The Local Ensemble Transform Kalman Filter

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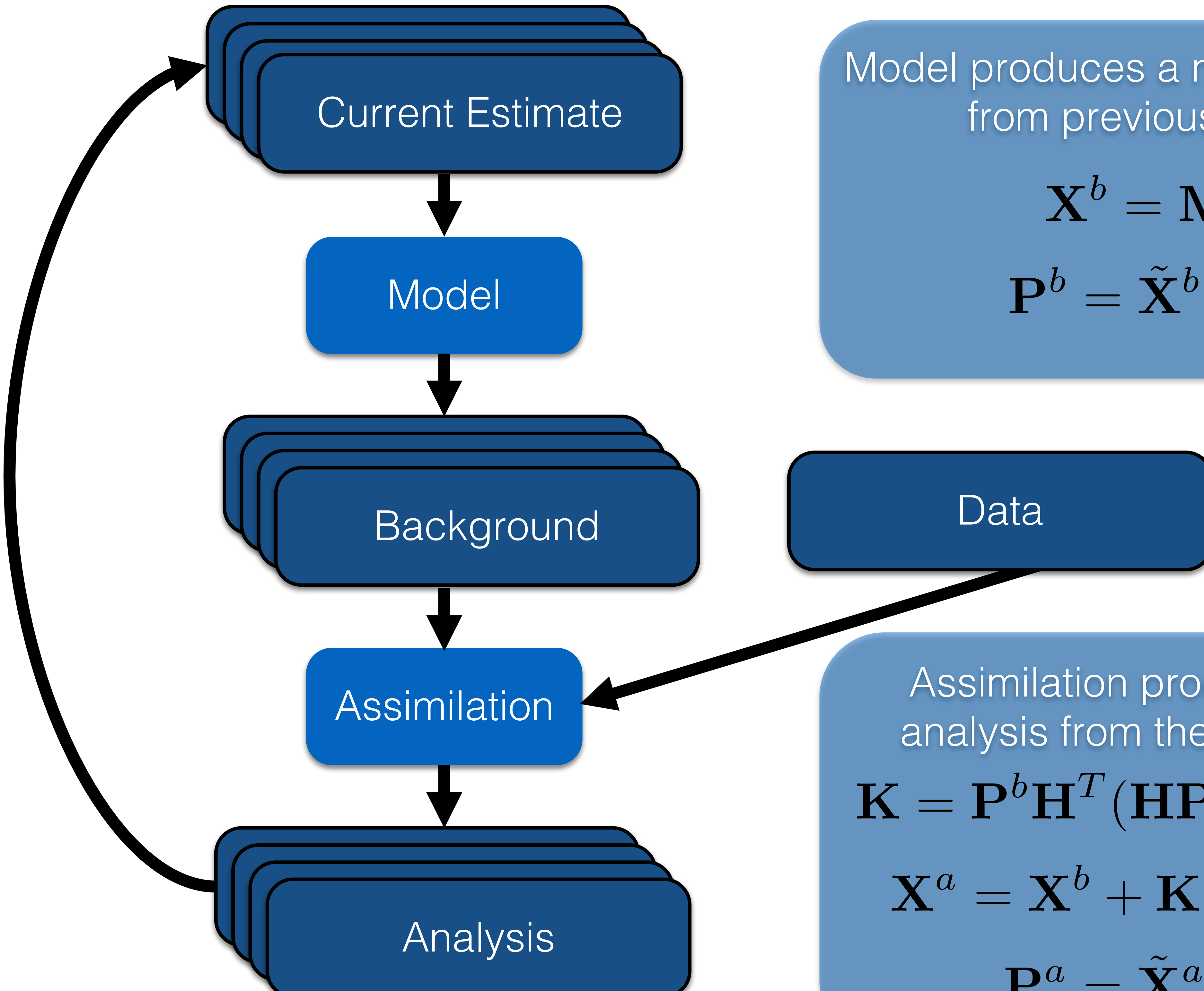
Ensemble Kalman Filter (EnKF)



