Math 128b – Spring 2014 – Homework set 10

Due Tuesday 4/15 before the lecture starts.

Problem 1

Download the image "Snowboarder.tiff" from the class website and construct a low rank approximation of the image.

Hints. You can read a tiff file into matlab using the command

```
X=imread('Snowboarder.tiff')
```

and you can look at it using the command

```
imagesc(X)
```

X is an object that contains the three channels (red, blue and green) of the image. You can get to each channel by using the commands

```
Red = double(X(:,:,1));
Green = double(X(:,:,2));
Blue = double(X(:,:,3));
```

Now Red, Green and Blue are matrices. Compute the SVD of each matrix so that $A = USV^T$, and compute a rank p approximation for each one using

$$\hat{A} = \sum_{i=1}^{p} s_i u_i v_i^T. \tag{1}$$

You can either use your SVD code from homework 9 or Matlab's svd command to compute the SVD. You can figure out how to choose p by looking at the singular values of A (neglect small singular values). You can then assemble the low rank approximations of the color image from the low rank approximations of the three channels using the commands

```
Xlr(:,:,1) = RedLr;
Xlr(:,:,2) = GreenLr;
Xlr(:,:,3) = BlueLr;
Xlr = uint8(Xlr);
```

where BlueLr, RedLr and GreenLr are the low rank approximations of Blue, Red and Green. Then you can look at your low rank approximation using

```
imagesc(Xlr)
```

You should hand in: your code, the low rank approximation of the image (it is fine if you do not have a color printer) and an answer to the question: what rank did you choose in each channel and why?

Problem 2

(a) Show (using integration by parts) that if $u(x,t)$ solves the heat equation

$$\begin{aligned} u_t &= u_{xx}, \\ u(x,0) &= f(x) \text{ for } 0 \le x \le 1, \\ u_x(0,t) &= 0 \text{ for } t \ge 0, \\ u_x(1,t) &= 0 \text{ for } t \ge 0, \end{aligned}$$

where $f(x)$ is a smooth function, then

$$\frac{d}{dt}\int_0^1 u^2(x,t)dx \le 0.$$

This implies that $u(x,t) \to 0$ as $t \to \infty$.

(b) Show that the heat equation with Neumann boundary conditions

$$\begin{aligned} u_t &= u_{xx}, \\ u(x,0) &= f(x) \text{ for } 0 \le x \le 1, \\ u_x(0,t) &= a \text{ for } t \ge 0, \\ u_x(1,t) &= b \text{ for } t \ge 0, \end{aligned}$$

where $f(x)$ is a smooth function, has a unique solution.

Hints. Suppose u and $\tilde{u}$ are both solutions of the heat equation, and define $w = u - \tilde{u}$, which also solves the above heat equation, but with slightly different Neumann boundary conditions and zero initial conditions. Then use the results from (a).

(c) Show that that the solution $u(x,t)$ of the heat equation

$$\begin{aligned} u_t &= u_{xx}, \\ u(x,0) &= f(x) \text{ for } 0 \le x \le 1, \\ u_x(0,t) &= u_x(1,t) = a \text{ for } t \ge 0, \end{aligned}$$

approaches a steady state, i.e.

$$u(x,t) \to v(x) = ax, \text{ as } t \to \infty.$$

Hint: try a solution of the form $u(x,t) = v(x) + w(x,t)$ and use the results from (a) and (b).

Problem 3

Write a program to solve the heat equation

$$\begin{aligned} u_t &= u_{xx}, \\ u(x,0) &= x^6 + x \text{ for } 0 \le x \le 1, \\ u(0,t) &= 0 \text{ for } t \ge 0, \\ u(1,t) &= 2 \text{ for } t \ge 0, \end{aligned}$$

with the forward finite difference method.

(a) What do you expect the solution to look like for large t? What is the "steady state solution"?

(b) Choose $h = 0.1$, $\sigma = 0.6$ and $k = \sigma h^2$. Make a plot of the solution after 20 time steps.

(c) Choose $h = 0.1$, $\sigma = 0.4$ and $k = \sigma h^2$. How many time steps are needed so that the solution you find via finite differences is "close" to the steady state solution? Plot the approximate solution and the steady state solution into the same figure.

Hints. You can define "close" by computing the 2-norm of the difference of your approximate solution and the steady state solution evaluated on the (space) grid. You can use the command "hold on" to plot more than one graph on one figure. For example:

```
plot(x,us,'b','LineWidth',2)
hold on, plot(x,u,'r--','LineWidth',2)
```