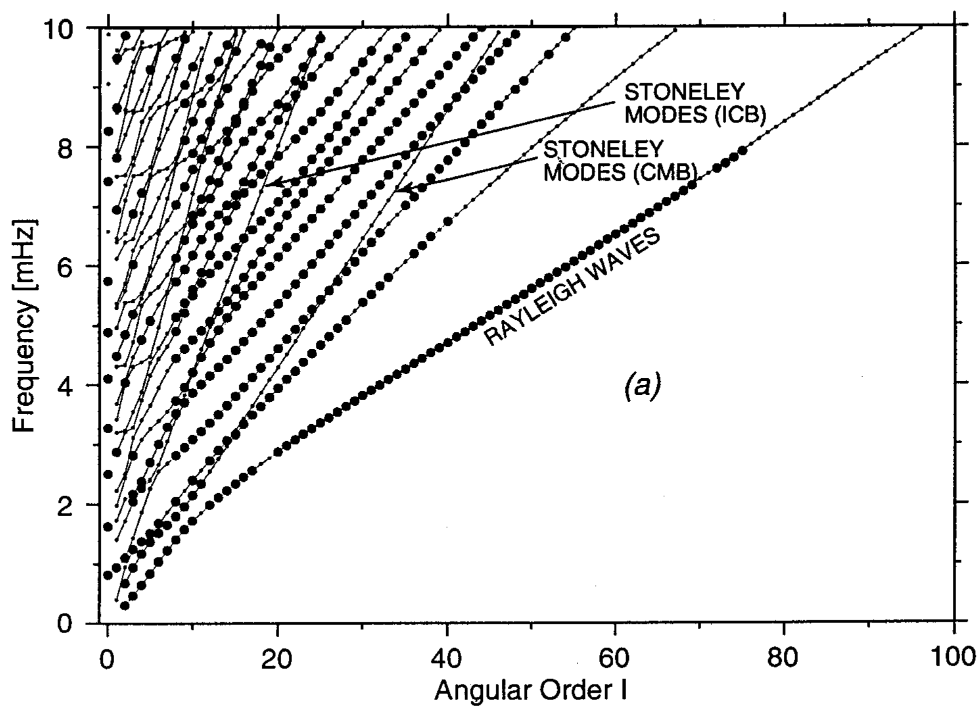

Figure 3.3 ω/ℓ plot for spheroidal modes. Large symbols indicate modes which have been observed

The labeling of spheroidal modes, ${}_nS_l$ follows that of the toroidal and radial modes and the frequencies of these modes for a particular Earth model are shown as a function of l in Figure 3.3. As with toroidal modes, there are $2l + 1$ free oscillations with the frequency ${}_n\omega_l$ (because m does not appear in the coefficient matrix of 3.20). Each of these oscillations has a displacement field of the form

$${}_n\mathbf{u}_l^{Sm} = \left[\hat{\mathbf{r}}_n U_l(r) Y_l^m(\theta,\phi) + \hat{\boldsymbol{\theta}}_n V_l(r) \frac{\partial Y_l^m}{\partial \theta}(\theta,\phi) + \operatorname{cosec}\theta \hat{\boldsymbol{\phi}}_n V_l(r) \frac{\partial Y_l^m}{\partial \phi}(\theta,\phi)\right] e^{i_n\omega_l t}$$

Remember that $-l \le m \le l$ and that the $2l + 1$ degeneracy stems from the spherical symmetry of the model.

3.7 Some Simple Solutions . We can find analytic forms for U, V, and W in 3.12 if we have some suitably chosen form of the model parameters as a function of radius. Homogeneous shells are a convenient approximation though physically implausible. Of course, a number of thin homogeneous shells can be used to approximate a continuously varying structure. For simplicity we look first at toroidal modes of an elastically isotropic body, *i.e.*,

$$\frac{d}{dr}(\mu Z) = \frac{dT}{dr} = -\frac{\mu}{r}\left(3Z - \frac{W}{r}(l+2)(l-1)\right) - \rho_0\omega_k^2 W$$

$$\text{where} \quad Z = \frac{dW}{dr} - \frac{W}{r}$$

If μ is a constant this becomes (3.29)

$$\frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} + W\left(\frac{\omega_k^2}{V_s^2} - \frac{l(l+1)}{r^2}\right) = 0$$

This is the equation for spherical Bessel functions with solutions: $j_l(kr)$ and $y_l(kr)$ where $k = \omega/V_s$ (Appendix B). In general, W is a linear combination of j_l and y_l, i.e.,

$$W(r) = Aj_l(kr) + By_l(kr)$$

If we consider a homogeneous *sphere*, $B = 0$ because the solutions must be regular at the origin. If the sphere is of radius a, the boundary conditions are satisfied if $T(a) = 0$. Now $T = \mu Z$ so equivalently we have $Z(a) = 0$ or

$$\frac{dW}{dX}(ka) = \frac{W}{X}(ka) \quad \text{where} \quad X = kr$$

Because $W \propto j_l(X)$ we have

$$\frac{dj_l}{dX} = \frac{j_l}{X} \quad \text{at} \quad X = ka = \frac{\omega}{V_s}a \tag{3.36}$$

Some explicit expressions for $j_l(X)$ for low values of l are given in Appendix B. Substitution into 3.35 gives a condition for roots, *i.e.*,

$$l = 1 \quad \rightarrow \quad \tan X = \frac{3X}{3 - X^2}$$

$$l = 2 \quad \rightarrow \quad \tan X = \frac{X^3 - 12X}{5X^2 - 12}$$

Because W is proportional to j_l we can also find the values of X at which W is zero. From Appendix B we get

$$l = 1 \quad \rightarrow \quad \tan X = X$$

$$l = 2 \quad \rightarrow \quad \tan X = \frac{3X}{3 - X^2}$$

We solve these equations graphically (Figures 3.4 and 3.5). We also indicate some calculated periods for $\overline{V_s} = 3.54$ km/sec and $a = 1220$ km which are appropriate values for the inner core. Consider the graph for $l = 1$ first. The first root occurs at $X \simeq 6$ while the first zero occurs at $X \simeq 4.5$. This root has an eigenfunction (W) with one zero so it is labeled ${}_1T_1$. The next root occurs at $X \simeq 9$ and has an eigenfunction with 2 zeroes. This is called ${}_2T_1$ and so on. As you can see, the modes are labeled ${}_nT_l$ where n is a counting index and, for toroidal modes, corresponds to the number of internal nodes in displacement. The situation for $l = 2$ is a little different. The first root has no nodes in radius (except the origin) and is called ${}_0T_2$. Note that ${}_0T_1$ corresponds to a rigid body rotation and has $\omega = 0$. We probably won't consider it any further (except to note that it can couple into ${}_0S_2$ through the rotation of the Earth and so can't be completely ignored).

Figure 3.6 shows the modes of oscillation of a homogeneous sphere where X (at a root) is plotted as a function of l. For each l there are an infinite number of modes. Also plotted are the roots for a homogeneous *shell* which is a better approximation to the mantle. If the inner radius of the shell is b and the outer radius is a then we must have

$$T(a) = T(b) = 0 \quad \text{where} \quad T = \mu\left(\frac{dW}{dr} - \frac{W}{r}\right) \quad \text{and} \quad W = Aj_l + By_l$$

Writing $j_l(ka)$ as j_l^a, etc., we find that

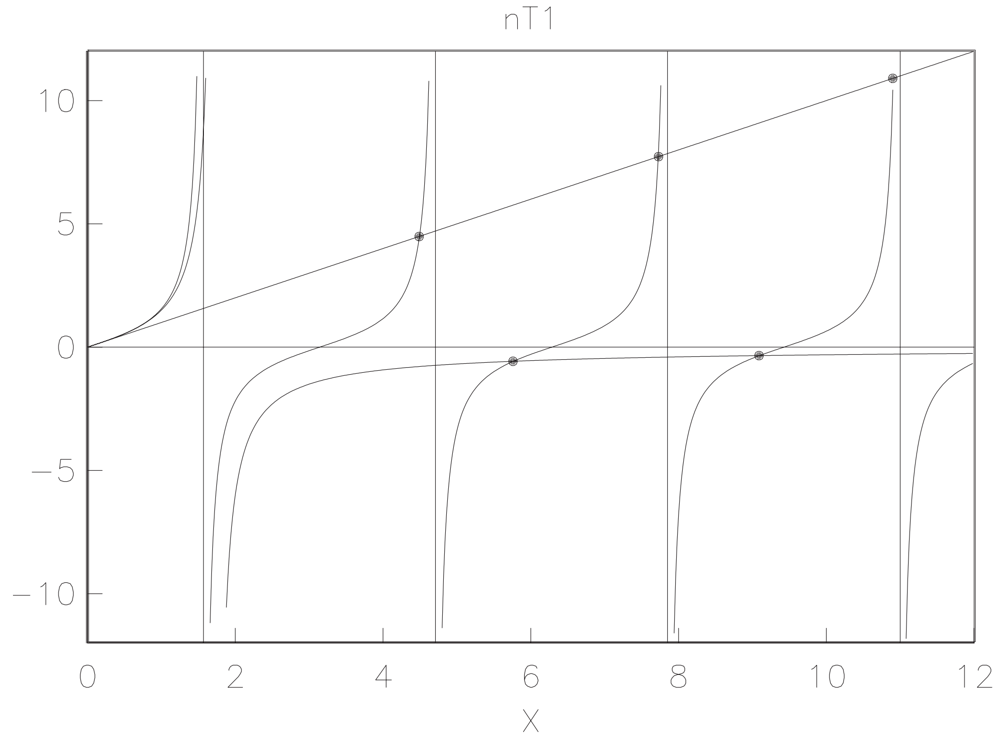

Figure 3.4 Graphical solutions for $\ell = 1$. The frequency roots are given by the equation $\tan X = 3X/(3 - X^2)$. For a mean shear velocity of 3.54 km/s and a radius of 1220 km we find that ${}_1T_1$ has a period of 375.5s; ${}_2T_1$ has a period of 238.2 s and ${}_3T_1$ has a period of 175 s. The equation $\tan X = X$ gives the location of the zeroes.

$$0 = A\left(\frac{dj_l^a}{dr} - \frac{j_l^a}{a}\right) + B\left(\frac{dy_l^a}{dr} - \frac{y_l^a}{a}\right)$$

and

$$0 = A\left(\frac{dj_l^b}{dr} - \frac{j_l^b}{b}\right) + B\left(\frac{dy_l^b}{dr} - \frac{y_l^b}{b}\right)$$

Eliminating A and B gives

$$\Delta = \left(\frac{dj_l^a}{dr} - \frac{j_l^a}{a}\right)\left(\frac{dy_l^b}{dr} - \frac{y_l^b}{b}\right) - \left(\frac{dj_l^b}{dr} - \frac{j_l^b}{b}\right)\left(\frac{dy_l^a}{dr} - \frac{y_l^a}{a}\right)$$

and when $\Delta = 0$, we have a solution which matches the boundary conditions. Δ is obviously a determinant and we plot it as a function of ω in Figure 3.7. We chose $V_s = 6.5$ km/s, $a = 6371$ km and $b = 3485$ km. The roots we obtain are quite close to the roots for a realistic mantle model though we see from Figure 3.6 that the results are quite different from the homogeneous sphere. The difference is that we now have a group of modes (low l, high n) which have very low group velocity, $(d\omega/dl)$. If we add up such modes we get a seismogram of arrivals which correspond to rays which travel near to vertical incidence across the mantle (ScS). We shall look at this in more detail in later sections.

