Math 128b – Spring 2014 – Homework set 12

Due Thursday 5/1 in class before the lecture starts.

Problem 1
Solve the baby wave equation

$$\begin{aligned} u_t + u_x &= 0, \\ u(x,0) &= \frac{1}{\sqrt{2\pi}} \exp(-x^2/2), \end{aligned}$$

on $-\infty \le x \le \infty$ and $t \ge 0$.

(a) Use the Crank-Nicolson scheme with $h = 0.1$ and $k = 0.1$ and simulate for 10 time units. Use the computational domain $-10 \le x \le 20$ and assume zero boundary conditions. Plot the initial conditions, and the exact solution and your approximation at time $t = 10$ (in the same figure, using different colors and linestyles).

Hints. You should use Matlab's sparse package to make use of the tridiagonal matrices in this method. You can also animate your simulation and see your wave travelling by doing something like this:

```
figure
set(gcf,'Color','w')
for kk=1:Nk
  u = myCrankNicolsonStep(u);
  plot(x,u,'LineWidth',2)
  drawnow
end
```

(b) Repeat the simulation with ($h = 0.1$, $k = 0.1$), ($h = 0.01$, $k = 0.01$), and ($h = 0.001$, $k = 0.001$). For each set of h and k compute the error

$$e(10, h, k) = ||f(x - 10) - w(x, 10, h, k)||_2$$

where $f(x - 10)$ is the exact solution evaluated on the numerical domain and $w(x, 10, h, k)$ is your approximation. Plot the error as a function of h on a logarithmic scale (use Matlab's "loglog" to do it). What is the slope of the line?

(c) Use the Lax-Friedrichs scheme with $h = 0.1$ and $k = 0.09$ and simulate for 10 time units. Use the computational domain $-10 \le x \le 20$. Plot the initial conditions, and the exact solution and your approximation at time $t = 10$ (in the same figure, using different colors and linestyles). How do these results compare to those obtained with the Crank-Nicolson scheme? Explain why this scheme is so good with $h = k$ (for this equation).

Problem 2
Plot the real and imaginary parts of the largest (in absolute value) eigenvalue of the matrices A of the

(a) Lax-Wendroff,

(b) Lax-Friedrichs,

(c) Crank-Nicolson,

for the baby wave equation as a function of σ, for $0 \leq \sigma \leq 2$ and a mesh with 100 grid points in space. Also plot the unit circle. What does this tell you about the stability of the schemes?

Hints. Loop over $\sigma = 0 : .01 : 2$, compute the matrices for each value of σ, compute the largest eigenvalee eigenvalues and plot it using, for eaxmple,

```
L = eig(A);
[~,ii] = max(abs(L));
eval = L(ii);
hold on, plot(real(eval), imag(eval),'r.','MarkerSize',20)
```

To plot the unit circle, you can do something like this:

```
x = -1:.01:1;
y = sqrt(1-x.^2);
hold on, plot(x, y,'r','LineWidth',2)
hold on, plot(x,-y,'r','LineWidth',2)
axis equal % this makes the circle look like a circle
```

Problem 3
Consider the wave equation

$$\begin{aligned} u_{tt} &= 4\, u_{xx}, \\ u(x,0) &= 0 \text{ for } 0 \leq x \leq 1, \\ u_t(x,0) &= 2\pi\, \sin(\pi x) \text{ for } 0 \leq x \leq 1, \\ u(0,t) &= 0 \text{ for } t \geq 0, \\ u(1,t) &= 0 \text{ for } t \geq 0. \end{aligned}$$

(a) Show that $u(x,t) = \sin(\pi x)\sin(2\pi t)$ solves this wave equation. Make a plot of the solution for $0 \leq t \leq 2$. The code below can help you make a nice looking plot.

```
dt = 1e-2; t=0:dt:2;
dx = 1e-2; x=0:dx:1;
[X,T] = meshgrid(x,t);
ue = sin(pi*X).*sin(2*pi*T),
surf(X,T,ue)
colormap nicecolormap
shading interp
view([0 90])
set(gcf,'Color','w')
title('Exact solution')
```

(b) Solve the wave equation for $0 \leq t \leq 2$ numerically using the Leap-Frog scheme discribed in the book (pp. 393–395) with $h = 0.1$, $k = h/c$. Plot the solution and compare to the exact solution.

(c) Solve the wave equation with the Leap-Frog scheme and $h = 0.1$, $h = 0.01$, $h = 0.001$ (use sparse matrices!) and for each value of h set $k = h/c$. Then compute the error at $x = 0.5$ and $t = 1.5$

$$e(h) = |w(0.5, 1.5, h) - u(0.5, 1.5)|,$$

where u is the exact solution and w is the approximate solution. Plot the error as a function of h on a logarithmic scale (use Matlab's loglog command). What is the slope of the line?