Data assimilation and inverse problems – Homework Set 1

Problem 1
Is the probability measure $P(H) = 1/2$, $P(T) = 1/2$ a good model for a coin toss? How often do you observe H (or T) when you toss a coin 10, 50, 100 or 500 times?

Problem 2
Let x and x be two random variables. Show that if x and y are independent, then x and y are uncorrelated. Find (or read about) a counter example that shows that if x and y are uncorrelated, they might not be independent. Show that if x and y are jointly Gaussian, then x and y are uncorrelated if and only if x and y are independent.

Problem 3
Let x be a multivariate random variable with probability density function $p(x) \propto \exp(-F(x))$, where $F(x)$ is a quadratic function with positive definite Hessian. Show that x is Gaussian with mean $\mu = \arg\min F(x)$ and that the covariance matrix is the inverse of the Hessian of F evaluated at its minimizer.

Problem 4
(Box-Muller algorithm). Let x and y be two independent uniform random variables on $[0, 1]$. Show that $\eta_1 = \sqrt{-2\sigma^2 \log x}\cos(2\pi y)$ and $\eta_2 = \sqrt{-2\sigma^2 \log x}\sin(2\pi y)$ are independent Gaussians with mean zero and variance σ^2.