As a final example for toroidal modes, we consider how we might solve the equations if we divide the mantle up into many homogeneous shells (rather than just one, as above). Equation 3.29 is still valid but V_s, μ and k will change from shell to shell. The solution in each shell will be of the form

$$W(r) = Aj_l(kr) + By_l(kr)$$

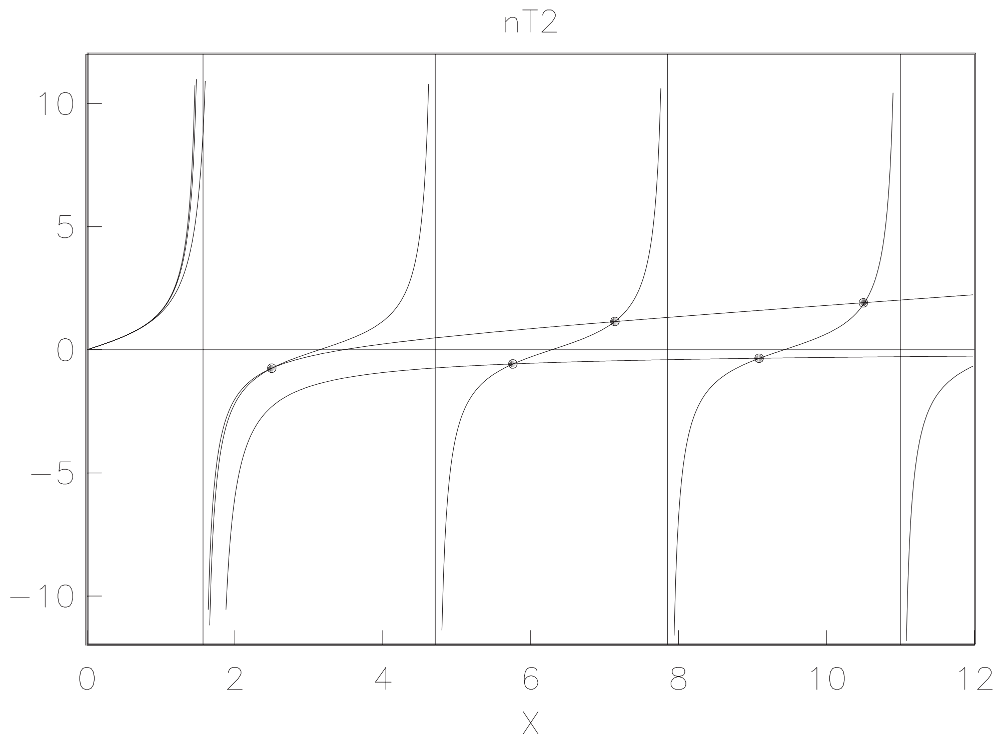

Figure 3.5 Graphical solutions for $\ell = 2$. The frequency roots are given by the equation $\tan X = (X^3 - 12X)/(5X^2 - 12)$. For a mean shear velocity of 3.54 km/s and a radius of 1220 km we find that ${}_0T_2$ has a period of 865.5s; ${}_1T_2$ has a period of 302.1 s and ${}_2T_2$ has a period of 205.8 s. The equation $\tan X = 3X/(3 - X^2)$ gives the location of the zeroes.

$$T(r) = A\mu\left(\frac{dj_l(kr)}{dr} - \frac{j_l(kr)}{r}\right) + B\mu\left(\frac{dy_l(kr)}{dr} - \frac{y_l(kr)}{r}\right)$$

Suppose r_b is the radius of the bottom of the shell and r_t is the radius of the top of the shell. Then we can write

$$\begin{bmatrix} W(r_b) \\ \\ T(r_b) \end{bmatrix} = \begin{bmatrix} j_l(kr_b) & y_l(kr_b) \\ \\ \mu\left(\frac{dj_l(kr_b)}{dr} - \frac{j_l(kr_b)}{r_b}\right) & \mu\left(\frac{dy_l(kr_b)}{dr} - \frac{y_l(kr_b)}{r_b}\right) \end{bmatrix} \quad \begin{bmatrix} A \\ \\ B \end{bmatrix} = \mathbf{F}(r_b) \cdot \begin{bmatrix} A \\ \\ B \end{bmatrix}$$

and

$$\begin{bmatrix} W(r_t) \\ \\ T(r_t) \end{bmatrix} = \begin{bmatrix} j_l(kr_t) & y_l(kr_t) \\ \\ \mu\left(\frac{dj_l(kr_t)}{dr} - \frac{j_l(kr_t)}{r_t}\right) & \mu\left(\frac{dy_l(kr_t)}{dr} - \frac{y_l(kr_t)}{r_t}\right) \end{bmatrix} \quad \begin{bmatrix} A \\ \\ B \end{bmatrix} = \mathbf{F}(r_t) \cdot \begin{bmatrix} A \\ \\ B \end{bmatrix}$$

so that

$$\begin{bmatrix} W(r_t) \\ \\ T(r_t) \end{bmatrix} = \mathbf{F}(r_t) \cdot \mathbf{F}^{-1}(r_b) \cdot \begin{bmatrix} W(r_b) \\ \\ T(r_b) \end{bmatrix} = \mathbf{P}(r_t, r_b) \cdot \begin{bmatrix} W(r_b) \\ \\ T(r_b) \end{bmatrix} \tag{3.37}$$

F is called a *fundamental* matrix and **P** is called a *propagator* matrix because it propagates the solution from one radius to another. If we were to divide the mantle up into many homogeneous shells, we could start with a solution $(W, T) = (1, 0)$ at the core mantle boundary and a trial frequency, ω_k. We then use 3.36 to propagate the solution through all the shells until we reach the surface. Once more, if T at the surface is

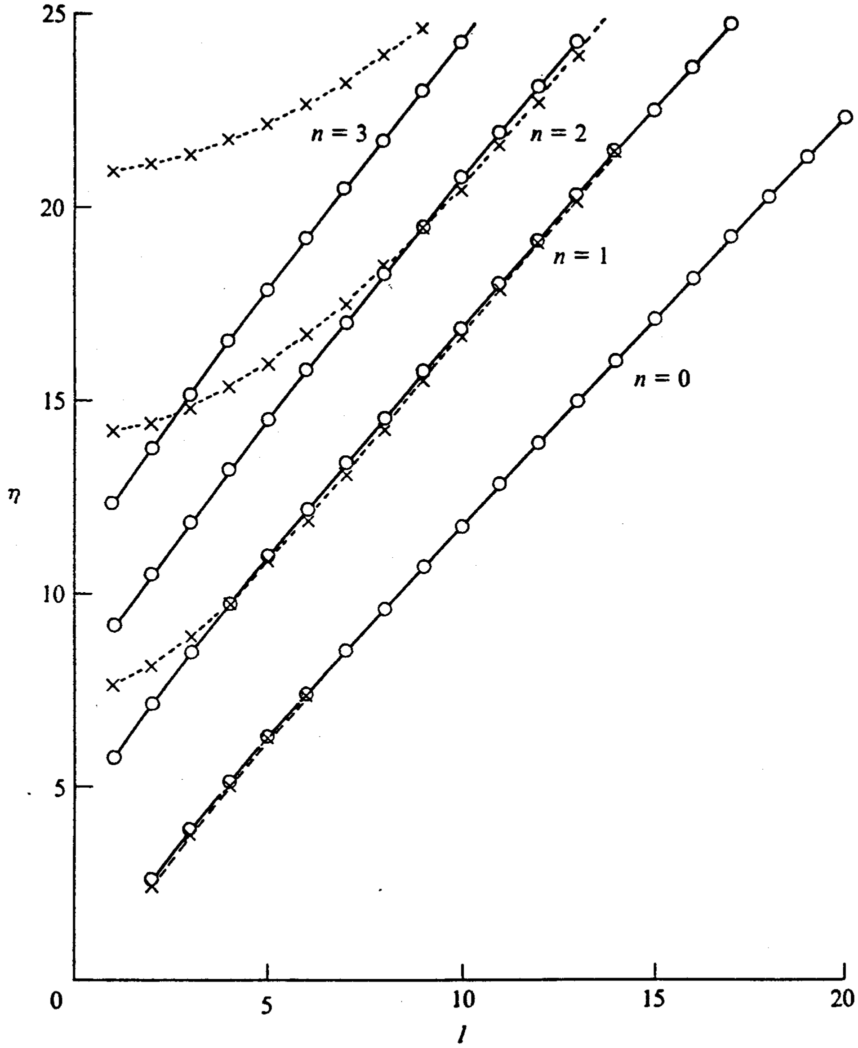

Figure 3.6 Non-dimensional frequency (our X) of toroidal oscillations as a function of ℓ. Open circles and solid curves refer to a uniform sphere and crosses and dashed curves to a uniform shell.

zero then we have a root – otherwise we have to modify our trial frequency and propagate through all the shells again.

As we have seen from equation 3.30, radial modes of a homogeneous sphere also have eigenfunctions which are spherical Bessel functions (proportional to $j_1(kr)$) which allows you to do the following problem:

Problem 3.1

What is the frequency spacing of radial mode overtones of a homogeneous sphere as $n \to \infty$ in terms of the compressional velocity of the sphere? (Finite n of order 10 should suffice!)

The spheroidal mode frequencies of a homogeneous sphere can be found in a similar way to the toroidal mode case but the algebra is much heavier. It turns out that g_0/r appears in the equations and is a constant for a sphere. Of course it may vary in a homogeneous shell so we cannot then get an exact analytic solution. We do not have to restrict ourselves to homogeneous shells. You might like to try and find how the material properties must vary as a function of radius if the eigenfunction is to be proportional to trigonometric functions or exponentials. This is possible for toroidal modes and almost possible for spheroidal modes

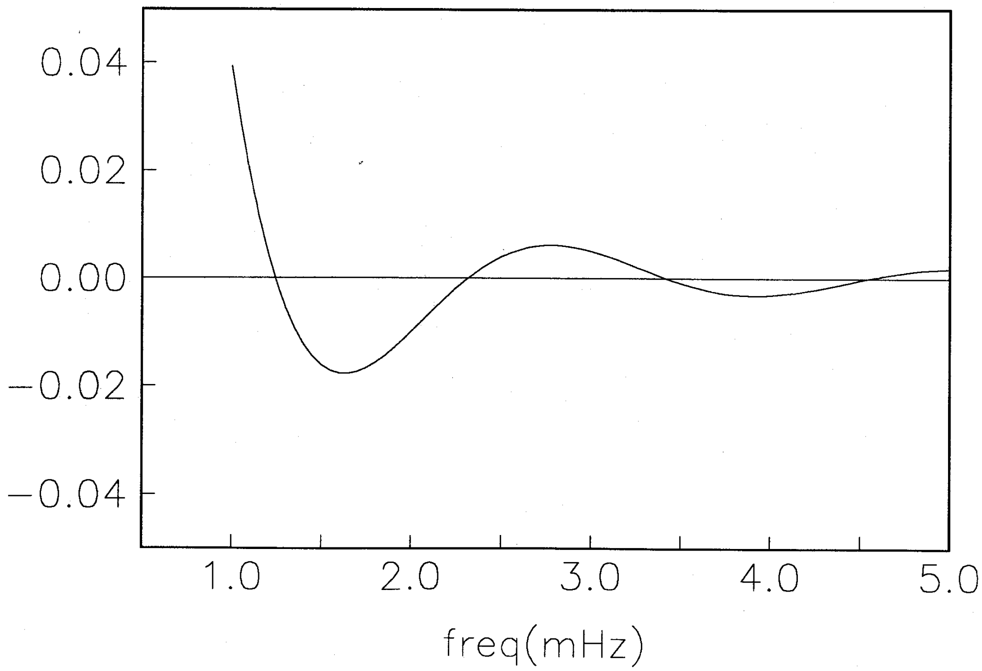

Figure 3.7 Determinant for $\ell = 1$ for a homogeneous shell with inner radius 3485km, outer radius 6371km and mean shear velocity of 6.5 km/s.

(with some assumptions about g). In any case, for a realistic Earth model, we will probably end up resorting to numerical calculation so some points about numerical solution are now discussed.

3.8 Numerical problems and the method of minors . Firstly, we are dealing with mixed dimensions so it makes sense to non-dimensionalize everything. A natural set of scales is

length $\rightarrow$ radius of the Earth, a
density $\rightarrow$ mean density of the Earth, $\bar{\rho}$
time $\rightarrow 1/\sqrt{\pi G \bar{\rho}}$ where G is Newton's constant.

