Math 128b – Spring 2014 – Homework set 3

Due Tuesday 2/11 in class before the lecture starts.

1. (Similar to p. 130, Computer problems 2.6: 5) Code the conjugate gradient method as on p. 122, but add the following feature: the code finishes (successfully) if the ∞-norm of the residual is smaller than a tolerance (specified by you). Then use your code to solve the sparse system (2.45) on p. 114. Use the code on p. 114 to create the sparse matrix A in Matlab as well as the right-hand-side b. Then solve the system for $n = 100,000$ and with the tolerance equal to 10^{-6}. Plot the ∞-norm of the residual as a function of the iteration number. Hand in this plot as well as your code that produced this plot.

2. Let A be an $n \times n$ matrix and let $A(k)$ be the matrix you obtain after k successfull steps of Gaussian elimination (GE) without pivoting. Show that A is singular if the nth row of $A(n-1)$ is zero.

3. Let U be an $n \times n$ upper triangular matrix.

   (a) Show that U is invertible if all its diagonal elements are non-zero. Hint: back-substituion.

   (b) Show that if one of the diagonal elements is zero, then U is singular. Hint: Suppose the ith diagonal element of U is zero. Show that there exists a non-zero vector x such that $Ux = 0$.

   Conclude that an upper triangular matrix is singular if and only if one of its diagonal elements is zero.

4. Let A be an $n \times n$ matrix with elements a_{ij}, $i, j = 1, \ldots, n$. Let A_k denote the upper-left minor of A:

$$A_k = \begin{pmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{k1} & \ldots & a_{kk} \end{pmatrix}.$$

   Note that doing k steps of GE without pivoting to the $k \times k$ upper minor of A yields the same result as taking the $k \times k$ upper-left minor after k steps of GE without pivoting (using the notation of problem 2, this means that $A(k)_k = A_k(k)$).

   (a) Use the above results and your results from problem 2 and problem 3 to show that if all $k \times k$ upper-left minors of A are invertible, then GE without pivoting finds the solution of $Ax = b$.

   (b) Use the above results to show that GE without pivoting always succeeds if A is SPD. Hint: also recall that an SPD matrix is invertible and that all its principal submatrices are also SPD.

5. Let A and B be $n \times n$ symmetric matrices.

   (a) Show that if A and B commute ($AB = BA$), then AB is also symmetric.

   (b) Construct an example of two symmetric matrices A and B such that AB is not symmetric.