Model produces a new background from previous analysis

$$\mathbf{X}^b = \mathbf{M}\mathbf{X}^a$$

$$\mathbf{P}^b = \tilde{\mathbf{X}}^b(\tilde{\mathbf{X}}^b)^T$$

Assimilation produces a new analysis from the background

$$\mathbf{K} = \mathbf{P}^b\mathbf{H}^T(\mathbf{H}\mathbf{P}^b\mathbf{H}^T + \mathbf{R})^{-1}$$

$$\mathbf{X}^a = \mathbf{X}^b + \mathbf{K}(\mathbf{Y} - \mathbf{H}\mathbf{X}^b)$$

$$\mathbf{P}^a = \tilde{\mathbf{X}}^a(\tilde{\mathbf{X}}^a)^T$$

(Bergers et al., 1998)

Square Root Filters

- No longer apply assimilation to each member individually
- Instead update ensemble mean and covariance
- Then update entire ensemble to have analysis mean and covariance

$$\mathbf{P}^a = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}^b$$

$$\mathbf{P}^a = \tilde{\mathbf{X}}^b(\mathbf{I} - \tilde{\mathbf{H}}^T(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^T + \mathbf{R})^{-1}\tilde{\mathbf{H}})(\tilde{\mathbf{X}}^b)^T$$

$$\tilde{\mathbf{H}} = \mathbf{H}\tilde{\mathbf{X}}^b$$

$$(\mathbf{P}^a)^{1/2} = \tilde{\mathbf{X}}^b(\mathbf{I} - \tilde{\mathbf{H}}^T(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^T + \mathbf{R})^{-1}\tilde{\mathbf{H}})^{1/2}$$

$$\mathbf{P}^a = \tilde{\mathbf{X}}^a(\tilde{\mathbf{X}}^a)^T$$

$$\tilde{\mathbf{X}}^a = \tilde{\mathbf{X}}^b(\mathbf{I} - \tilde{\mathbf{H}}^T(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^T + \mathbf{R})^{-1}\tilde{\mathbf{H}})^{1/2}$$

Note: $\mathbf{P}^a = \tilde{\mathbf{X}}^a(\tilde{\mathbf{X}}^a)^T = \tilde{\mathbf{X}}^a\mathbf{U}\mathbf{U}^T(\tilde{\mathbf{X}}^a)^T = \tilde{\mathbf{X}}^a\mathbf{U}(\tilde{\mathbf{X}}^a\mathbf{U})^T$

(Tippett et al., 2002)

Square Root Filters

- Using Woodbury Identity, change the form of the square root
- This removes the need to calculate $\widetilde{H}\widetilde{H}^T$ which can be expensive

$$(\mathbf{A} + \mathbf{UCV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1}$$

$$\tilde{\mathbf{X}}^a = \tilde{\mathbf{X}}^b(\mathbf{I} - \tilde{\mathbf{H}}^T(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^T + \mathbf{R})^{-1}\tilde{\mathbf{H}})^{1/2}$$

$$\tilde{\mathbf{X}}^a = \tilde{\mathbf{X}}^b(\mathbf{I} + \tilde{\mathbf{H}}^T\mathbf{R}^{-1}\tilde{\mathbf{H}})^{-1/2}$$

Ensemble Transform Kalman Filter (ETKF)

