Math 128b – Spring 2014 – Homework set 7

Due Tuesday 3/18 in class before the lecture starts.

1. Code:

   (a) the power iteration method to find the leading eigenvalue and the corresponding eigenvector of an $n \times n$ matrix;

   (b) the inverse power iteration method to find the smallest eigenvalue matrix and the corresponding eigenvector of an $n \times n$ matrix;

   (c) the Rayleigh quotient iteration method to find the smallest eigenvalues and corresponding eigenvector of an $n \times n$ matrix.

   Test your codes on the matrix

$$A = \begin{pmatrix} 7 & -33 & -15 \\ 2 & 26 & 7 \\ -4 & -50 & -13 \end{pmatrix},$$

   and compare with Matlab's command "eig(A)".

   You should hand in (i) your code for all three methods; (ii) the results of all three methods applied to A.

2. Let A be an $n \times n$ matrix and define the Rayleigh quotient for any $x \neq 0$ by

$$R(x) = \frac{x^T A x}{x^T x}.$$

   Let λ be an eigenvalue of A with corresponding eigenvalue v. Show that

$$\nabla R(v) = 0,$$

   and that

$$\lambda = R(v),$$

   i.e. the eigenvectors are the stationary points of the Rayleigh quotient and the eigenvalues are the local maxima, minima or saddle points. Hint: note that

$$\nabla R(x) = 2\frac{1}{x^T x} A x - 2\frac{x^T A x}{(x^T x)^2} x.$$

3. Show that the determinant of an $n \times n$ matrix A equals the product of its eigenvalues. Hint: recall that the characteristic polynomial is $\det(A - \lambda I)$ and that its roots are the eigenvalues. Thus we can factorize the characteristic polynomial $\det(A-\lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$. Now find a clever choice for λ.

4. Let $a \neq 0$ and $\epsilon > 0$ and

$$A = \begin{pmatrix} a & \epsilon \\ \epsilon & a \end{pmatrix}.$$

   (a) Find the eigenvalues and eigenvectors of A as functions of ϵ.
   (b) Take the limit of as $\epsilon \to 0$ of the eigenvalues and eigenvectors and compare with the eigenvalues and eigenvectors of

$$B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(a) Compute the eigenvalues of A.
(b) Show that $v = (1,0)^T$ is an eigenvector of A with eigenvalue $\lambda = 1$.
(c) Show that the power method applied to this A will converge for any starting vector $x_0 \neq 0$ to the eigenvector $v = (1,0)^T$.

Hints. Recall that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Let $x_0 = (a,b)^T$ be your initial guess. After k steps of power iteration you have

$$x_{k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} a \\ b \end{pmatrix}.$$

Now normalize x_{k+1}, by computing $x_{k+1}/||x_{k+1}||_2$ and let $k \to \infty$.