Math 128b – Spring 2014 – Homework set 6

Due Tuesday 3/11 in class before the lecture starts.

1. p. 229, Exercise 3: let

$$A = \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

   Prove that for any initial guess x_0 and any right hand side b, GMRES converges to the exact solution after two steps.

2. Let x be a real $n \times 1$ vector. Minimize

$$f(x) = \frac{1}{2}x^T x,$$

   with Newton's method and show that it converges after one step for any initial guess x_0.

3. Let A be an $n \times n$ SPD matrix and let x and b be real $n \times 1$ vectors (b is given). Minimize

$$f(x) = \frac{1}{2}x^T Ax + b^T x + c,$$

   with Newton's method and show that it converges after one step for any initial guess x_0.

4. Let x_1 be an $n \times 1$ vector that represents the weather today.

   (a) Suppose you have one measurement y (a scalar), indicative of today's weather. Find the "best" estimate of today's weather given the data y by minimizing

$$f(x) = \frac{1}{2}||y - Hx_1||_2^2,$$

   where $H = (1, 0, 0, \dots, 0)$ is a $1 \times n$ (row) vector, with Newton's method. What can you say about $(x_1)_j$, $j + 1, \dots, n$, (the elements of x_1) given this one measurement?

   (b) Let x_0 be an $n \times 1$ vector that represents the weather yesterday. Suppose you have some knowledge about yesterday's weather, which you describe by x_0 being "close" to μ, a given $n \times 1$ vector. Further suppose that you have a numerical model that computes today's weather from yesterday's weather by

$$x_1 = Ax_0,$$

   where A is a given, invertible $n \times n$ matrix. Update your estimate of yesterday's weather by minimizing

$$f(x) = \frac{1}{2}||HAx_0 - y||_2^2 + \frac{1}{2}||x_0 - \mu||_2^2.$$

   To do the minimization, compute the gradient ∇f and then solve $\nabla f = 0$. What is your forecast for today's and tomorrow's weather? Suggestion: compute

$$x_1 = Ax_0,$$

   and

$$x_2 = Ax_1 = A^2 x_0,$$

where x_0 is your best guess at the weather yesterday, and x_1 and x_2 represent the weather today and tomorrow.

(c) You are given an improved, nonlinear model of the weather. The model relates the weather at day j (described by a 3×1 vector $x^j = (x_1^j, x_2^j, x_3^j)$) to the weather at day $j+1$ (x^{j+1}) by

$$\begin{aligned} x_1^{j+1} &= x_1^j + \sigma(x_2^j - x_1^j)\Delta t, \\ x_2^{j+1} &= x_2^j + x_1^j(\rho - x_3^j)\Delta t, \\ x_3^{j+1} &= x_3^j + (x_1^j x_2^j - \beta x_3^j)\Delta t, \end{aligned} \tag{1}$$

where $\sigma = 10$, $\beta = 8/3$, $\rho = 28$ and $\Delta t = 0.05$ (this is a discretization of the Lorenz attractor). You also have measurements of today's weather in the form of the 3×1 vector

$$y = (-3.37,\ -4.01,\ 19.62)^T.$$

Find the best estimate of yesterday's weather by approximately minimizing

$$f(x_0) = \frac{1}{2}(\mathcal{M}(x_0) - y)^T(\mathcal{M}(x_0) - y) + \frac{1}{2}(x_0 - \mu)^T(x_0 - \mu),$$

where

$$\mu = (-3.1,\ -3.7,\ 24)^T,$$

is your best guess at yesterday's weather before you collected the data y and $\mathcal{M}(x_0)$ is the vector you obtain by plugging x_0 into the model (1). To do so, first formulate this problem as a nonlinear least squares problem, i.e. find the (vector-valued) function $r(x_0)$ such that

$$f(x_0) = \frac{1}{2}r(x_0)^T r(x_0);$$

then perform one step of Gauss-Newton to obtain x_0^*.

Find your best estimate of today's weather by computing $x_1^* = \mathcal{M}(x_0^*)$, and make a forecast for tomorrow's weather by computing $x_2^* = \mathcal{M}(x_1^*)$.

Find the errors of your estimation as follows. Yesterday's weather really was

$$x_0^{\text{true}} = (-3.6,\ -3.2,\ 22.1)^T.$$

Today's and tomorrow's true weathers are $x_1^{\text{true}} = \mathcal{M}(x_0^{\text{true}})$ and $x_1^{\text{true}} = \mathcal{M}(x_0^{\text{true}})$. Compute the errors

$$\begin{aligned} e_0 &= \frac{||x_0^* - x_0^{\text{true}}||_2}{||x_0^{\text{true}}||_2}, \\ e_1 &= \frac{||x_1^* - x_1^{\text{true}}||_2}{||x_1^{\text{true}}||_2}, \\ e_2 &= \frac{||x_2^* - x_2^{\text{true}}||_2}{||x_2^{\text{true}}||_2}, \end{aligned}$$

which represent the errors in your estimates of yesterday's, today's and tomorrow's weather.

Compare these with the errors you would have made without the Gauss-Newton minimization and the data as follows. Compute $\mu_1 = \mathcal{M}(\mu)$ and $\mu_2 = \mathcal{M}(\mu_1)$; these would be your estimates of today's and tomorrow's weather without using Gauss-Newton and the data y. Compute the errors

$$\begin{aligned}
\hat{e}_0 &= \frac{||\mu - x_0^{\text{true}}||_2}{||x_0^{\text{true}}||_2}, \\
\hat{e}_1 &= \frac{||\mu_1 - x_1^{\text{true}}||_2}{||x_1^{\text{true}}||_2}, \\
\hat{e}_2 &= \frac{||\mu_2 - x_2^{\text{true}}||_2}{||x_2^{\text{true}}||_2},
\end{aligned}$$

and compare with the errors e_0, e_1 and e_2 you found above.