Recast state space in terms of ensemble

$$\mathbf{x}^b = \bar{\mathbf{x}}^b + \tilde{\mathbf{X}}^b \mathbf{w}^b$$

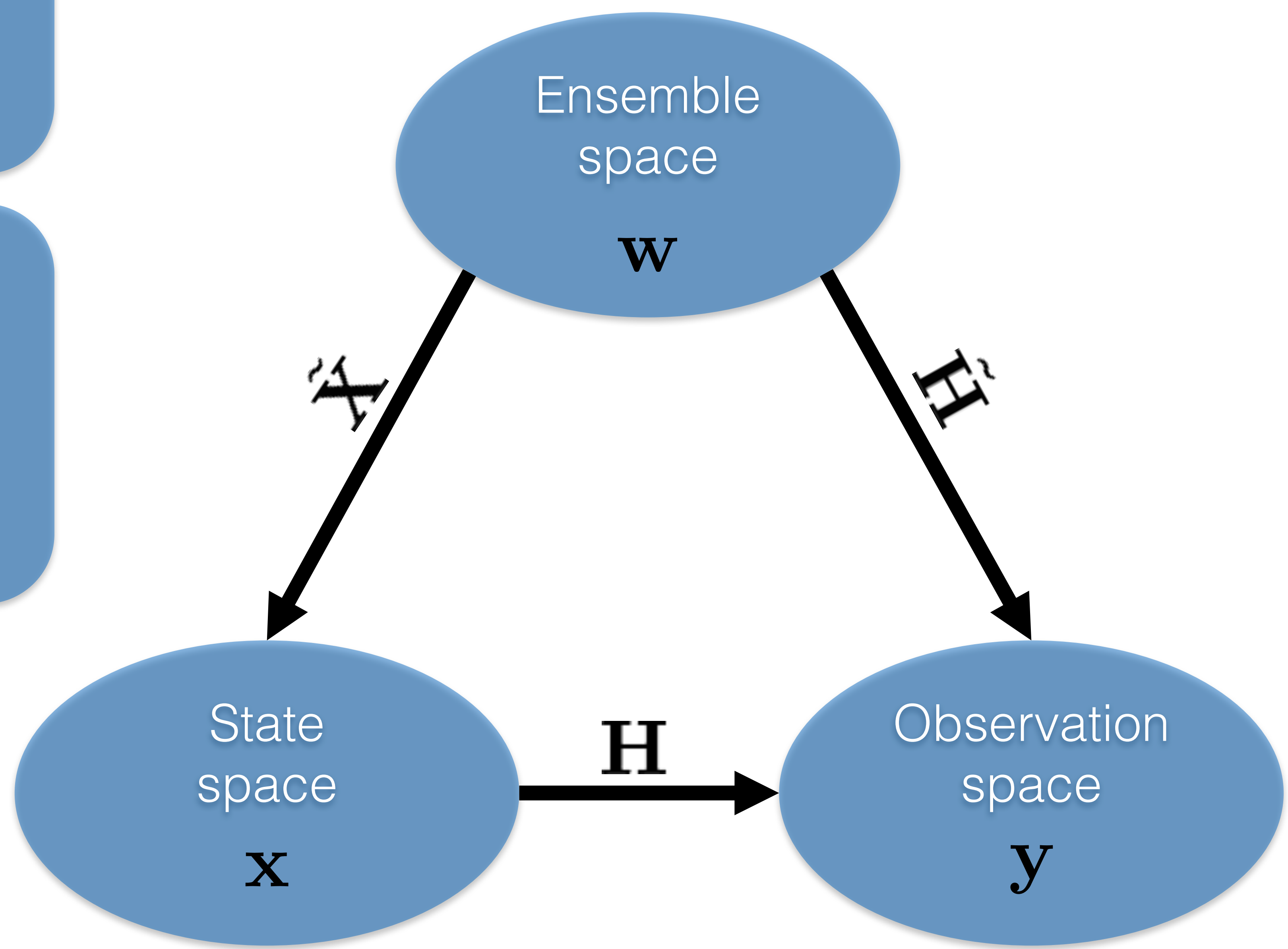
$$\mathbf{w}^b \sim N(0, \mathbf{I})$$

Dimension is greatly reduced

$$\dim(\mathbf{x}) \sim 10^4$$

$$\dim(\mathbf{w}) \sim 10$$

$$\tilde{\mathbf{H}} = \mathbf{H}\tilde{\mathbf{X}}^b$$



(Hunt et al., 2007)(Bishop et al. 2001)