This time unit is about 930 seconds which is reasonable for long-period calculation. Note that $g_0(a) = 4/3$ with these units.

We should also take care that $\mathbf{y}$ is chosen so that all the elements of the matrix $\mathbf{A}$ are similar in magnitude. This accounts for the $\sqrt{l(l+1)}$ scaling in y_2 and y_5 in equation 3.20.

The numerical solution for the radial and toroidal modes is simple and has already been described. The numerical solution for the spheroidal modes is complicated and suffers from some numerical instabilities. One problem is that the three solutions $\mathbf{y}^1$, $\mathbf{y}^2$, and $\mathbf{y}^3$ tend to become parallel (i.e., $\mathbf{y}^1 \propto \mathbf{y}^2$, etc.) thus the determinant is singular (numerically). In fact we end up differencing large, nearly equal numbers when we evaluate the determinant (3.34) which can end up giving us nonsense for the secular equation.

We get around this by working with the minors of the solutions. In a solid, we work with third-order minors. In a fluid, where we have only two independent solutions, we work with second-order minors. We illustrate the technique with a solid, but using Cowling's approximation so that we need only work with second-order minors. We denote the two independent solutions which are regular at the origin as $\mathbf{y}^1$ and $\mathbf{y}^2$. Each $\mathbf{y}$ consists of (y_1, y_2, y_3, y_4) and the last two elements of the vector (i.e., those proportional to R and S – see 3.27) must be zero at the free surface at a root. Thus if $\mathbf{y}^1$ and $\mathbf{y}^2$ are the two solutions we have at the surface at a root

$$y_3 = a_1 y_3^1 + a_2 y_3^2 = 0$$
$$y_4 = a_1 y_4^1 + a_2 y_4^2 = 0$$

or

$$\det \begin{bmatrix} y_3^1 & y_3^2 \\ y_4^1 & y_4^2 \end{bmatrix} = y_3^1 y_4^2 - y_4^1 y_3^2 = 0$$

This last step, the formation of the determinant, is the one that causes us numerical problems. To avoid this, we work directly with a vector of second-order minors, *i.e.*, let the minor vector of $[\mathbf{y}^1, \mathbf{y}^2]$ be

$$\begin{bmatrix} y_1^1, y_1^2 \\ y_2^1, y_2^2 \\ y_3^1, y_3^2 \\ y_4^1, y_4^2 \end{bmatrix} = \begin{bmatrix} y_1^1 y_2^2 - y_2^1 y_1^2 \\ y_1^1 y_3^2 - y_3^1 y_1^2 \\ y_1^1 y_4^2 - y_4^1 y_1^2 \\ y_2^1 y_3^2 - y_3^1 y_2^2 \\ y_2^1 y_4^2 - y_4^1 y_2^2 \\ y_3^1 y_4^2 - y_4^1 y_3^2 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}$$

$\mathbf{m}$ is the vector of all independent second-order minors of $\mathbf{y}^1$ with $\mathbf{y}^2$. Now $d\mathbf{y}/dr = \mathbf{A}\mathbf{y}$ where, from 3.25, $\mathbf{A}$ can be written:

$$\mathbf{A} = \begin{bmatrix} T_{11} & T_{12} & C_{11} & 0 \\ T_{21} & T_{22} & 0 & C_{22} \\ S_{11} & S_{12} & -T_{11} & -T_{21} \\ S_{12} & S_{22} & -T_{12} & -T_{22} \end{bmatrix}$$

Now differentiate $\mathbf{m}$ and substitute from the above equation giving

$$\frac{d\mathbf{m}}{dr} = \mathbf{B}\mathbf{m}$$

where

$$\mathbf{B} = \begin{bmatrix} T_{11}+T_{22} & 0 & C_{22} & -C_{11} & 0 & 0 \\ S_{12} & 0 & -T_{21} & T_{12} & 0 & 0 \\ S_{22} & -T_{12} & T_{11}-T_{22} & 0 & T_{12} & C_{11} \\ -S_{11} & T_{21} & 0 & -(T_{11}-T_{22}) & -T_{21} & -C_{22} \\ -S_{12} & 0 & T_{21} & -T_{12} & 0 & 0 \\ 0 & -S_{12} & S_{11} & -S_{22} & S_{12} & -(T_{11}+T_{22}) \end{bmatrix}$$

This system has some interesting properties (note the peculiar symmetry about the back diagonal). In particular we find that $dm_2/dr = -dm_5/dr$ and consideration of the starting solutions indicates that $m_2 = -m_5$ throughout the structure. We need therefore only propagate a 5-vector and vary ω until we find the values at which m_6 is zero at the surface. (Note that $m_6(r = a)$ is zero at a root.) There is an algebraic relationship between minors valid at all depths, *i.e.*,

$$m_1 m_6 - m_2 m_5 + m_3 m_4 \equiv m_1 m_6 + m_2^2 + m_3 m_4 = 0 \tag{3.38}$$

Unfortunately, this is nonlinear and so cannot be used to reduce the system of equations further. It does, however, serve as a basis for a mode counter which we discuss later.

The numerical procedure is now almost identical to the procedure for radial and toroidal modes except we no longer know what the eigenfunction is. In what follows we use some results from Woodhouse (1988). Central to the theory are the "spanning" matrices, $\mathbf{M}$ and $\tilde{\mathbf{M}}$ and Woodhouse gives a recipe for their construction. For the second order minors they are

$$\mathbf{M} = \begin{bmatrix} 0 & m_1 & m_2 & m_3 \\ -m_1 & 0 & m_4 & m_5 \\ -m_2 & -m_4 & 0 & m_6 \\ -m_3 & -m_5 & -m_6 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{M}} = \begin{bmatrix} 0 & m_6 & -m_5 & m_4 \\ -m_6 & 0 & m_3 & -m_2 \\ m_5 & -m_3 & 0 & m_1 \\ -m_4 & m_2 & -m_1 & 0 \end{bmatrix}$$

Note that $\mathbf{M}^T \cdot \tilde{\mathbf{M}} = 0$ which leads the the quadratic identity given above. Now form $\mathbf{N} = \mathbf{M}^T \boldsymbol{\Sigma}^{\mathbf{T}}$ and $\tilde{\mathbf{N}} = \boldsymbol{\Sigma}\tilde{\mathbf{M}}$ where

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

It turns out that, for the second order minors, $\mathbf{N} = \tilde{\mathbf{N}}$ where

$$\mathbf{N} = \begin{bmatrix} -m_2 & -m_3 & 0 & m_1 \\ -m_4 & m_2 & -m_1 & 0 \\ 0 & -m_6 & -m_2 & -m_4 \\ m_6 & 0 & -m_3 & m_2 \end{bmatrix}$$

and, in general,$\mathbf{N} \cdot \tilde{\mathbf{N}} = 0$. Algebraically, it is easy to verify that $\mathbf{N} \cdot \mathbf{y} = 0$ where $\mathbf{y}$ is an eigenfunction that satisfies the boundary conditions. In fact, a direct way to find the expression for $\mathbf{N}$ is to use the fact that a_1 and a_2 can be eliminated in four different ways from $\mathbf{y} = a_1\mathbf{y}^1 + a_2\mathbf{y}^2$ which results in $\mathbf{N} \cdot \mathbf{y} = 0$.

We now integrate a solution to the original equations downwards with an arbitrary starting value. Label this solution as $\mathbf{x}$ and let $\mathbf{x}(a) = (1, 0, 0, 0)$ say. At every depth form

$$\mathbf{b} = \mathbf{N} \cdot \mathbf{x} \quad \text{where} \quad \frac{d\mathbf{x}}{dr} = \mathbf{A}\mathbf{x}$$

Using results from Woodhouse (1988) we find that for the second order system

$$\frac{d\mathbf{N}}{dr} = \mathbf{A}\mathbf{N} - \mathbf{N}\mathbf{A}$$

so differentiating the expression for $\mathbf{b}$ gives

$$\frac{d\mathbf{b}}{dr} = \frac{d\mathbf{N}}{dr}\mathbf{x} + \mathbf{N}\frac{d\mathbf{x}}{dr} = \mathbf{A}\mathbf{N}\mathbf{x} - \mathbf{N}\mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{N}\mathbf{x} = \mathbf{A}\mathbf{b}$$

Thus $\mathbf{b}$ satisfies the differential equation and $\mathbf{N} \cdot \mathbf{b} = 0$ because $\mathbf{N} \cdot \tilde{\mathbf{N}} = 0$. This condition means that all boundary conditions are satisfied. $\mathbf{b}$ is therefore our desired eigenvector. The merit of this method is that it is numerically stable. Since $\mathbf{N}$ is made up of upgoing minor vector elements, it is proportional to squares of the original vector so the product $\mathbf{b} = \mathbf{N} \cdot \mathbf{x}$ has the correct numerical behavior to capture solutions which are exponentially growing upwards at depth. Certain modes that are trapped on interfaces can be exponentially growing downwards (Stoneley modes and inner core modes) and this algorithm can fail for these at high enough frequency. Woodhouse (1988) gives an alternative construction of the eigenfunction which performs better in these cases.

The calculation of minor vectors for the solid system with self gravitation is much more complicated (see Gilbert and Backus, 1966). The $\mathbf{m}$ vector is now 20 elements long, *i.e.*,

$$m_1 = y_1^1 y_2^2 y_3^3 + y_2^1 y_3^2 y_1^3 + y_3^1 y_1^2 y_2^3 - y_1^1 y_3^2 y_2^3 - y_2^1 y_1^2 y_3^3 - y_3^1 y_2^2 y_1^3$$
$$\vdots$$
$$m_{20} = y_4^1 y_5^2 y_6^3 + y_5^1 y_6^2 y_4^3 + y_6^1 y_4^2 y_5^3 - y_4^1 y_6^2 y_5^3 - y_5^1 y_4^2 y_6^3 - y_6^1 y_5^2 y_4^3$$

This vector is all possible independent third-order minors of the three solutions, *i.e.*,

$$\begin{matrix} Row1 \\ Row2 \\ Row3 \\ Row4 \\ Row5 \\ Row6 \end{matrix} \quad \begin{bmatrix} y_1^1 & y_1^2 & y_1^3 \\ y_2^1 & y_2^2 & y_2^3 \\ y_3^1 & y_3^2 & y_3^3 \\ y_4^1 & y_4^2 & y_4^3 \\ y_5^1 & y_5^2 & y_5^3 \\ y_6^1 & y_6^2 & y_6^3 \end{bmatrix}$$

Now compute third-order determinants using the following row combinations: 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456. Note that m_{20} is the minor vector element that should be zero at the free surface if we are at a root. There are only 14 independent minors because

$$\begin{aligned} m_2 &= m_{13} \\ m_3 &= -m_7 \\ m_5 &= -m_{12} \\ m_8 &= m_{19} \\ m_9 &= -m_{16} \\ \text{and} \quad m_{14} &= -m_{18} \end{aligned}$$

