

Data assimilation and inverse problems – Homework Set 2

Build a Kalman filter for the following system. The model is a discretization of a single-degree-of-freedom, viscously damped linear oscillator governed by the second-order differential equation

$$\frac{d^2z}{dt^2} + 2\xi\omega \frac{dz}{dt} + \omega^2z = 0. \quad (1)$$

You can use $\omega = 1$ and $\xi = 10^{-3}$, write your code so that it is easy to change these parameters. Recall that you can transform a second-order differential equation into a set of two first-order differential equations and use an explicit Euler scheme to discretize (1). You should end up with a discrete equation of this type:

$$x_{k+1} = Mx_k, \quad k = 0, 1, 2, \dots,$$

with initial condition $x(0) \sim \mathcal{N}(0, I)$, and where x is 2×1 vector (a discrete-time approximation of $(z, dz/dt)^T$), $M = I + \Delta t A$, I is the identity matrix, and A is a 2×2 matrix that depends on ξ and ω . You can pick $\Delta t = 0.001$, but write your code so that it is easy to change the time-step Δt . We assume that we can get noisy measurements of the position z , but the velocity is hidden, i.e.,

$$y_{k+1} = Hx_{k+1} + \eta_k, \quad \eta_k \sim \mathcal{N}(0, R), \text{ iid}, \quad (2)$$

where $H = [1 \ 0]$, $R = 5$.

We will now do what is often called a “synthetic data experiment”, or “twin-experiment”: we invent a “true state”, by simulating the model for a given time T , and then perturb the true state by noise (see equation (2)) to create data. We then use these data and a Kalman filter to recover the “true state”.

The following three steps are important in a synthetic data experiment.

- Create a true state by simulating (1), starting from the initial condition $x_0 = [1, 2]^T$ for $T = 100$ dimensionless time units. Make a plot of the true state as a function of time. Use a subplot command and have one of the subplots show the velocity, the other the position. You should find that both the position and the velocity oscillate and that both slowly decrease (the rate at which the oscillations decrease is controlled by ξ , the oscillation frequency is controlled by ω , if you use a large ω , you will need to decrease your time-step Δt).
- Create “synthetic data” by perturbing the true state from (a) by Gaussian noise as in equation (2). Make a plot of the position, the position data, and the velocity, again using two subplots, one for position and position data, one for velocity. Use dots (or some other marker) for the observation. You should find that the observations are noisy and that the noise is large.
- Build a Kalman filter based on equations (1) and (2) and use the synthetic data from (b). Make plots of (i) the “true state” position, position observations, and its KF reconstruction; (ii) the “true state” velocity, and its KF reconstruction; (iii) the two components of the Kalman gain. You can use $x_0 \sim \mathcal{N}(0, I)$. What happens when you make other (more “wrong”) assumptions about the initial state?
- Compute the mean square error (MSE) at time k ,

$$\text{MSE}_k = \frac{1}{2} \sum_{j=1}^2 ([x_k^t]_j - [\bar{x}_k^a]_j)^2,$$

where $[x_k^t]_j$, $j = 1, 2$, is the j th component of the true state at time k , $[\bar{x}_k^a]_j$, $j = 1, 2$, is the j th component of the analysis mean at time k . Compute the normalized trace of the analysis covariance

$$\text{trP}_k = \frac{1}{2} \text{trace}(P_k^a).$$

Plot MSE_k and trP_k vs. time in the same window.