Recast the Kalman Gain in terms of $\widetilde{\mathbf{P}}^{\mathbf{a}}$

$$\mathbf{K} = \mathbf{P}^b\mathbf{H}^T(\mathbf{R} + \mathbf{H}\mathbf{P}^f\mathbf{H}^T)^{-1}$$

$$\mathbf{K} = \mathbf{P}^b\mathbf{H}^T(\mathbf{R}^{-1} + \mathbf{R}^{-1}\mathbf{H}((\mathbf{P}^b)^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1})$$

$$\mathbf{K} = \mathbf{P}^b\mathbf{H}^T(\mathbf{R}^{-1} + \mathbf{R}^{-1}\mathbf{H}\mathbf{P}^a\mathbf{H}^T\mathbf{R}^{-1})$$

$$\mathbf{K} = \mathbf{P}^b((\mathbf{P}^a)^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})\mathbf{P}^a\mathbf{H}^T\mathbf{R}^{-1}$$

$$\mathbf{K} = \mathbf{P}^b(\mathbf{P}^b)^{-1}\mathbf{P}^a\mathbf{H}^T\mathbf{R}^{-1}$$

$$\mathbf{K} = \mathbf{P}^a\mathbf{H}^T\mathbf{R}^{-1}$$

$$\tilde{\mathbf{P}}^a = (\mathbf{I} + \tilde{\mathbf{H}}^T\mathbf{R}^{-1}\tilde{\mathbf{H}})^{-1}$$

$$\mathbf{K} = \tilde{\mathbf{X}}^b\tilde{\mathbf{P}}^a(\tilde{\mathbf{X}}^b)^T\mathbf{H}^T\mathbf{R}^{-1}$$

$$\mathbf{K} = \tilde{\mathbf{X}}^b\tilde{\mathbf{P}}^a\tilde{\mathbf{H}}^T\mathbf{R}^{-1}$$

(Hunt et al., 2007)(Bishop et al. 2001)

Ensemble Transform Kalman Filter (ETKF)

Recast state space in terms of ensemble

$$\mathbf{x}^b = \bar{\mathbf{x}}^b + \tilde{\mathbf{X}}^b \mathbf{w}^b$$

$$\mathbf{w}^b \sim N(0, \mathbf{I})$$

Dimension is greatly reduced

$$\dim(\mathbf{x}) \sim 10^4$$

$$\dim(\mathbf{w}) \sim 10$$

Perform assimilation in ensemble space

$$\tilde{\mathbf{P}}^a = (\mathbf{I} + \tilde{\mathbf{H}}^T \mathbf{R}^{-1} \tilde{\mathbf{H}})^{-1}$$

$$\bar{\mathbf{w}}^a = \tilde{\mathbf{P}}^a \mathbf{H}^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}\bar{\mathbf{x}}^b)$$

$$\bar{\mathbf{x}}^a = \bar{\mathbf{x}}^b + \tilde{\mathbf{X}}^b \bar{\mathbf{w}}^a$$

$$\tilde{\mathbf{P}}^a = \mathbf{C}(\mathbf{\Lambda} + \mathbf{I})^{-1} \mathbf{C}^T$$

$$\mathbf{W}^a = \mathbf{C}(\mathbf{\Lambda} + \mathbf{I})^{-1/2} \mathbf{C}^T + \bar{\mathbf{w}}^a$$

$$\bar{\mathbf{X}}^a = \bar{\mathbf{x}}^b + \tilde{\mathbf{X}}^b \mathbf{W}^a$$

KF

$$\mathbf{P}^a = ((\mathbf{P}^b)^{-1} + \mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1}$$

$$\bar{\mathbf{x}}^a = \bar{\mathbf{x}}^f + \mathbf{P}^a \mathbf{H}^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}\bar{\mathbf{x}}^f)$$

ETKF

$$\tilde{\mathbf{P}}^a = (\mathbf{I} + \tilde{\mathbf{H}}^T \mathbf{R}^{-1} \tilde{\mathbf{H}})^{-1}$$

$$\bar{\mathbf{w}}^a = \tilde{\mathbf{P}}^a \tilde{\mathbf{H}}^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}\bar{\mathbf{x}}^f)$$

(Hunt et al., 2007)(Bishop et al. 2001)

Localization in ETKF (LETKF)

- Rank of P is equal to the ensemble size - 1
- The assimilation step will be performed in a lower dimensional space
- Spurious correlations exist over long distances
- Apply localization through domain and observation localization
- Truncate state and observation vector to include 'nearby' points.

