4. PSD of Discrete Processes

Every result for a continuous-time stationary stochastic process has an analog in the discrete theory, and there are no surprises. I will state the results here without derivation. Here is the definition of the PSD as a limit of finite Fourier transforms:

$$S_X(f) = \lim_{N \to \infty} \frac{1}{2N} \mathcal{E}\,[\,|\sum_{n=-N}^{N} X_n e^{-2\pi i n f}\,|^2], \quad -\tfrac{1}{2} \le f \le \tfrac{1}{2}. \tag{4.1}$$

The alternative definition through the autocovariance is

$$S_X(f) = \sum_{n=-\infty}^{\infty} R_X(n)\, e^{-2\pi i n f}, \quad -\tfrac{1}{2} \le f \le \tfrac{1}{2} \tag{4.2}$$

and of course one can obtain the autocovariance from the PSD with the coefficients of the Fourier series expansion in (4.2)

$$R_X(n) = \int_{-1/2}^{1/2} S_X(f)\, e^{2\pi i n f}\, df\,. \tag{4.3}$$

Setting $n = 0$ in (4.3) gives

$$\sigma_X^2 = \int_{-1/2}^{1/2} S_X(f)\, df \tag{4.4}$$

so that the variance is again the integral of the PSD over frequency. If we restrict ourselves to convolution filters, then filtering a discrete sequence gives the power spectrum

$$S_{g * X}(f) = |\,\hat{g}(f)\,|^2\, S_X(f) \tag{4.5}$$

where

$$\hat{g}(f) = \sum_{n=-\infty}^{\infty} g_n\, e^{-2\pi i n f}\,. \tag{4.6}$$

5. Aliasing in the PSD

Although we have seen how aliasing affects sampled data earlier, we should see how sampling modifies the PSD for stochastic processes. This is an important issue in the real world because sampled data may be the only record we have of an underlying continuous physical signal whose power spectrum we would like to know. We call a continuous-time signal $Y(t)$, and we derive from it the discrete process:

$$X_n = Y(n\Delta t), \;\; n = 0, \pm 1, \pm 2, \cdots \tag{5.1}$$

where Δt is the sampling interval. As I asserted earlier it is almost always easier to derive properties of the spectrum from the autocovariance. So the autocovariance of the sampled series is

$$R_X(n) = \mathcal{E}\,[X_j\,X_{j+n}] \tag{5.2}$$

$$= \mathcal{E}\,[Y(j\Delta t)\,Y((j+n)\Delta t) = R_Y(j\Delta t)\,. \tag{5.3}$$

Thus the autocovariance of X_n is simply the sampled autocovariance of Y.

The PSD of the discrete process is given by (4.2), which we modify by including Δt in the exponent to scale the frequencies, replacing the Nyquist frequency of ½ in (4.2) by $1/2\Delta t$, and also scaling the expression by the same quantity:

$$S_X(f) = \Delta t \sum_{n=-\infty}^{\infty} R_X(n)\,e^{-2\pi i n f \Delta t}, \quad -1/2\Delta t \le f \le 1/2\Delta t\,. \tag{5.4}$$

Now substitute (5.3)

$$S_X(f) = \Delta t \sum_{n=-\infty}^{\infty} R_Y(n\Delta t)\,e^{-2\pi i n f \Delta t}\,. \tag{5.5}$$

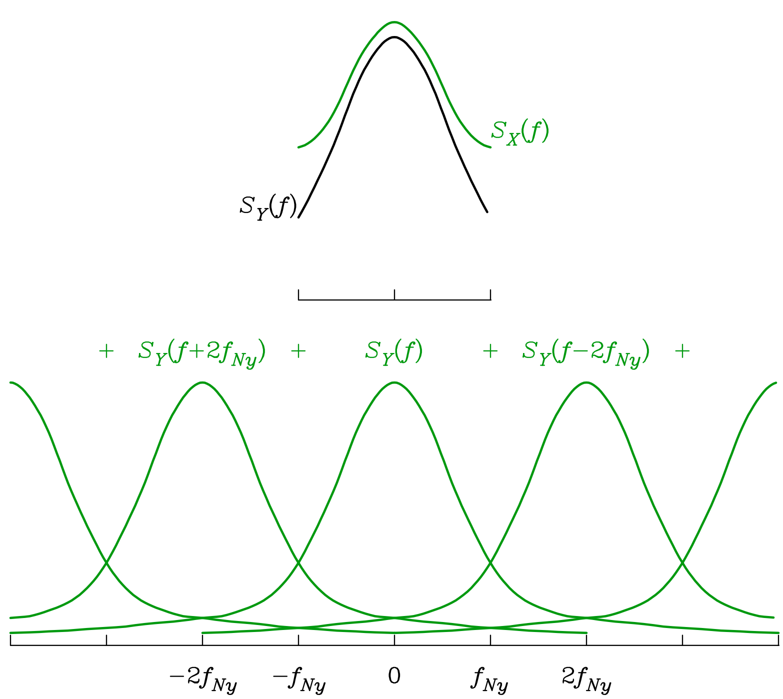
To sum this series we appeal to the **Poisson Sum Rule** given in our treatment of Fourier Theory

$$S_X(f) = \Delta t \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} R_Y(n\Delta t)\,e^{-2\pi i n f \Delta t}\,e^{-2\pi i n m}\,dn\,. \tag{5.6}$$

We change variables in the integral: set $t = n\Delta t$ and then recognize the definition of the continuous process PSD:

$$S_X(f) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} R_Y(t)\,e^{-2\pi i t(f + m/\Delta t)}\,dt \tag{5.7}$$

Figure 3: Aliasing in the PSD



$$= \sum_{m=-\infty}^{\infty} S_Y(f + m/\Delta t)\,. \tag{5.8}$$

Thus the discrete process PSD is a sum of spectra of the original continuous process, shifted by multiples of **twice the Nyquist frequency.** If the power in the continuous process has fallen off to low levels at the Nyquist frequency, the PSD of S_X will be a good approximation to S_Y, although in general S_X will be a factor of two or more above the "true" PSD at $f = 1/2\Delta t$. The moral is that to get a good PSD one must set the sampling rate high enough to avoid aliasing.