

Math 128b – Spring 2014 – Homework set 5

Due Tuesday 2/25 in class before the lecture starts.

1. Why is the solution of the normal equations $A^T Ax = A^T b$ called the “least squares” solution of $Ax = b$, where A is $m \times n$, x is $n \times 1$, b is $m \times 1$, $m > n$? Hint: consider the function $F(x) = \|r\|_2^2 = \sum r_j^2 = (b - Ax)^T (b - Ax)$, where $r = b - Ax$ is the residual. This function is minimized when its gradient ∇F is zero.
2. Write Matlab code to find the QR-factorization of an $m \times n$ matrix A using
 - (a) the Gram-Schmidt method;
 - (b) Householder reflectors.
 - (c) Let x_1, \dots, x_{11} be 11 evenly spaced points in $[2, 4]$ (i.e. $x_i = 2 + (i - 1)0.2$) and let

$$y_i = 1 + x_i + x_i^2 + x_i^3 + x_i^4 + x_i^5 + x_i^6 + x_i^7, \quad i = 1, \dots, 11,$$

be 11 data values. Use your codes from (a) - (c) to find the coefficients of the degree 7 polynomial that best fits these 11 data points (in the least squares sense). What is the “exact” solution for the coefficients?

(d) Use Matlab’s backslash operator to solve the normal equations $A^T Ax = A^T b$ directly (i.e. by inverting $A^T A$). Compare to the results from (c).

You should hand in: (i) your codes for QR-factorization using Gram-Schmidt and using Householder reflectors; (ii) your code that solves $A^T Ax = A^T b$ using the QR-factorization of A ; (iii) estimate of x (the least squares solution) obtained with these codes, as well as the exact solution.

3. Let v_j , $j = 1, \dots, m$ be m non-zero vectors of size $n \times 1$ with $m \leq n$. Show that if v_j are mutually orthogonal (i.e. $v_i^T v_j = 0$ if $i \neq j$), then they are also linearly independent.
4. Let A be an $m \times n$ matrix, $m \geq n$. Show that $A^T A$ is invertible if A has rank n .
5. Let Q be an $n \times n$ orthogonal matrix and let x be a vector of size n . Show with an example that it is possible that $\|Qx\|_1 \neq \|x\|_1$ or $\|Qx\|_\infty \neq \|x\|_\infty$. Why is this *not* an issue with least squares? Hint: try

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and $x = (1, 1)^T$.

6. p.198, exercise 6: Let A be an $n \times n$ invertible matrix.
 - (a) Show that $(A^T)^{-1} = (A^{-1})^T$.
 - (b) Let b be an $n \times 1$ vector; then $Ax = b$ has exactly one solution. Show that this solution satisfies the normal equations.