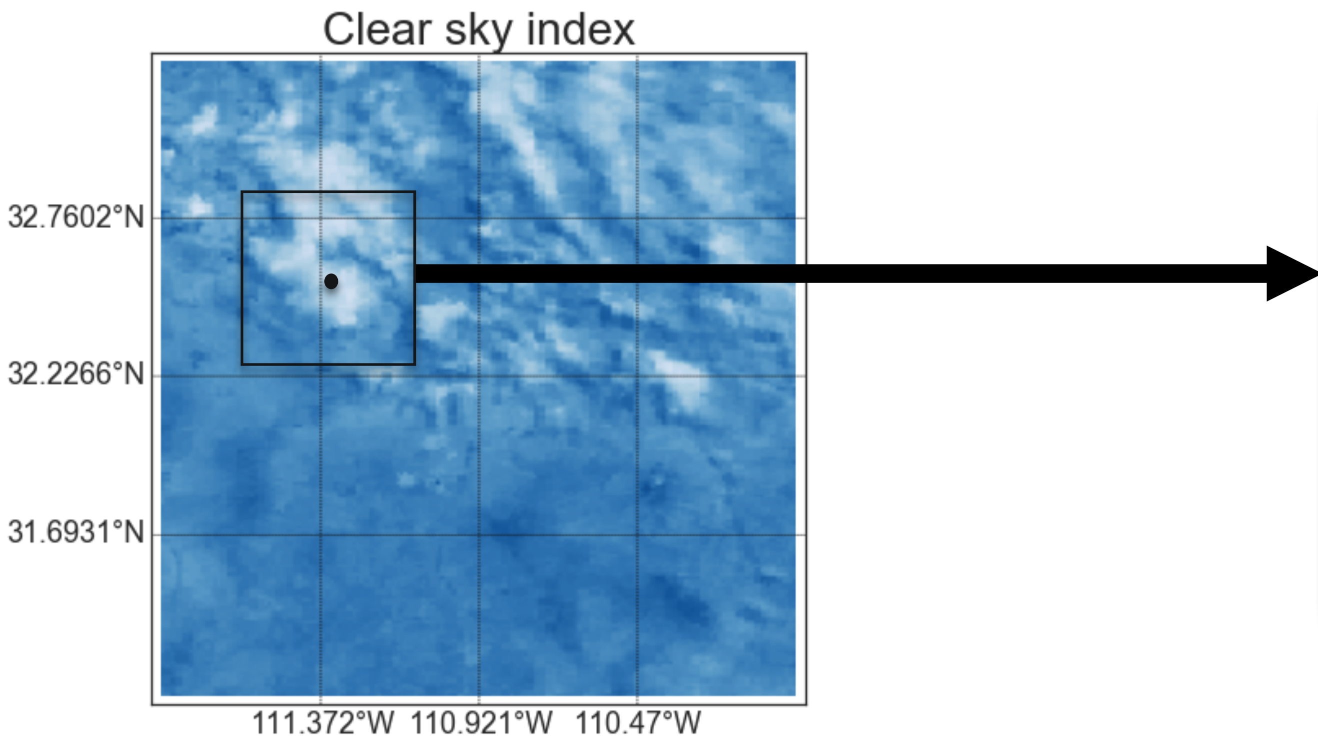
Domain localization

$$\mathbf{x}_l = \mathbf{x}_{[-l:l]}$$

$$\mathbf{y}_l = \mathbf{y}_{[-l:l]}$$

Observation localization

$$\mathbf{R}_l = \tilde{\mathbf{L}} \circ \mathbf{R}$$



- Assimilate data for each point in our state individually
- Only include those elements of our state vector and observation vector that ought to correlate with the assimilation point in question

(Hunt et al., 2007)

