~~Review~~ Review

Lecture 9

$\boxed{1.1}$ Linear equations

- What is a linear equation
- Matrix notation
- What is a solution or solution set.
- Key: row-equivalence: elementary row operations do not change the solution
- Consistent => has one or more solutions

$\boxed{1.2}$ Row-reduction

↳ The workhorse algo

↳ row-echelon form of the enough to see if the system has a solution

↳ reduced row-echelon form is unique.
(every matrix has only one reduced row echelon form

$\boxed{1.3}$ Vector equations

↳ Vector = matrix with one column.

↳ Know how to handle them

↳ linear combination

↳ Span $\{\underline{v}_1 \dots \underline{v}_p\}$

$\boxed{1.4}$ Matrix Eq $\underline{\underline{A}}\underline{x} = \underline{b}$

↳ $\underline{\underline{A}}\,\underline{x} = x_1\underline{a}_1 + x_2\underline{a}_2 + \dots + x_n\underline{a}_n$
or "row x column"

↳ When is $\underline{\underline{A}}\,\underline{x} = \underline{b}$ consistent for all $\underline{b}$?

↳ How to solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$?

↳ Vector eqn = $\underline{\underline{A}}\,\underline{x} = \underline{b}$ = augmented matrix of linear system

Homogeneous & Inhomogeneous eqns

↳ $\underline{\underline{A}}\,\underline{x} = \underline{0}$ ≙ homogeneous problem

⇝ has solutions only if there is a free variable

⇝ always has trivial solution

⇝ solutions define planes through origin

↳ $\underline{\underline{A}}\,\underline{x} = \underline{b}$ ≙ inhomogeneous problem.

⇝ Any solution can be written as $\underline{x} = \underline{p} + \underline{x}_h$

⇝ Solutions are shifted planes

$\boxed{1.7}$ Linear independence

↳ $\{\underline{v}_1 \dots \underline{v}_p\}$ lin. indep. $\Rightarrow$ $x_1\underline{v}_1 + \dots + x_p\underline{v}_p = \underline{0}$ has only trivial solution

↳ $\{\underline{v}_1 \dots \underline{v}_p\}$ are dependent

↳ one vector is a lin. combination of the other

↳ one vector is $\underline{0}$.

↳ if there are more vectors than elements in each vector, then they are lin. dependent.

$\boxed{1.8}$ $\boxed{1.9}$ Linear transformations

↳ Think of matrices as fcts! $T(\underline{x}) = \underline{\underline{A}}\,\underline{x}$

↳ $\underline{\underline{A}}\,\underline{x}$ is a linear fct

↳ Range, domain etc are connected to $\underline{\underline{A}}\,\underline{x} = \underline{b}$ solutions.

↳ every linear fct ≙ $\underline{\underline{A}}\,\underline{x} = \underline{b}$

2

the three planes ✓

$$x_1 + 2x_2 + x_3 = 4$$
$$x_2 - x_3 = 1$$
$$x_1 + 3x_2 + \quad = 0$$

have at least one common point of intersection?

Solve system!

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{pmatrix} \underset{\mathrm{III}-\mathrm{I}}{\sim} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 4 \end{pmatrix} \underset{\mathrm{III}-\mathrm{II}}{\sim} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

↳ System is not consistent

↳ There is no x_1, x_2, x_3 such that all eqns are true

↳ There is no point common to all 3 planes.

Chose h, k such that
a) the system has no solution
b) — " — has a unique solution
c) — " — has many solutions

$$\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned} \rightarrow \begin{pmatrix} 1 & h & 2 \\ 4 & 8 & k \end{pmatrix} \underset{\mathrm{II}-4\cdot\mathrm{I}}{\sim} \begin{pmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{pmatrix}$$

a) no solution → system is inconsistent ⇒ rightmost column is a pivot column

↳ $h = 2, \; k \neq 8$.

b) unique solution → ~~System is …~~
2nd column must be pivot column ⇒ $h \neq 2$

$$\boxed{x_2 = \frac{k-8}{8-4h}}$$

$$x_1 = 2 - h \cdot x_2$$

$$\boxed{x_1 = 2 - h \cdot \frac{k-8}{8-4h}}$$

c) many solutions ⇒ we must have a free variable

$$\Rightarrow \quad 8 - 4h = k - 8$$

$$k + 4h = 16 \qquad \text{e.g.} \quad \begin{aligned} k &= 4 \\ h &= 2 \end{aligned}$$

... a 3×3 matrix $\underline{\underline{A}}$ with non-zero entries and a $\underline{b}$ in $\mathbb{R}^3$ such that $\underline{b}$ ~~in $\mathbb{R}^3$~~ ... not in the span of the columns of $\underline{\underline{A}}$.

...ution:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Why? Columns of $\underline{\underline{A}}$ are linearly dependent!

$$\text{Span}(\underline{a}_1, \underline{a}_2, \underline{a}_3) = c \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

There is no c such that $c \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Give a geometric description of Span $\{\underline{v}_1, \underline{v}_2\}$ for $\underline{v}_1 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$.

Solution: For every $\underline{w}$ in span $(\underline{v}_1, \underline{v}_2)$

$$\hookrightarrow \underline{w} = c_1 \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3c_1 - 2c_2 \\ 0 \\ 2c_1 + 3c_2 \end{pmatrix}$$

$\hookrightarrow$ 2nd element is always zero

$\hookrightarrow$ span $\{\underline{v}_1, \underline{v}_2\}$ is the x-z-plane

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Let $\underline{\underline{A}}$ be an $m \times n$ matrix and let $\underline{u}, \underline{v}$ be vectors in $\mathbb{R}^n$ with the property $\underline{\underline{A}}\underline{u} = \underline{0}$, $\underline{\underline{A}}\underline{v} = \underline{0}$. Show that $\underline{\underline{A}}(c \cdot \underline{u} + d\underline{v}) = 0$ for every scalar constants c, d.

proof:

$$\underline{\underline{A}}(c\underline{u} + d\underline{v}) = \underline{\underline{A}}(c\underline{u}) + \underline{\underline{A}}(d\underline{v}) = c \cdot \underline{\underline{A}}\underline{u} + d \cdot \underline{\underline{A}}\underline{v}$$

$$= c \cdot \underline{0} + d \cdot \underline{0} = \underline{0} \checkmark$$

Are the vectors: $\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ linearly independent?

Find value(s) of h such that the following three vectors are linearly ✓ dependent:

$$\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ h \end{pmatrix}$$

Solve the system:

$$\begin{pmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{pmatrix} \begin{matrix} \\ \text{I}+\text{II} \\ \text{III}-4\cdot\text{I} \end{matrix} \sim \begin{pmatrix} 1 & 3 & -1 \\ 0 & -2 & +4 \\ 0 & -5 & h+4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & \frac{h+4}{-5} \end{pmatrix}$$

For linear dependence we must have free variables

$$\hookrightarrow \frac{h+4}{(-5)} = -2$$

$$h+4 = 10$$

$$h = 6$$