Elimination of these redundant elements gives a differential system:

$$\begin{aligned}
m_1' &= (T_{11} + T_{22} + T_{33})m_1 + C_{33}m_4 - C_{22}m_6 + C_{11}m_{11} \\
m_2' &= S_{13}m_1 + T_{22}m_2 - T_{21}m_3 - T_{31}m_4 - C_{22}m_8 \\
m_3' &= S_{23}m_1 - T_{12}m_2 + T_{11}m_3 + C_{11}m_{18} \\
m_4' &= (T_{11} + T_{22} - T_{33})m_4 + C_{22}m_{10} - C_{11}m_{15} \\
m_5' &= -S_{12}m_1 + T_{31}m_3 + T_{33}m_5 - T_{21}m_6 - C_{33}m_9 + T_{12}m_{11} \\
m_6' &= -S_{22}m_1 - 2T_{12}m_5 + (T_{11} - T_{22} + T_{33})m_6 - C_{33}m_{10} - C_{11}m_{17} \\
m_8' &= -S_{22}m_2 + S_{12}m_3 - S_{23}m_5 + S_{13}m_6 - T_{22}m_8 + T_{31}m_{10} - T_{12}m_{18} \\
m_9' &= -S_{23}m_2 - S_{13}m_3 + S_{12}m_4 - T_{33}m_9 - T_{21}m_{10} + T_{12}m_{15} \\
m_{10}' &= -2S_{23}m_3 + S_{22}m_4 - 2T_{12}m_9 + (T_{11} - T_{22} - T_{33})m_{10} + C_{11}m_{20} \\
m_{11}' &= S_{11}m_1 - 2T_{31}m_2 + 2T_{21}m_5 + (T_{33} + T_{22} - T_{11})m_{11} - C_{33}m_{15} + C_{22}m_{17} \\
m_{15}' &= 2S_{13}m_2 - S_{11}m_4 + 2T_{21}m_9 + (T_{22} - T_{11} - T_{33})m_{15} - C_{22}m_{20} \\
m_{17}' &= 2S_{12}m_5 - S_{11}m_6 + 2T_{31}m_8 + S_{22}m_{11} + (T_{33} - T_{11} - T_{22})m_{17} + C_{33}m_{20} \\
m_{18}' &= -S_{12}m_2 + S_{11}m_3 + S_{13}m_5 - T_{21}m_8 + T_{31}m_9 + S_{23}m_{11} - T_{11}m_{18} \\
m_{20}' &= 2S_{13}m_8 - 2S_{12}m_9 + S_{11}m_{10} - S_{22}m_{15} - 2S_{23}m_{18} - (T_{11} + T_{22} + T_{33})m_{20}
\end{aligned}$$

The minor vectors in a fluid must be converted to a minor vector in the solid when integrating from fluid to solid (and vice versa). The relationships needed to do this are

Fluid	$\rightarrow$	Solid
m_1	$\rightarrow$	$-m_6$
m_2	$\rightarrow$	$-m_8$
m_3	$\rightarrow$	m_{10}
m_4	$\rightarrow$	$-m_{17}$
m_6	$\rightarrow$	m_{20}

$$\begin{array}{ccc} \text{Solid} & \rightarrow & \text{Fluid} \\ m_1 & \rightarrow & m_1 \\ m_2 & \rightarrow & m_2 \\ m_4 & \rightarrow & m_3 \\ m_{11} & \rightarrow & -m_4 \\ m_{15} & \rightarrow & -m_6 \end{array}$$

and all others are zero. The recovery of the eigenfunction proceeds in an analogous way to the case without self-gravitation but now the system of equations we integrate down must be second order to get the correct numerical behavior. We work with the vector of all second order minors of two solutions:

$$\begin{array}{c} Row1 \\ Row2 \\ Row3 \\ Row4 \\ Row5 \\ Row6 \end{array} \begin{bmatrix} y_1^1 & y_1^2 \\ y_1^1 & y_2^2 \\ y_1^1 & y_3^2 \\ y_1^1 & y_4^2 \\ y_1^1 & y_5^2 \\ y_1^1 & y_6^2 \end{bmatrix}$$

using the row combinations: 12,13, 14, 15 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56. Thus the $\mathbf{x}$ vector has 15 elements with

$$x_1 = y_1^1 y_2^2 - y_2^1 y_1^2$$
$$\vdots$$
$$x_{15} = y_5^1 y_6^2 - y_6^1 y_5^2$$

Differentiation gives the following system of equations:

$$\begin{aligned}
x_1' =&(T_{11} + T_{22})x_1 + C_{22}x_4 - C_{11}x_7 \\
x_2' =&(T_{11} + T_{33})x_2 + C_{33}x_5 + T_{12}x_6 - C_{11}x_{10} \\
x_3' =&S_{12}x_1 + S_{13}x_2 - T_{21}x_4 - T_{31}x_5 + T_{12}x_7 \\
x_4' =&S_{22}x_1 + S_{23}x_2 + (T_{11} - T_{22})x_4 + T_{12}(x_8 - x_3) + C_{11}x_{13} \\
x_5' =&S_{23}x_1 + (T_{11} - T_{33})x_5 + T_{12}x_9 + C_{11}x_{14} \\
x_6' =&- T_{31}x_1 + T_{21}x_2 + (T_{22} + T_{33})x_6 + C_{33}x_9 - C_{22}x_{11} \\
x_7' =&- S_{11}x_1 - T_{21}(x_8 - x_3) + S_{13}x_6 - (T_{11} - T_{22})x_7 - T_{31}x_9 - C_{22}x_{13} \\
x_8' =&- S_{12}x_1 + T_{21}x_4 + S_{23}x_6 - T_{12}x_7 \\
x_9' =&- S_{13}x_1 + T_{21}x_5 + (T_{22} - T_{33})x_9 + C_{22}x_{15} \\
x_{10}' =&- S_{11}x_2 - S_{12}x_6 + (T_{33} - T_{11})x_{10} - T_{21}x_{11} - T_{31}(x_{12} - x_3) - C_{33}x_{14} \\
x_{11}' =&- S_{12}x_2 + T_{31}x_4 - S_{22}x_6 - T_{12}x_{10} + (T_{33} - T_{22})x_{11} - C_{33}x_{15} \\
x_{12}' =&- S_{13}x_2 + T_{31}x_5 - S_{23}x_6 \\
x_{13}' =&S_{11}x_4 - S_{22}x_7 + S_{12}(x_8 - x_3) - S_{23}x_{10} + S_{13}x_{11} - (T_{11} + T_{22})x_{13} + T_{31}x_{15} \\
x_{14}' =&S_{11}x_5 - S_{23}x_7 + S_{12}x_9 + S_{13}(x_{12} - x_3) - (T_{11} + T_{33})x_{14} - T_{21}x_{15} \\
x_{15}' =&- S_{13}x_4 + S_{12}x_5 + S_{22}x_9 + S_{23}(x_{12} - x_8) - T_{12}x_{14} - (T_{22} + T_{33})x_{15}
\end{aligned}$$

Inspection of these equations shows that $x_3 = -x_8 - x_{12}$ so a system of only 14 equations needs to be integrated downwards. We write this system of equations as

$$\frac{d\mathbf{x}}{dr} = \mathbf{X}\mathbf{x}$$

Following the recipe of Woodhouse (1988), we now construct the spanning matrices, $\mathbf{M}$ and $\tilde{\mathbf{M}}$ where $\mathbf{M}$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & -m_3 & m_8 & m_9 & m_{10} \\ 0 & -m_1 & -m_2 & -m_3 & -m_4 & 0 & 0 & 0 & 0 & m_{11} & -m_5 & m_2 & -m_{18} & m_{15} & -m_9 \\ m_1 & 0 & -m_5 & -m_6 & m_3 & 0 & -m_{11} & m_5 & -m_2 & 0 & 0 & 0 & m_{17} & m_{18} & m_8 \\ m_2 & m_5 & 0 & -m_8 & -m_9 & m_{11} & 0 & m_{18} & -m_{15} & 0 & -m_{17} & -m_{18} & 0 & 0 & m_{20} \\ m_3 & m_6 & m_8 & 0 & -m_{10} & -m_5 & -m_{18} & 0 & m_9 & m_{17} & 0 & -m_8 & 0 & -m_{20} & 0 \\ m_4 & -m_3 & m_9 & m_{10} & 0 & m_2 & m_{15} & -m_9 & 0 & m_{18} & m_8 & 0 & m_{20} & 0 & 0 \end{bmatrix}$$

and $\tilde{\mathbf{M}}$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & m_{20} & -m_8 & m_{18} & -m_{17} & -m_9 & -m_{15} & -m_{18} & m_2 & m_5 & m_{11} \\ 0 & -m_{20} & m_8 & -m_{18} & m_{17} & 0 & 0 & 0 & 0 & -m_{10} & m_9 & -m_8 & m_3 & m_6 & -m_5 \\ m_{20} & 0 & m_9 & m_{15} & m_{18} & 0 & m_{10} & -m_9 & m_8 & 0 & 0 & 0 & m_4 & -m_3 & m_2 \\ -m_8 & -m_9 & 0 & -m_2 & -m_5 & -m_{10} & 0 & -m_3 & -m_6 & 0 & -m_4 & m_3 & 0 & 0 & -m_1 \\ m_{18} & -m_{15} & m_2 & 0 & -m_{11} & m_9 & m_3 & 0 & m_5 & m_4 & 0 & -m_2 & 0 & m_1 & 0 \\ -m_{17} & -m_{18} & m_5 & m_{11} & 0 & -m_8 & m_6 & -m_5 & 0 & -m_3 & m_2 & 0 & -m_1 & 0 & 0 \end{bmatrix}$$

The relationship $\mathbf{M}^T \cdot \tilde{\mathbf{M}} = 0$ leads to the following 24 quadratic identities

$$\begin{aligned}
0 =& m_{20}m_1 - m_8m_2 + m_{18}m_3 - m_{17}m_4 \\
0 =& -m_8m_5 + m_{18}m_6 + m_{17}m_3 \\
0 =& -m_{20}m_5 + m_{18}m_8 - m_{17}m_9 \\
0 =& -m_{20}m_6 + m_8^2 - m_{17}m_{10} \\
0 =& m_{20}m_3 + m_8m_9 - m_{18}m_{10} \\
0 =& -m_8m_{11} - m_{18}m_5 - m_{17}m_2 \\
0 =& -m_{20}m_{11} - m_{18}^2 - m_{17}m_{15} \\
0 =& -m_{20}m_2 + m_8m_{15} + m_{18}m_9 \\
0 =& -m_9m_2 - m15m_3 - m_{18}m_4 \\
0 =& m_{20}m_1 - m_9m_5 - m_{15}m_6 + m_{18}m_3 \\
0 =& m_{20}m_4 + m_9^2 + m_{15}m_{10} \\
0 =& -m_9m_{11} + m_{15}m_5 - m_{18}m_2 \\
0 =& m_9m_1 + m_2m_3 + m_5m_4 \\
0 =& -m_8m_1 + m_2m_6 - m_5m_3 \\
0 =& -m_8m_3 - m_9m_6 + m_5m_{10} \\
0 =& -m_8m_4 + m_9m_3 - m_2m_{10} \\
0 =& m_{15}m_1 - m_2^2 + m_{11}m_4 \\
0 =& m_{18}m_1 - m_2m_5 - m_{11}m_3 \\
0 =& m_{18}m_3 - m_{15}m_6 + m_2m_8 + m_{11}m_{10} \\
0 =& -m_{17}m_1 - m_5^2 - m_{11}m_6 \\
0 =& -m_{17}m_4 + m_{18}m_3 + m_5m_9 + m_{11}m_{10} \\
0 =& m_{20}m_1 - m_{10}m_{11} - m_9m_5 - m_8m_2 \\
0 =& m_{10}m_1 + m_3^2 + m_6m_4 \\
0 =& -m_{17}m_4 - m_8m_2 + m_6m_{15} + m_5m_9
\end{aligned}$$

