Math 529

Lorenz '96 : ODE Solvers & Linearization

L96: $$\frac{dx_i}{dt} = (x_{i+1} - x_{i-2})x_{i-1} - x_i + F$$

I write this in vector form $\dot{\bar{x}} = \bar{F}(\bar{x})$

where $$\boxed{\bar{F}(\bar{x}) = A\bar{x} \circ B\bar{x} - \bar{x} + \bar{F}},$$

"$\circ$" is the hadamard product,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 0 & 1 & \ddots & & \vdots \\ 0 & -1 & 0 & 0 & \ddots & & \vdots \\ & & & \ddots & \ddots & & \\ & & & & & 0\,1 & 0 \\ 0 & 0 & & & \cdots & 0\,0 & 1 \\ 1 & 0 & & & \cdots & -1\,0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & & & 0 \\ 0 & 1 & 0 & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & & 0\,0 & 0 \\ & & & & 1\,0 & 0 \\ 0 & & \cdots & & 0\,1 & 0 \end{bmatrix}$$

We then have $$\boxed{\nabla \bar{F}(\bar{x}) = \mathrm{diag}(Ax)B + \mathrm{diag}(Bx)A - I}$$

(proof left to reader).

I will now drop vector notation.

Solver #1: RK2 (explicit trapezoid)

$$x_1 = x_0 + \tfrac{1}{2}\Delta t\, F(x_0) + \tfrac{1}{2}\Delta t\, F(x_0 + \Delta t\, F(x_0))$$

$$\vdots$$

$$x_{20} = x_{19} + \tfrac{1}{2}\Delta t\, F(x_{19}) + \tfrac{1}{2}\Delta t\, F(x_{19} + \Delta t\, F(x_{19}))$$

Linearized M:

$$M = \frac{\partial x_{20}}{\partial x_0} = \frac{\partial x_{19}}{\partial x_0} + \tfrac{1}{2}\Delta t\, \nabla F(x_{19}) \cdot \frac{\partial x_{19}}{\partial x_0} + \cdots$$

$$\cdots + \tfrac{1}{2}\Delta t\, \nabla F(x_{19} + \Delta t\, F(x_{19}))\left(\frac{\partial x_{19}}{\partial x_0} + \Delta t\, \nabla F(x_{19})\frac{\partial x_{19}}{\partial x_0}\right)$$

$$= \left(I + \tfrac{1}{2}\Delta t\, \nabla F(x_{19}) + \tfrac{1}{2}\Delta t\, \nabla F(x_{19} + \Delta t\, F(x_{19}))(I + \Delta t\, \nabla F(x_{19}))\right)\frac{\partial x_{19}}{\partial x_0}$$

$$\frac{\partial x_{19}}{\partial x_0} = \cdots\left[\quad\cdots\cdots\quad\right]\frac{\partial x_{18}}{\partial x_0}$$

$$\vdots$$

$$\frac{\partial x_1}{\partial x_0} = \left[I + \tfrac{1}{2}\Delta t\, \nabla F(x_0) + \tfrac{1}{2}\Delta t\, \nabla F(x_0 + \Delta t\, f(x_0))(I + \Delta t\, \nabla f(x_0))\right]$$

$\boxed{\text{Solver 2: RK3}}$

Butcher tableau:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & -1 & 2 & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \tfrac{1}{2}\Delta t, y_n + \tfrac{1}{2}\Delta t k_1\right)$$

$$k_3 = f\left(t_n + \Delta t, y_n + \Delta t(2k_2 - k_1)\right).$$

$$y_{n+1} = y_n + \Delta t\left(\tfrac{1}{6}k_1 + \tfrac{2}{3}k_2 + \tfrac{1}{6}k_3\right)$$

L96 RHS has no time dependence, so we have

$$x_1 = x_0 + \tfrac{1}{6}\Delta t f(x_0) + \tfrac{2}{3}\Delta t f\left(x_0 + \tfrac{1}{2}\Delta t f(x_0)\right) + \dots$$

$$\dots + \tfrac{1}{6}\Delta t\, f\left[x_0 + 2\Delta t f\left(x_0 + \tfrac{1}{2}\Delta t f(x_0)\right) - \Delta t f(x_0)\right]$$

RK3

Linearized M: $$M = \frac{\partial x_{20}}{\partial x_0} = \frac{\partial x_{19}}{\partial x_0} + \frac{1}{6}\Delta t \nabla f(x_{19})\frac{\partial x_{19}}{\partial x_0} + \dots$$

$$\dots + \frac{2}{3}\Delta t\, \nabla f\left(x_{19} + \tfrac{1}{2}\Delta t f(x_{19})\right)\left[\frac{\partial x_{19}}{\partial x_0} + \frac{1}{2}\Delta t \nabla f(x_{19})\frac{\partial x_{19}}{\partial x_0}\right] + \dots$$

$$\dots + \frac{1}{6}\Delta t\, \nabla f\left[x_{19} + 2\Delta t f\left(x_{19} + \tfrac{1}{2}\Delta t f(x_{19})\right) - \Delta t f(x_{19})\right] \cdot \dots$$

$$\dots \left[\frac{\partial x_{19}}{\partial x_0} + 2\Delta t \nabla f\left(x_{19} + \tfrac{1}{2}\Delta t f(x_{19})\right)\left(\frac{\partial x_{19}}{\partial x_0} + \frac{1}{2}\Delta t \nabla f(x_{19})\frac{\partial x_{19}}{\partial x_0}\right) - \dots\right.$$

$$\left.\dots - \Delta t \nabla f(x_{19})\frac{\partial x_{19}}{\partial x_0}\right]$$