Inflation

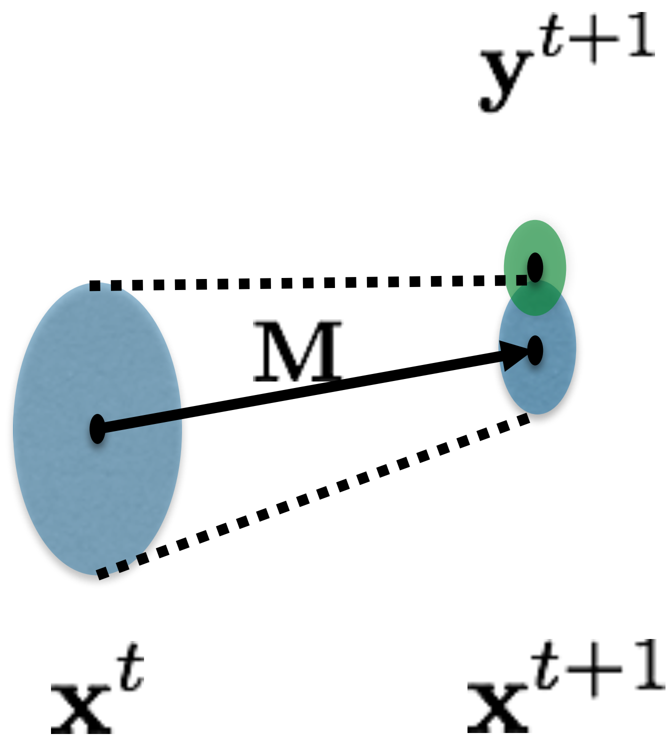
- Model, representation, and sampling errors result in state error underestimation
- Error in ensemble mean is too low compared to observation error
- Observations are under valued in assimilation step
- Inflation allows us to increase ensemble error covariance each step

Inflation

$$\tilde{\mathbf{P}}^a = ((\mathbf{I})^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}$$

$$\tilde{\mathbf{P}}^a_i = ((\mathbf{I}\rho)^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}$$

$$\rho > 1$$



(Hunt et al., 2007)

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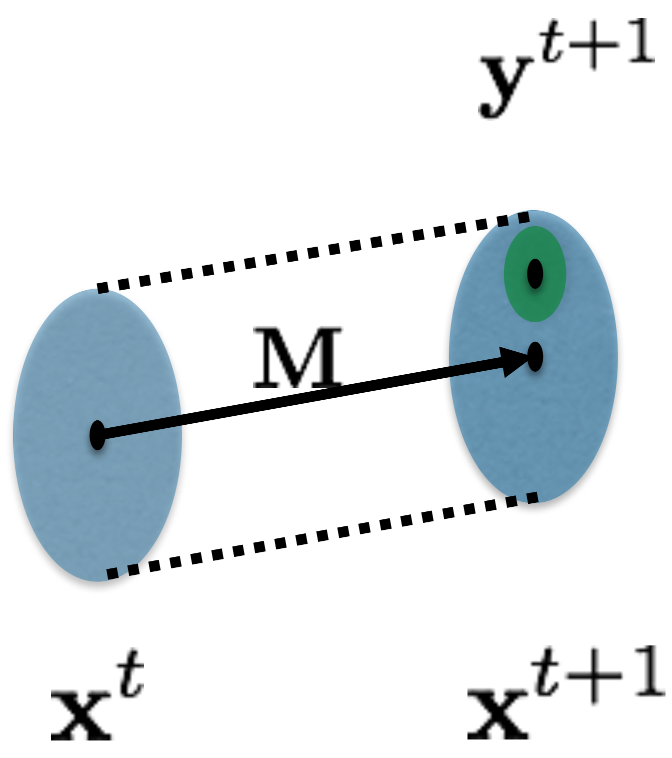
Inflation

$$\tilde{\mathbf{P}}^a = ((\mathbf{I})^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}$$

$$\tilde{\mathbf{P}}_i^a = ((\mathbf{I}\rho)^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}$$

$$\rho > 1$$



(Hunt et al., 2007)