Now $\mathbf{N} = \mathbf{M}^T \mathbf{\Sigma}^{\mathbf{T}}$ is given by

$$
\mathbf{N} = \begin{bmatrix}
m_2 & m_3 & m_4 & 0 & 0 & -m_1 \\
m_5 & m_6 & -m_3 & 0 & m_1 & 0 \\
0 & m_8 & m_9 & 0 & m_2 & m_5 \\
-m_8 & 0 & m_{10} & 0 & m_3 & m_6 \\
-m_9 & -m_{10} & 0 & 0 & m_4 & -m_3 \\
m_{11} & -m_5 & m_2 & -m_1 & 0 & 0 \\
0 & -m_{18} & m_{15} & -m_2 & 0 & m_{11} \\
m_{18} & 0 & -m_9 & -m_3 & 0 & -m_5 \\
-m_{15} & m_9 & 0 & -m_4 & 0 & m_2 \\
0 & m_{17} & m_{18} & -m_5 & -m_{11} & 0 \\
-m_{17} & 0 & m_8 & -m_6 & m_5 & 0 \\
-m_{18} & -m_8 & 0 & m_3 & -m_2 & 0 \\
0 & 0 & m_{20} & -m_8 & m_{18} & -m_{17} \\
0 & -m_{20} & 0 & -m_9 & -m_{15} & -m_{18} \\
m_{20} & 0 & 0 & -m_{10} & m_9 & -m_8
\end{bmatrix}
$$

and $\tilde{\mathbf{N}} = \mathbf{\Sigma}\tilde{\mathbf{M}}$ is

$$
\begin{bmatrix}
m_8 & m_9 & 0 & m_2 & m_5 & m_{10} & 0 & m_3 & m_6 & 0 & m_4 & -m_3 & 0 & 0 & m_1 \\
-m_{18} & m_{15} & -m_2 & 0 & m_{11} & -m_9 & -m_3 & 0 & -m_5 & -m_4 & 0 & m_2 & 0 & -m_1 & 0 \\
m_{17} & m_{18} & -m_5 & -m_{11} & 0 & m_8 & -m_6 & m_5 & 0 & m_3 & -m_2 & 0 & m_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & m_{20} & -m_8 & m_{18} & -m_{17} & -m_9 & -m_{15} & -m_{18} & m_2 & m_5 & m_{11} \\
0 & -m_{20} & m_8 & -m_{18} & m_{17} & 0 & 0 & 0 & 0 & -m_{10} & m_9 & -m_8 & m_3 & m_6 & -m_5 \\
m_{20} & 0 & m_9 & m_{15} & m_{18} & 0 & m_{10} & -m_9 & m_8 & 0 & 0 & 0 & m_4 & -m_3 & m_2
\end{bmatrix}
$$

and, as before, $\mathbf{N} \cdot \mathbf{y} = 0$ and $\mathbf{N} \cdot \tilde{\mathbf{N}} = 0$. Note that $\mathbf{\Sigma}$ is now

$$
\mathbf{\Sigma} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
$$

Now suppose we integrate a vector $\mathbf{x}$ down from the surface with an arbitrary starting solution and at every depth form

$$
\mathbf{b} = \tilde{\mathbf{N}} \cdot \mathbf{x} \quad \text{where} \quad \frac{d\mathbf{x}}{dr} = \mathbf{X}\mathbf{x}
$$

Using results from Woodhouse (1988) we find that for the third order system

$$
\frac{d\mathbf{N}}{dr} = \mathbf{XN} - \mathbf{NA}
$$

$$
\frac{d\tilde{\mathbf{N}}}{dr} = \mathbf{A}\tilde{\mathbf{N}} - \tilde{\mathbf{N}}\mathbf{X}
$$

so differentiating the expression for $\mathbf{b}$ gives

$$
\frac{d\mathbf{b}}{dr} = \frac{d\tilde{\mathbf{N}}}{dr}\mathbf{x} + \tilde{\mathbf{N}}\frac{d\mathbf{x}}{dr} = \mathbf{A}\tilde{\mathbf{N}}\mathbf{x} - \tilde{\mathbf{N}}\mathbf{X}\mathbf{x} + \tilde{\mathbf{N}}\mathbf{X}\mathbf{x} = \mathbf{A}\tilde{\mathbf{N}}\mathbf{x} = \mathbf{A}\mathbf{b}
$$

Thus $\mathbf{b}$ satisfies the original system of equations and, because $\mathbf{N} \cdot \tilde{\mathbf{N}} = 0$, it follows that $\mathbf{N} \cdot \mathbf{b} = 0$ so $\mathbf{b}$ satisfies the boundary conditions and is our eigenfunction. Since $\tilde{\mathbf{N}}$ is made up of (upgoing) minor vector elements, it is proportional to cubes of the original vector while the $\mathbf{x}$ vector is made up of squares, so the product $\mathbf{b} = \tilde{\mathbf{N}} \cdot \mathbf{x}$ has the correct numerical behavior to capture solutions which are exponentially growing upwards at depth. Certain modes that are trapped on interfaces can be exponentially growing downwards and this algorithm can fail at high enough frequency. Woodhouse (1988) gives an alternative construction of the eigenfunction which involves integrating a third-order minor vector down which performs better in these cases.

3.9 Properties of mode solutions . We shall now consider some general properties of mode solutions (particularly the asymptotic forms for high frequency). First we shall consider the toroidal mode case because this is an example of a Sturm-Liouville equation and so we can use many of the results from Sturm-Liouville theory (cf Birkhoff and Rota, 1969, chapter 10). We define an operator $L(y_k)$ such that

$$L(y_k) = (py_k')' - qy_k = -\lambda\tilde{\rho}y_k \tag{3.39}$$

where prime denotes differentiation with respect to r and p, q, and $\tilde{\rho}$ are functions of r. The equation governing toroidal modes is

$$\frac{d}{dr}T + \frac{\mu}{r}(3Z - \frac{W}{r}(l+2)(l-1)) + \rho_0\omega_k^2 W = 0 \tag{3.40}$$

If we let $y_k = W/r$, $p = \mu r^4$, $q = \mu r^2(l+2)(l-1)$, $\tilde{\rho} = \rho_0 r^4$ and $\lambda = \omega_k^2$, we obtain the Sturm-Liouville form. The boundary condition that $T = 0$ becomes $dy/dr = 0$.

The operator is self-adjoint which allows us to prove that the eigenvalues (ω_k^2) are real and the eigenfunctions orthogonal (in a well-defined way). We postpone discussion of these results to a later section. The Sturm-Liouville system can be further transformed to a form which is suitable for analysis in the high frequency limit. Let

$$s = \int_b^r \frac{dr}{V_s} \quad \text{so} \quad \frac{d}{dr} = \frac{1}{V_s}\frac{d}{ds}$$

and let $M = r^2\sqrt{\rho V_s}$ and $X = My = MW/r$. The toroidal mode equation becomes

$$\frac{d^2X}{ds^2} + [\omega^2 - \tilde{q}(s)]X = 0 \tag{3.41}$$

$$\text{where} \quad \tilde{q}(s) = \frac{1}{M}\frac{d^2M}{ds^2} + \frac{(l+2)(l-1)}{r^2}V_s^2$$

and the boundary conditions are

$$\frac{dX}{ds}M - \frac{dM}{ds}X = 0 \quad \text{at} \quad s = 0 \quad \text{and} \quad s = \gamma = \int_b^a \frac{dr}{V_s}$$

When l is small and the model varies smoothly with radius, the equation for large ω becomes

$$\frac{d^2X}{ds^2} + \omega^2 X \simeq 0$$

The solution is $X = C\sin(\omega s + \chi)$ where C and χ are constants. Application of the boundary conditions give

$$\tan(\omega s + \chi) = \frac{\omega M(s)}{M'(s)} \quad \text{at} \quad s = 0 \quad \text{and} \quad s = \gamma$$

When $s = 0$ and for large ω we have

$$\tan\chi = \frac{\omega M(0)}{M'(0)} \to \infty \quad \text{hence} \quad \chi \to \frac{\pi}{2}$$

When $s = \gamma$ and for large ω we have

$$\tan\,(\omega\gamma + \frac{\pi}{2}) = \frac{\omega M(\gamma)}{M'(\gamma)} \to \infty \quad \text{hence} \quad \omega \to \frac{n\pi}{\gamma}$$

where n is an integer. The frequency separation between modes then becomes

$$\delta\omega = \frac{2\pi}{2\gamma} = \frac{2\pi}{T_{ScS}}$$

where T_{ScS} is the two-way travel time for vertically incident shear waves across the mantle. This result is valid for large ω and small l and assumes that ρ and V_s vary smoothly with radius. We can generalize this result to allow large l as well as large ω. First we need the behavior of P_l^m's in the large l limit. From appendix B we find that, when $m \ll l$ and $l \to \infty$

$$P_l^m(\cos\theta) = (-l)^m \left(\frac{2}{\pi l \sin\theta}\right)^{\frac{1}{2}} \cos\,[(l + \tfrac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2}]$$

If we put the source at the pole of the coordinate system then θ is just the angular epicentral distance. It turns out that a localized source is incapable of exciting large m modes (we shall show that a point source excites only modes with $|m| \leq 2$) so the restriction to small m is valid. Inspection of the asymptotic form for the P_l^m indicates that we have a traveling wave with a phase factor (ignoring some multiple of $\pi/2$) of $e^{\pm ikX} = e^{\pm i(l+\frac{1}{2})\theta}$ where X is range. If a is the radius of the Earth then $a\theta = X$ so

$$ka = l + \tfrac{1}{2}$$

Now $k = \omega/c$ where c is the horizontal phase velocity and $a/c = p$ is the horizontal angular phase slowness. Thus

$$l + \tfrac{1}{2} = \omega p$$

We shall also see that p can also be interpreted in terms of a "ray parameter." The solutions to the Sturm-Liouville problem, 3.38, are "exponential" in behavior if $-q + \lambda\tilde{\rho} < 0$ and the solutions are oscillatory if $-q + \lambda\tilde{\rho} > 0$. The point at which $\lambda\tilde{\rho} = q$ is called a "turning point." Substituting in the forms for q and $\tilde{\rho}$ for toroidal modes gives

$$\rho_0\omega_k^2 r^2 - \mu(l+2)(l-1) = 0 \quad \text{at the turning point}$$

For large l, $(l+2)(l-1) \simeq (l + \frac{1}{2})^2$ so

$$\rho_0\omega_k^2 r^2 - \mu\omega_k^2 p^2 = 0 \quad \text{hence} \quad p = \frac{r}{V_s} \quad \text{at the turning point}$$

This is also the value of the ray parameter in the ray equation at the turning point. The interpretation of this result is that, if we add up all modes with similar p we will construct body waves with a turning point at the depth at which the turning point occurs in the Sturm-Liouville equation. This result also implies that we have an exponentially decreasing function beneath the turning point so body waves can be slightly sensitive to structure just below the turning point.

Now, for large l the value of $\tilde{q}$ defined in 3.40 becomes

$$\tilde{q}(s) = \frac{(l+2)(l-1)V_s^2}{r^2} \simeq \frac{\omega^2 p^2 V_s^2}{r^2}$$

and the equation for X becomes

$$\frac{d^2X}{ds^2} + \omega^2 V_s^2 \left(\frac{1}{V_s^2} - \frac{p^2}{r^2} \right) X = 0$$

$$\text{or} \quad \frac{d^2X}{dr^2} + \omega^2 g^2 X = 0$$

where $g = \sqrt{1/V_s^2 - p^2/r^2}$. We now have the equation in the standard form for WKB analysis (see e.g., Bender and Orzag, p. 490). X can be approximated by sines and cosines above the turning point, by exponentials below the turning point, and by an Airy function around the turning point (the traditional WKB analysis is invalid at the turning point). The eigenvalues, ω_k^2, can be found by matching the boundary conditions on X. The solution above the turning point is of the form

$$\frac{1}{\sqrt{g}} \left[A \sin \left(\omega \int_b^r g(r)\, dr \right) + B \cos \left(\omega \int_b^r g(r)\, dr \right) \right] \tag{3.42}$$

where b is the turning point. [Note the similarity to the solution we would have had if g had been a constant. The effect of slowly varying g is to vary the argument to the trig functions as a function of radius.] To make life simple, we consider those modes which are oscillatory throughout the mantle so that b can be interpreted as the radius of the core-mantle boundary. For a slowly varying medium, the boundary conditions are that $X' = 0$ on both boundaries. These are satisfied when

$$\omega_n = \frac{n\pi}{\int_b^a g\, dr}$$

Thus, at fixed p, the spacing between modes is given by

$$\delta\omega(p) = \frac{2\pi}{2\int_b^a g\, dr} = \frac{2\pi}{\tau(p)}$$

where $\tau(p)$ is the delay time. On an ω/l diagram (*e.g.*, fig 3.2), modes with the same p lie on lines emanating from the origin, *i.e.*, $\omega/l + \frac{1}{2}$ = constant. In the limit that we have ScS-equivalent modes (ScS rays are close to vertical incidence), we find that $p \to 0$ and $\delta\omega(0) = 2\pi/T_{ScS}$ as before.