Solver 2: RK3

$$M = \Big\{ I + \frac{1}{6}\Delta t \nabla F(x_{19}) + \frac{2}{3}\Delta t \nabla F\left(x_{19} + \frac{1}{2}\Delta t F(x_{19})\right)\left(I + \frac{1}{2}\Delta t \nabla F(x_{19})\right) + \ldots$$

$$\ldots + \frac{1}{6}\Delta t \nabla F\left[x_{19} + 2\Delta t F\left(x_{19} + \frac{1}{2}\Delta t F(x_{19})\right) - \Delta t F(x_{19})\right] \cdot \ldots$$

$$\ldots \cdot \left[I + 2\Delta t \nabla F\left(x_{19} + \frac{1}{2}\Delta t F(x_{19})\right)\left(I + \frac{1}{2}\Delta t \nabla F(x_{19})\right) - \Delta t \nabla F(x_{19})\right]\Big\}\left(\frac{\partial x_{19}}{\partial x_0}\right)$$

$$\vdots$$

$$\frac{\partial x_1}{\partial x_0} = I + \frac{1}{6}\Delta t \nabla F(x_0) + \frac{2}{3}\Delta t \nabla F\left(x_0 + \frac{1}{2}\Delta t F(x_0)\right)\left(I + \frac{1}{2}\Delta t \nabla F(x_0)\right) + \ldots$$

$$\ldots + \frac{1}{6}\Delta t \nabla F\left[x_0 + 2\Delta t F\left(x_0 + \frac{1}{2}\Delta t F(x_0)\right) - \Delta t F(x_0)\right] \cdot \ldots$$

$$\ldots \cdot \left[I + 2\Delta t \nabla F\left(x_0 + \frac{1}{2}\Delta t F(x_0)\right)\left(I + \frac{1}{2}\Delta t \nabla F(x_0)\right) - \Delta t \nabla F(x_0)\right]$$

Solver 3 : RK4

should be $\frac{1}{3}$

$$x_1 = x_0 + \frac{1}{6}\Delta t\, k_1^0 + \frac{1}{3}\Delta t\, k_2^0 + \frac{1}{3}\Delta t\, k_3^0 + \frac{1}{6}\Delta t\, k_4^0$$

where
$$k_1^0 = F(x_0)$$
$$k_2^0 = F\left(x_0 + \tfrac{1}{2}\Delta t\, F(x_0)\right) = F\left(x_0 + \tfrac{1}{2}\Delta t\, k_1^0\right)$$
$$k_3^0 = F\left(x_0 + \tfrac{1}{2}\Delta t\, F\left(x_0 + \tfrac{1}{2}\Delta t\, F(x_0)\right)\right) = F\left(x_0 + \tfrac{1}{2}\Delta t\, k_2^0\right)$$
$$k_4^0 = F\left(x_0 + \Delta t\, F\left(x_0 + \tfrac{1}{2}\Delta t\, F\left(x_0 + \tfrac{1}{2}\Delta t\, F(x_0)\right)\right)\right) = F\left(x_0 + \Delta t\, k_3^0\right)$$

Then
$$M = \frac{\partial x_{20}}{\partial x_0} = \frac{\partial x_{19}}{\partial x_0} + \frac{1}{6}\Delta t\, \nabla k_1^{19} + \frac{1}{3}\Delta t\, \nabla k_2^{19} + \dots$$
$$\dots + \frac{1}{3}\Delta t\, \nabla k_3^{19} + \frac{1}{6}\Delta t\, \nabla k_4^{19}$$

where
$$\nabla k_1^{19} = \nabla F(x_{19})\frac{\partial x_{19}}{\partial x_0}$$
$$\nabla k_2^{19} = \nabla F\left(x_{19} + \tfrac{1}{2}\Delta t\, k_1^{19}\right)\left(\frac{\partial x_{19}}{\partial x_0} + \tfrac{1}{2}\Delta t\, \nabla k_1^{19}\right)$$
$$\nabla k_3^{19} = \nabla F\left(x_{19} + \tfrac{1}{2}\Delta t\, k_2^{19}\right)\left(\frac{\partial x_{19}}{\partial x_0} + \tfrac{1}{2}\Delta t\, \nabla k_2^{19}\right)$$
$$\nabla k_4^{19} = \nabla F\left(x_{19} + \Delta t\, k_3^{0}\right)\left(\frac{\partial x_{19}}{\partial x_0} + \Delta t\, \nabla k_3^{19}\right)$$

Solver 3 : RK4

then $M = \dfrac{\partial x_{20}}{\partial x_0} = \dots$

$$= \Big[I + \tfrac{1}{6}\Delta t\, \nabla F(x_{19}) + \dots$$

$$\dots + \tfrac{1}{3}\Delta t \boxed{\nabla F\left(x_{19} + \tfrac{1}{2}\Delta t\, k_1^{19}\right)\left(I + \tfrac{1}{2}\Delta t\, \nabla F(x_{19})\right)} + \dots$$

$$\dots + \tfrac{1}{3}\Delta t \boxed{\nabla F\left(x_{19} + \tfrac{1}{2}\Delta t\, k_2^{19}\right)\left(I + \tfrac{1}{2}\Delta t \cdot \boxed{}\right)} + \dots$$

$$+ \tfrac{1}{6}\Delta t\, \nabla F\left(x_{19} + \Delta t\, k_3^{19}\right)\left(I + \Delta t \cdot \boxed{}\right) \Big] \cdot \frac{\partial x_{19}}{\partial x_0}$$
