Math 128b – Spring 2014 – Homework set 11

Due Tuesday 4/22 in class before the lecture starts.

Problem 1
Somebody left a smelly piece of cheese in the first row on the very left side of 2 Evans (seen from my perspective, i.e. looking at the room from the black board), on Monday at 3:30 pm. Can you, who usually sits on the very right seat in the first row, smell the cheese on Tuesday at the beginning of class while sitting in your seat?

To answer this question, model this situation with the heat equation $u_t = Du_{xx}$ on $0 \leq x \leq 11$, $t \geq 0$ (assuming the classroom is about 11m wide). The diffusion coefficient of smell in air is $D = 4.2 \cdot 10^{-5}\,\mathrm{m}^2/\mathrm{s}$. The piece of cheese generates an initial smell distribution so that $u(x,0) = \exp(-x^2)$. The boundary conditions on the left are $u(0,t) = 1$, which means the cheese is such that a smell level of "1" is maintained; on the right, set $u_x(11,t) = 0$. The distance from your seat to the piece of cheese is $10\,\mathrm{m}$. The "threshold" for you being able to smell the cheese is that the solution at your seat is larger than 10^{-3} (in appropriate smell units). To answer the question you should simulate the heat equation for $24\,\mathrm{hrs}$ and check if the threshold is reached at your seat.

Use the Crank-Nicolson scheme to do the simulation. You should use Matlab's sparse package to form the matrices A and B (of the scheme) or else this calculation will take very long.

(a) Choose any h and k you find appropriate, and simulate for 24 hrs.

(b) To build up trust in your calculations, investigate the effect of various h and k. First, create a reference solution by choosing $h = 10^{-3}$ and $k = 10^{-3}$. Then plot: (i) the solution after 24 hrs; and (ii) the time evolution of the the smell at your seat, i.e. $u(10,t)$. Repeat the simulation with ($h = 0.1$, $k = 0.1$), ($h = 0.01$, $k = 0.1$), and ($h = 0.01$, $k = 0.01$) and plot $u(x, 24\,\mathrm{hrs})$ and $u(10\,\mathrm{m}, t)$ for $0 \leq t \leq 24\,\mathrm{hrs}$, for each approximation and in the same figure as the results of the reference calculation.

You should hand in: your code to solve the heat equation, a plot of the solution after 24 hrs (obtained with the various values for h and k), and a plot of the time evolution of the smell at your seat, i.e. $u(10\,\mathrm{m}, t)$ for $0 \leq t \leq 24\,\mathrm{hrs}$, also for the various choices of h and k.

Hint: use the "hold on" command to plot more than one graph on one figure and use different colors. For example:

```
plot(x,us,'b','LineWidth',2)
hold on, plot(x,u,'r--','LineWidth',2)
```

Problem 2

(a) Show that $u(x,t) = \exp(-t)\sin(\pi x)$ solves the heat equation

$$\pi^2 u_t = u_{xx},$$
$$u(x,0) = \sin(\pi x) \text{ for } 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0 \text{ for } 0 \le t \le 1.$$

Plot the solution on $0 \le x \le 1$, $0 \le t \le 1$, using Matlab's "surf" command. Hint: to make a nice figure, you can download the file "nicecolormap.m" and use code similar to the following.

```
dt = 1e-2; t=0:dt:1;
dx = 1e-2; x=0:dx:1;
[X,T] = meshgrid(x,t);
ue = exp(-T).*sin(pi*X);
surf(X,T,ue)
colormap nicecolormap
shading interp
view([0 90])
set(gcf,'Color','w')
title('Exact solution')
```

You should hand in your plot, as well as the code that generates the plot.

(b) Use the backward difference method with $k = 0.01$, $h = 0.01$ to solve the heat equation. Plot the solution and compare with the exact solution. Does it look similar to the exact solution?

You should hand in your plot, as well as the code that generates the plot.

(c) Use the backward difference method with $(h = 0.1,\ k = 0.1)$, $(h = 0.01,\ k = 0.01)$, $(h = 0.001,\ k = 0.001)$. For each choice, compute the error

$$e(h,k) = |u(0.5,1) - w(0.5,1,k,h)|,$$

where $u(x,t)$ is the exact solution and where $w(x,t,h,k)$ is the backward difference approximation with step sizes k and h. For each h and k, plot $w(x,1)$ on the same figure as the exact solution $u(x,1) = \exp(-1)\sin(\pi x)$, and fill out the following table

Step size	Error $e(h,k)$
$h = 0.1,\ k = 0.1$	
$h = 0.01,\ k = 0.01$	
$h = 0.001,\ k = 0.001$	

You should hand in your plot, as well as the code that generates the plot.

Problem 3

Show that if $f(x)$ is a degree 3 polynomial, then $u(x,t) = f(x) + ctf''(x)$ solves the initial value problem $u_t = cu_{xx}$, $u(x,0) = f(x)$.