Our analysis has assumed that the model parameters are slowly varying with radius (*i.e.*, d^2M/ds^2 is small) and so is invalid when there are first-order discontinuities such as the 670-km or 400-km discontinuities. The analysis proceeds as above except we find WKB solutions for each layer between discontinuities and match boundary conditions at the interface. The result is that the asymptotic spacing of modes is no longer constant but oscillates about a mean spacing. This effect is called the "solotone" effect (see Lapwood and Usami, chapter 8 for more details).

We can apply the same kind of analysis to the equations governing radial and spheroidal mode oscillation. Consider first the radial mode equation, 3.26. The trick is to realize that the tractions are of order ω times the displacements so that y_4 is of order ωy_1. (The tractions are proportional to derivatives of the displacement and it is easy to see that differentiation of a form such as equation 3.41 will bring out a factor of ω.) Consequently, for large ω and slowly varying material properties we obtain

$$\frac{dy_1}{dr} \simeq \frac{y_4}{\sigma} \quad \text{and} \quad \frac{dy_4}{dr} \simeq -\rho_0 \omega_k^2 y_1$$

($\sigma = \lambda + 2\mu = \rho_0 V_p^2$). Combining these gives

$$\frac{d^2y_1}{dr^2} + \frac{\omega_k^2}{V_p^2} y_1 = 0$$

Now define

$$\gamma = \int_0^a \frac{dr}{V_p} \quad \text{and} \quad s = \frac{1}{\gamma}\int_0^r \frac{dr}{V_p} \quad \text{so} \quad \frac{d}{dr} = \frac{1}{V_p\gamma}\frac{d}{ds}$$

and let $y_1 = (V_p/\omega)^{\frac{1}{2}}\, X$ and assume that V_p is slowly varying finally giving

$$\frac{d^2X}{ds^2} + \omega^2\gamma^2 X \simeq 0$$

The solution of this is

$$X = B\sin(\gamma s\omega + \chi)$$

where B and χ are constants. The boundary condition ($y_4 = 0$ at the surface) is satisfied when $\cos(\omega\gamma+\chi) = 0$ so $\omega_n = ((n+\frac{1}{2})\pi + \chi)/\gamma$ and the asymptotic frequency separation between radial modes is given by

$$\delta\omega = \frac{\pi}{\gamma} = \frac{2\pi}{T_{PKIKP}}$$

where T_{PKIKP} is the diametrical travel time for a P-wave crossing the Earth. We can apply a similar kind of analysis to the spheroidal mode equations. First, you will have noticed that the effect of gravity gives terms in the equation of motion with a characteristic squared frequency of $\sim 4\pi G\rho_0$. This corresponds to a period of about 3000 seconds so we anticipate that the effects of gravity can be ignored for periods much shorter than this. (For example, see the argument to the spherical Bessel function for the radial mode problem, 3.32, where gravity can clearly be ignored if $\omega_k^2 \gg 16/3\pi G\rho_0$). This means that we can neglect terms in ϕ_1 and $d\phi_1/dr$ and 3.20 reduces to a fourth-order system. If we futher note that y_4 and y_5 are of order ω times y_1 and y_2, we obtain (for the isotropic case and for large ω)

$$\begin{aligned}\frac{dy_1}{dr} &\simeq \frac{y_4}{\sigma}\\ \frac{dy_2}{dr} &\simeq \frac{y_5}{\mu}\\ \frac{dy_4}{dr} &\simeq -\rho_0\omega_k^2 y_1\\ \frac{dy_5}{dr} &\simeq -\rho_0\omega_k^2 y_2\end{aligned}$$

Note that the equations for (y_1, y_4) have now decoupled from the equations for (y_2, y_5). The equations for (y_1, y_4) are exactly the same as for radial modes and clearly have something to do with P-equivalent modes. The equations for (y_2, y_5) differ only in that the shear velocity, V_s, appears instead of V_p. Thus we find shear dominated modes with frequency spacing $\delta\omega = 2\pi/T_J$ and $\delta\omega = 2\pi/T_{ScS}$ where T_J is the diametrical travel time of shear waves in the inner core.

***3.10 The oscillation theorem*.** Much of the original research by Sturm was devoted to counting the number of zeroes in the eigenfunction y_k for a particular eigenvalue, λ_k. This has application in seismology because it allows us to count how many modes exist between two frequencies at fixed l. For the toroidal mode case, we can show that the zeroes of W and T must interlace. A simple way of looking at this feature of the solution is by making a Prüfer substitution (Birkoff and Rota, 1969 p288). Our original equation can be written

$$(py_k')' + (\lambda\tilde{\rho} - q)y_k = 0$$

The Prüfer substitution is

$$p(r)y_k' = A(r)\cos\theta(r) \quad \text{and} \quad y_k(r) = A(r)\sin\theta(r)$$

After some algebra, we obtain two first order differential equations: one for $\theta(r)$ and one for $A(r)$:

$$\frac{d\theta}{dr} = (\lambda\tilde{\rho} - q)\sin^2\theta + \frac{1}{p}\cos^2\theta \tag{3.43}$$

and

$$\frac{1}{A}\frac{dA}{dr} = \left[\frac{1}{p} - \lambda\tilde{\rho} + q\right]\sin\theta\cos\theta$$

The most interesting of these is the equation for θ. Once we are above the turning point, all terms on the right hand side of 3.42 are positive so θ always increases as r increases. In terms of our original variables for the toroidal mode equation (T and W) we have

$$\tan\theta = \frac{W}{r^4 T}$$

so the boundary conditions for mantle toroidal modes are $\tan\theta(r) = \infty$ at $r = a$ and b. The surface condition is satisfied when $\theta(a) = \frac{\pi}{2} + n\pi$ and $\theta(a)$ is a monotonic function of ω. Suppose at each radius we plot r^4T as a function of W, *i.e.*,

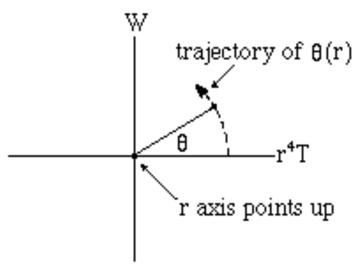

Fig 3.8

As r increases, θ increases. Thus, the zeroes of T and W interlace and the number of times we go round the spiral as we integrate up in radius determines the mode count. Alternatively, we can integrate the equation for θ numerically and use the result for $\theta(a)$ to determine the overtone number, n, directly.
[A similar result is desirable for spheroidal modes and is based upon the interlacing properties of combinations of minor vectors. In a self-gravitating fluid, the roles of T and W above are taken by m_6 and $m_3 - m_4$. From equation 3.37, we infer that $m_3 - m_4$ and m_6 cannot simultaneously be zero without generating a null solution (actually, this is not strictly true) so the interlacing behavior is reasonable and a counter can be constructed.]

3.11 Variational Principles . By virtue of Poisson's equation, we can regard ϕ_1 as a function of the displacement field s, *i.e.*, $\phi_1 = \phi_1(\mathbf{s})$. We can then write our basic equations 2.53 as

$$\rho_0\frac{\partial^2\mathbf{s}}{\partial t^2} = \mathbf{L}(\mathbf{s}) + \mathbf{f} \tag{3.44}$$

where

$$\mathbf{L}(\mathbf{s}) = \nabla\cdot\mathbf{T} - \nabla(s_r\rho_0 g_0) - \rho_0\nabla\phi_1 + \hat{\mathbf{r}}g_0\nabla\cdot(\rho_0\mathbf{s})$$

For a solution of the form $\mathbf{s} = \mathbf{s}_k(\mathbf{r})e^{i\omega_k t}$ when $\mathbf{f} = 0$, we have

$$\mathbf{L}(\mathbf{s}_k) + \rho_0\omega_k^2\mathbf{s}_k = 0 \tag{3.45}$$

We show in the appendix to this chapter that $\mathbf{L}$ is self-adjoint. We obtain for two differentiable fields, s and s′ (which don't necessarily satisfy the boundary conditions) the following result:

$$\int_V \mathbf{s}' \cdot \mathbf{L}(\mathbf{s})\, dV + \int_\Sigma [\mathbf{s}' \cdot \mathbf{T} \cdot \hat{\mathbf{r}}]_-^+\, d\Sigma = \int_V \mathbf{s} \cdot \mathbf{L}(\mathbf{s}')\, dV + \int_\Sigma [\mathbf{s} \cdot \mathbf{T}' \cdot \hat{\mathbf{r}}]_-^+\, d\Sigma \tag{3.46}$$

The surface integral is over all discontinuities and $[\,]_-^+$ indicates the difference in the bracketed quantity across a discontinuity found by subtracting the value on the lower (smaller radius) side from that on the upper side. For displacement fields which satisfy the boundary conditions, s and $\mathbf{T} \cdot \hat{\mathbf{r}}$ are continuous so we get

$$\int_V \mathbf{s}' \cdot \mathbf{L}(\mathbf{s})\, dV = \int_V \mathbf{s} \cdot \mathbf{L}(\mathbf{s}')\, dV \tag{3.47}$$

i.e., $\mathbf{L}$ is self-adjoint. Let $\mathbf{s}_k$ and $\mathbf{s}_j$ be two solutions with associated eigenvalues ω_k^2 and ω_j^2. Then

$$\begin{aligned} \mathbf{L}(\mathbf{s}_k) &= -\,\rho_0 \omega_k^2 \mathbf{s}_k \\ \mathbf{L}(\mathbf{s}_j^\star) &= -\,\rho_0 \omega_j^{\star 2} \mathbf{s}_j^\star \end{aligned}$$

where $\star$ denotes complex conjugation. (Note that $\mathbf{L}$ is a real operator.) Thus

$$-\omega_k^2 \int_V \rho_0 \mathbf{s}_j^\star \cdot \mathbf{s}_k\, dV = -\omega_j^{\star 2} \int_V \rho_0 \mathbf{s}_j^\star \cdot \mathbf{s}_k\, dV$$

$$\text{or} \quad (\omega_k^2 - \omega_j^{\star 2}) \int_V \rho_0 \mathbf{s}_j^\star \cdot \mathbf{s}_k\, dV = 0$$

so if $k = j$, $\omega_k^2 = \omega_j^{\star 2}$ (*i.e.*, the eigenvalues are real) and if $k \neq j$

$$\int_V \rho_0 \mathbf{s}_j^\star \cdot \mathbf{s}_k dV = 0 \tag{3.48}$$

(assuming no degeneracy). In fact we usually normalize the solutions so that

$$\int_V \rho_0 \mathbf{s}_j^\star \cdot \mathbf{s}_k dV = \delta_{jk} \tag{3.49}$$

which is a statement of orthogonality. Suppose now we define the function $\omega^2(\mathbf{s})$ by

$$-\omega^2(\mathbf{s}) \int_V \rho_0 \mathbf{s}^\star \cdot \mathbf{s}\, dV = \int_V \mathbf{s}^\star \cdot \mathbf{L}(\mathbf{s})\, dV + \int_\Sigma [\mathbf{s}^\star \cdot \mathbf{T} \cdot \hat{\mathbf{r}}]_-^+\, d\Sigma \tag{3.50}$$

If $\mathbf{s} = \mathbf{s}_k$ (an eigenfunction) then $\omega^2(\mathbf{s}_k) = \omega_k^2$ (an eigenvalue) and

$$\omega_k^2 \int_V \rho_0 \mathbf{s}^\star \cdot \mathbf{s}\, dV = -\int_V \mathbf{s}^\star \cdot \mathbf{L}(\mathbf{s})\, dV$$

An important property of $\omega^2(\mathbf{s})$ is that $\omega^2(\mathbf{s}_k)$ is stationary to small perturbations in $\mathbf{s}_k$. To illustrate this, we let $\mathbf{s} \to \mathbf{s} + \delta\mathbf{s}$ and define

$$\delta\omega^2 = \omega^2(\mathbf{s} + \delta\mathbf{s}) - \omega^2(\mathbf{s})$$

Substitution into equation 3.49 and keeping terms up to first order gives

$$\delta\omega^2 \int\limits_V \rho_0 \mathbf{s}^\star \cdot \mathbf{s}\, dV + \omega^2 \int\limits_V \rho_0 \delta\mathbf{s}^\star \cdot \mathbf{s}\, dV + \omega^2 \int\limits_V \rho_0 \mathbf{s}^\star \cdot \delta\mathbf{s}\, dV$$
$$= - \int\limits_V \delta\mathbf{s}^\star \cdot \mathbf{L}(\mathbf{s})\, dV - \int\limits_V \mathbf{s}^\star \cdot \mathbf{L}(\delta\mathbf{s})\, dV$$
$$- \int\limits_\Sigma [\mathbf{s}^\star \cdot \delta\mathbf{T} \cdot \hat{\mathbf{r}}]_-^+ \, d\Sigma - \int\limits_\Sigma [\delta\mathbf{s}^\star \cdot \mathbf{T} \cdot \hat{\mathbf{r}}]_-^+ \, d\Sigma$$

Using equation 3.46 gives

$$\delta\omega^2 \int\limits_V \rho_0 \mathbf{s} \cdot \mathbf{s}^\star\, dV = - \int\limits_V \delta\mathbf{s}^\star \cdot \left(\mathbf{L}(\mathbf{s}) + \rho_0\omega^2\mathbf{s}\right)\, dV - \int\limits_\Sigma [\delta\mathbf{s}^\star \cdot \mathbf{T} \cdot \hat{\mathbf{r}}]_-^+ \, d\Sigma$$
$$- \int\limits_V \delta\mathbf{s} \cdot \left(\mathbf{L}(\mathbf{s}^\star) + \rho_0\omega^2\mathbf{s}^\star\right)\, dV - \int\limits_\Sigma [\delta\mathbf{s} \cdot \mathbf{T}^\star \cdot \hat{\mathbf{r}}]_-^+ \, d\Sigma$$

Now, if $\mathbf{s}^\star$ is an eigenfunction and ω^2 is an eigenvalue *and* δs is continuous at interfaces we have

$$\delta\omega^2 \int\limits_V \rho_0 \mathbf{s}^\star \cdot \mathbf{s}\, dV = 0$$

i.e., $\delta\omega^2 = 0$ so ω^2 is stationary. (This result is only true if both s and $\mathrm{s} + \delta\mathrm{s}$ satisfy the boundary conditions). This result is sometimes called “Rayleigh’s Principle” and is extremely useful as we shall see below. When we have an eigenvalue and eigenvector, we have

$$\omega_k^2 \int\limits_V \rho_0 \mathbf{s}_k^\star \cdot \mathbf{s}_k\, dV = - \int\limits_V \mathbf{s}^\star \cdot \mathbf{L}(\mathbf{s})\, dV$$

The term on the left-hand side clearly has something to do with kinetic energy. The total kinetic energy is

$$KE = \tfrac{1}{2} \int\limits_V \rho_0 \mathbf{v} \cdot \mathbf{v}\, dV$$

where $\mathbf{v}$ is velocity. For an oscillator with sinusoidal behavior [$\mathbf{s} \propto \mathbf{s}_k \sin(\omega_k t)$] we have

$$KE = \tfrac{1}{2}\, \omega_k^2 \cos^2(\omega_k t) \int\limits_V \rho_0 \mathbf{s}_k^\star \cdot \mathbf{s}_k\, dV$$

and if we average over a cycle (the mean value of $\cos^2(\omega_k t)$ over a cycle is $\frac{1}{2}$) we get

$$\overline{KE} = \tfrac{1}{4}\, \omega_k^2 \int\limits_V \rho_0 \mathbf{s}_k^\star \cdot \mathbf{s}_k\, dV = \omega_k^2 \mathcal{T} \quad \text{say}$$

In a similar way, the mean potential energy is

$$\mathcal{V} = -\tfrac{1}{4} \int\limits_V \mathbf{s}^\star \cdot \mathbf{L}(\mathbf{s})\, dV$$

and can be thought of as being made up of elastic potential energy and gravitational potential energy. We can write the potential energy in many ways using equation 3.43 for $\mathbf{L}$ and Poisson’s equation. From the appendix, we note that one way to write it is

$$\mathcal{V} = \tfrac{1}{4} \int\limits_V \{\nabla \mathbf{s}^\star \cdot \cdot \mathbf{C} : \nabla \mathbf{s} + \tfrac{1}{2}\, \rho_0 \nabla \phi \cdot [\mathbf{s}^\star \cdot \nabla \mathbf{s} - \mathbf{s}^\star (\nabla \cdot \mathbf{s}) + \mathbf{s} \cdot \nabla \mathbf{s}^\star - \mathbf{s}(\nabla \cdot \mathbf{s}^\star)]$$

$$+ \rho_0 [\mathbf{s}^\star \nabla \phi_1 + \mathbf{s} \nabla \phi_1^\star + \mathbf{s}^\star \cdot \mathbf{s} \nabla (\nabla \phi_1)] + \frac{1}{4\pi G} \nabla \phi_1 \cdot \nabla \phi_1^\star \,\} \, dV$$

where $\phi_1^\star = \phi_1(\mathbf{s}^\star)$ and $\phi_1 = \phi_1(\mathbf{s})$. The first term on the right-hand side is clearly the elastic work ($\mathbf{C}$ is the fourth-order elastic tensor) while the rest of the right-hand side is "gravitational". This form of $\mathcal{V}$ emphasizes the quadratic form of the potential energy. If we now form the "Lagrangian", $\mathcal{L}$, where

$$\mathcal{L} = \omega^2 \mathcal{T} - \mathcal{V} \tag{3.51}$$

we find that $\mathcal{L}$ is minimized at an eigensolution and its value is zero because of the quadratic form of both $\mathcal{T}$ and $\mathcal{V}$. This means that there is equality of the time-averaged kinetic and potential energies.

Rayleigh's principle states that $\omega^2(\mathbf{s}) = \mathcal{V}(\mathbf{s})/\mathcal{T}(\mathbf{s})$ is stationary if $\mathbf{s}$ is an eigenvector and is the basis of the Rayleigh-Ritz method for computing eigenfrequencies and eigenfunctions. Suppose we expand $\mathbf{s}$ in a set of known basis functions, $\Psi_i(\mathbf{r})$, *e.g.*,

$$\mathbf{s} = \sum_i a_i \Psi_i(\mathbf{r})$$

Then $\mathcal{T}$, the time-averaged kinetic energy becomes

$$\mathcal{T} = \tfrac{1}{4} \int\limits_V \rho_0 \mathbf{s}^\star \cdot \mathbf{s} \, dV = \mathbf{a}^T \cdot \Upsilon \cdot \mathbf{a}$$

$$\text{where} \quad \Upsilon_{ij} = \tfrac{1}{4} \int\limits_V \rho_0 \Psi_i^\star \cdot \Psi_j \, dV$$

and can be computed. Similarly, $\mathcal{V}$ can be written as

$$\mathcal{V} = \mathbf{a}^T \cdot \mathbf{V} \cdot \mathbf{a}$$

where $\mathbf{V}$ has complicated (but computable) matrix elements. We now have

$$\omega^2 \mathbf{a}^T \cdot \Upsilon \cdot \mathbf{a} = \mathbf{a}^T \cdot \mathbf{V} \cdot \mathbf{a}$$

ω^2 can be regarded as a function of $\mathbf{a}$ and is stationary with respect to small perturbations in $\mathbf{a}$ at an eigensolution. Differentiation with respect to $\mathbf{a}$ gives

$$\mathbf{V} \cdot \mathbf{a} = \omega^2 \Upsilon \cdot \mathbf{a}$$

which is a generalized eigenvalue problem for the eigenvalues, ω^2, and the eigenvectors, $\mathbf{a}$. In practice, we apply the Rayleigh-Ritz technique to the separated equations. We substitute the vector spherical harmonic form for $\mathbf{s}$ and do the integral over θ and ϕ analytically. We then get forms for the kinetic and potential energy integrals which are integrals over r alone and are independent of the azimuthal order number, m. For example, if we substitute in the form for $\mathbf{s}$ for toroidal motion and we look at a single l, m component:

$${}_n\mathbf{s}_l^m = -\hat{\mathbf{r}} \times \nabla_1 [W_l^m(r) Y_l^m(\theta, \phi)]$$

we find that

$$\omega^2 \int\limits_b^a \rho_0 W^2 r^2 \, dr = \int\limits_b^a \left[\mathrm{L} r^2 \left(\frac{dW}{dr} - \frac{W}{r} \right)^2 + \mathrm{N}(l+2)(l-1) W^2 \right] dr$$

which is independent of m. We now let

$$W = \sum_i a_i \Psi_i(r)$$

and get

$$\mathbf{B} \cdot \mathbf{a} = \omega^2 \mathbf{A} \cdot \mathbf{a}$$

where

$$A_{ij} = \int_b^a \rho_0 \Psi_i \Psi_j r^2 \, dr$$

$$B_{ij} = \int_b^a \left[\mathrm{L} r^2 \left(\frac{d\Psi_i}{dr} - \frac{\Psi_i}{r} \right) \left(\frac{d\Psi_j}{dr} - \frac{\Psi_j}{r} \right) + \mathbf{N}(l+2)(l-1)\Psi_i \Psi_j \right] dr$$

It is sensible to choose the Ψ_i so that they individually match the boundary conditions on W so that any linear combination of them will also match the boundary conditions. It is also sensible to choose the Ψ_i so that they are localized polynomials in radius with limited overlap. Inspection of the above equations shows that if Ψ_i and Ψ_j don't overlap except when $i \simeq j$ then both $\mathbf{A}$ and $\mathbf{B}$ will be banded matrices. There are very fast algorithms for solving the generalized eigenvalue problem with banded matrices and we have all the machinery for isolating eigenvalues so ensuring that we compute a complete set of modes.

There are some disadvantages to this scheme. There are classes of oscillations which are low-frequency gravitational oscillations in the fluid core and ocean. A choice of basis functions which can model elastic-gravitational seismic modes is incapable of modelling these modes. Their eigenfunctions are therefore poorly represented and the corresponding eigenvalues are badly wrong. Unfortunately, the eigenvalues, which should be close to zero frequency, are now distributed through the seismic band and the algorithm will compute all these spurious modes as well as the real seismic modes. The method of choice for computing eigenvalues and eigenvectors remains the method of minors which we described in section 3.6.

12. Appendix: Self-adjointness of the operator $\mathbf{L}(\mathbf{s})$

This appendix follows a derivation first given by John Woodhouse. By virtue of Poisson's equation, we can regard ϕ_1 as a function of the displacement field s, *i.e.*, $\phi_1 = \phi_1(\mathbf{s})$. We can then write our basic equations 2.45 as

$$\rho_0 \frac{\partial^2 \mathbf{s}}{\partial t^2} = \mathbf{L}(\mathbf{s}) + \mathbf{f} \tag{3.52}$$

where

$$\mathbf{L}(\mathbf{s}) = \nabla \cdot \mathbf{T}_E - \nabla \cdot (\mathbf{s} \cdot \nabla \mathbf{T}_0) - \rho_0 \nabla \phi_1 - \rho_1 \nabla \phi_0 \tag{3.53}$$

which, for an isotropic prestress ($\mathbf{T}_0 = -p_0 \mathbf{I}$) and subtituting for $\rho_1 [= -\nabla \cdot (\rho_0 \mathbf{s})]$ becomes

$$\mathbf{L}(\mathbf{s}) = \nabla \cdot \mathbf{T}_E - \rho_0 \nabla \phi_1 + \nabla \cdot (\rho_0 \mathbf{s}) \nabla \phi_0 + \nabla(\mathbf{s} \cdot \nabla p_0) \tag{3.54}$$

We show in this appendix that $\mathbf{L}$ is self-adjoint. To do this, it turns out to be more convenient to write $\mathbf{L}(\mathbf{s})$ in terms of the Piola-Kirchoff incremental tensor, $\tilde{\mathbf{T}}$, introduced in chapter 2 (equation 2.94):

$$\mathbf{T}_0 + \tilde{\mathbf{T}} = \left[\mathbf{I} + (\nabla \cdot \mathbf{s})\mathbf{I} - (\nabla \mathbf{s})^T \right] [\mathbf{T}_E + \mathbf{T}_0]$$

Hence, to first order

$$\tilde{\mathbf{T}} = \mathbf{T}_E + (\nabla\cdot\mathbf{s})\mathbf{T}_0 - (\nabla\mathbf{s})^T\cdot\mathbf{T}_0 \tag{3.55}$$

which, for an isotropic prestress, becomes

$$\tilde{\mathbf{T}} = \mathbf{T}_E - p_0(\nabla\cdot\mathbf{s})\mathbf{I} + p_0(\nabla\mathbf{s})^T \tag{3.56}$$

Taking the divergence gives

$$\nabla\cdot\tilde{\mathbf{T}} = \nabla\cdot\mathbf{T}_E - \nabla p_0(\nabla\cdot\mathbf{s}) - p_0\nabla(\nabla\cdot\mathbf{s}) + \nabla p_0(\nabla\mathbf{s})^T + p_0\nabla\cdot(\nabla\mathbf{s})^T \tag{3.57}$$

It is easy to verify that $\nabla\cdot(\nabla\mathbf{s})^T = \nabla(\nabla\cdot\mathbf{s})$ so

$$\nabla\cdot\tilde{\mathbf{T}} = \nabla\cdot\mathbf{T}_E - \nabla p_0\cdot\left[(\nabla\cdot\mathbf{s})\mathbf{I} - (\nabla\mathbf{s})^T\right] \tag{3.58}$$

Thus, after some manipulation, 3.54 becomes

$$\mathbf{L}(\mathbf{s}) = \nabla\cdot\tilde{\mathbf{T}} - \rho_0\nabla\phi_1 - \rho_0\mathbf{s}\cdot\nabla(\nabla\phi_0) \tag{3.59}$$

Now consider two displacement fields, s and s′ so

$$\begin{aligned}\mathbf{s}'\cdot\mathbf{L}(\mathbf{s}) &= \mathbf{s}'\cdot\nabla\cdot\tilde{\mathbf{T}} - \rho_0\mathbf{s}'\cdot\nabla\phi_1 - \rho_0\mathbf{s}'\cdot\mathbf{s}\cdot\nabla(\nabla\phi_0)\\ &= \nabla\cdot(\mathbf{s}'\tilde{\mathbf{T}}) - \nabla\mathbf{s}'\cdot\cdot\tilde{\mathbf{T}} - \rho_0\mathbf{s}'\cdot\nabla\phi_1 - \rho_0\mathbf{s}'\cdot\mathbf{s}\cdot\nabla(\nabla\phi_0)\\ &\quad + \rho_0\mathbf{s}\cdot\nabla\phi_1' - \rho_0\mathbf{s}\cdot\nabla\phi_1'\end{aligned} \tag{3.60}$$

where $\phi_1' \equiv \phi_1(\mathbf{s}')$. Adding and subtracting $\rho_0\mathbf{s}\cdot\nabla\phi_1'$ is the first step in getting a symmetric form. Now

$$\begin{aligned}\rho_0\mathbf{s}\cdot\nabla\phi_1' &= \nabla\cdot(\rho_0\mathbf{s}\phi_1') - \phi_1'\nabla\cdot(\rho_0\mathbf{s})\\ &= \nabla\cdot(\rho_0\mathbf{s}\phi_1') + \rho_1\phi_1'\\ &= \nabla\cdot(\rho_0\mathbf{s}\phi_1') + \frac{\phi_1'\nabla^2\phi_1}{4\pi G} \qquad (\nabla^2\phi_1 = 4\pi G\rho_1)\\ &= \nabla\cdot(\rho_0\mathbf{s}\phi_1') + \frac{\phi_1'\nabla\cdot\nabla\phi_1}{4\pi G}\\ &= \nabla\cdot\left[\rho_0\mathbf{s}\phi_1' + \frac{\phi_1'\nabla\phi_1}{4\pi G}\right] - \frac{\nabla\phi_1'\cdot\nabla\phi_1}{4\pi G}\end{aligned}$$

Substituting into eqn 3.60 gives

$$\begin{aligned}\mathbf{s}'\mathbf{L}(\mathbf{s}) &= \nabla\cdot\left[\mathbf{s}'\cdot\tilde{\mathbf{T}} + \phi_1'(\rho_0\mathbf{s} + \frac{\nabla\phi_1}{4\pi G})\right]\\ &\quad - \left[\nabla\mathbf{s}'\cdot\cdot\tilde{\mathbf{T}} + \rho_0\mathbf{s}'\nabla\phi_1 + \rho_0\mathbf{s}\nabla\phi_1' + \rho_0\mathbf{s}'\cdot\mathbf{s}\cdot\nabla(\nabla\phi_0) + \frac{\nabla\phi_1'\cdot\nabla\phi_1}{4\pi G}\right]\end{aligned} \tag{3.61}$$

Now $\tilde{\mathbf{T}} = \mathbf{\Lambda} : \nabla\mathbf{s}$ so

$$\nabla\mathbf{s}'\cdot\cdot\tilde{\mathbf{T}} = \nabla\mathbf{s}'\cdot\cdot\mathbf{\Lambda} : \nabla\mathbf{s}$$

but $\Lambda_{ijkl} = \Lambda_{klij}$ so

$$\nabla\mathbf{s}'\cdot\cdot\mathbf{\Lambda} : \nabla\mathbf{s} = \nabla\mathbf{s}\cdot\cdot\mathbf{\Lambda} : \nabla\mathbf{s}'$$

thus

$$\nabla\mathbf{s}'\cdot\cdot\tilde{\mathbf{T}} = \nabla\mathbf{s}\cdot\cdot\tilde{\mathbf{T}}'$$

It therefore follows that the second term in equation 3.61 is unchanged if s and s′ are swapped. Thus

$$\mathbf{s}'\mathbf{L}(\mathbf{s}) - \mathbf{s}\mathbf{L}(\mathbf{s}') = \nabla \cdot \left[\mathbf{s}' \cdot \tilde{\mathbf{T}} - \mathbf{s} \cdot \tilde{\mathbf{T}}' + \phi_1'(\rho_0 \mathbf{s} + \frac{\nabla \phi_1}{4\pi G}) - \phi_1(\rho_0 \mathbf{s}' + \frac{\nabla \phi_1'}{4\pi G}) \right] \tag{3.62}$$

If we integrate over the volume of the Earth and apply Gauss' theorem, we must remember that there are internal surfaces on which properties may be discontinuous. We therefore write:

$$\int_V \nabla \cdot \mathbf{u} \, dV = \int_S \hat{\mathbf{n}} \cdot \mathbf{u} \, dS = -\int_\Sigma [\hat{\mathbf{r}} \cdot \mathbf{u}]_-^+ \, d\Sigma \tag{3.63}$$

where the last term is the sum of contributions from all interfaces (including the surface) and is evaluated by finding the difference in the bracketed quantity across a discontinuity (by subtracting the value on the lower (smaller radius) side from that on the upper side). The value of the quantity outside the Earth is taken to be zero.
Thus

$$\int_V [\mathbf{s}'\mathbf{L}(\mathbf{s}) - \mathbf{s}\mathbf{L}(\mathbf{s}')] \; dV = -\int_\Sigma \left[\mathbf{s}' \cdot \tilde{\mathbf{T}} - \mathbf{s} \cdot \tilde{\mathbf{T}}' + \phi_1'(\rho_0 \mathbf{s} + \frac{\nabla \phi_1}{4\pi G}) - \phi_1(\rho_0 \mathbf{s}' + \frac{\nabla \phi_1'}{4\pi G}) \right]_-^+ \cdot \hat{\mathbf{r}} \, d\Sigma$$

From the boundary conditions on ϕ_1, i.e., ϕ_1 is continuous everywhere and $(\nabla \phi_1 + 4\pi G \rho_0 \mathbf{s}) \cdot \hat{\mathbf{r}}$ is continuous at the undeformed boundary (for any s) then

$$\int_V [\mathbf{s}'\mathbf{L}(\mathbf{s}) - \mathbf{s}\mathbf{L}(\mathbf{s}')] \; dV = -\int_\Sigma \left[\mathbf{s}' \cdot \tilde{\mathbf{T}} \cdot \hat{\mathbf{r}} - \mathbf{s} \cdot \tilde{\mathbf{T}}' \cdot \hat{\mathbf{r}} \right]_-^+ \, d\Sigma \tag{3.64}$$

For a welded boundary, $\tilde{\mathbf{T}} \cdot \hat{\mathbf{r}}$ is continuous and s is continuous so the term on the right is zero if s and s′ satisfy the boundary conditions. Substituting in the form for the Piola-Kirchoff tensor (3.56) gives

$$\begin{aligned} \int_V [\mathbf{s}'\mathbf{L}(\mathbf{s}) - \mathbf{s}\mathbf{L}(\mathbf{s}')] \; dV &= -\int_\Sigma [\mathbf{s}' \cdot \mathbf{T}_E \cdot \hat{\mathbf{r}} - \mathbf{s} \cdot \mathbf{T}_E' \cdot \hat{\mathbf{r}}]_-^+ \; d\Sigma \\ &\quad - \int_\Sigma p_0 \left[\mathbf{s}'(\nabla \mathbf{s})^T - \mathbf{s}' \cdot (\nabla \cdot \mathbf{s})\mathbf{I} - \mathbf{s}(\nabla \mathbf{s}')^T + \mathbf{s} \cdot (\nabla \cdot \mathbf{s}')\mathbf{I} \right]_-^+ \cdot \hat{\mathbf{r}} \, d\Sigma \end{aligned}$$

The last term can be shown to vanish using Gauss' theorem (3.63) and the fact that $\nabla \cdot (\nabla \mathbf{s})^T = \nabla(\nabla \cdot \mathbf{s})$ so finally we obtain:

$$\int_V [\mathbf{s}' \cdot \mathbf{L}(\mathbf{s}) - \mathbf{s} \cdot \mathbf{L}(\mathbf{s}')] \; dV = -\int_\Sigma [\mathbf{s}' \cdot \mathbf{T}_E \cdot \hat{\mathbf{r}} - \mathbf{s} \cdot \mathbf{T}_E' \cdot \hat{\mathbf{r}}]_-^+ \, d\Sigma \tag{3.65}$$

We can get an explicit expression for $\int \mathbf{s}' \cdot \mathbf{L}(\mathbf{s}) dV$ by using equation 3.61. We integrate this equation over volume and use Gauss' theorem to get

$$\begin{aligned} \int_V \mathbf{s}' \cdot \mathbf{L}(\mathbf{s}) \, dV + \int_\Sigma [\mathbf{s}' \cdot \mathbf{T}_E \cdot \hat{\mathbf{r}}]_-^+ \, d\Sigma &= \int_V \mathbf{s} \cdot \mathbf{L}(\mathbf{s}') \, dV + \int_\Sigma [\mathbf{s} \cdot \mathbf{T}_E' \cdot \hat{\mathbf{r}}]_-^+ \, d\Sigma \\ = -\int_V \{ \nabla \mathbf{s}' \cdot \cdot \mathbf{C} : \nabla \mathbf{s} &+ \tfrac{1}{2} \rho_0 \nabla \phi_0 \cdot [\mathbf{s}' \cdot \nabla \mathbf{s} - \mathbf{s}'(\nabla \cdot \mathbf{s}) + \mathbf{s} \cdot \nabla \mathbf{s}' - \mathbf{s}(\nabla \cdot \mathbf{s}')] \\ + \rho_0[\mathbf{s}' \nabla \phi_1 + \mathbf{s} \nabla \phi_1' &+ \mathbf{s}' \cdot \mathbf{s} \nabla(\nabla \phi_1)] + \frac{1}{4\pi G} \nabla \phi_1 \cdot \nabla \phi_1' \; \} \; dV \end{aligned} \tag{3.66}$$

This particular algebraic form emphasizes the quadratic nature of the right hand side.