GEOPHYSICAL DATA ANALYSIS

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CHAPTER 2: SPECTRAL ANALYSIS OF STOCHASTIC PROCESSES

3.5. PSD of the Derivative

Another useful result that ought to have been derived in Section 3 is the form of the PSD of the time derivative of a continuous process. We get it the same way as we obtained the spectrum of a convolution, by starting at the autocovariance:

$$R_X(t) = \mathcal{E}\,[X(s)\,X(s+t)]\,. \tag{3.13}$$

Differentiate on t:

$$\frac{dR_X}{dt} = \mathcal{E}\,[X(s)\,X'(s+t)] \tag{3.14}$$

where the prime is the time derivative. Under the expectation change variables so that $u = s + t$:

$$\frac{dR_X}{dt} = \mathcal{E}\,[X(u-t))\,X'(u)]\,. \tag{3.15}$$

Differentiate again:

$$\frac{d^2R_X}{dt^2} = -\mathcal{E}\,[X'(u-t)\,X'(u)] = -R_{X'}(-t) \tag{3.16}$$

$$= -R_{X'}(t)\,. \tag{3.17}$$

This is quite a nice result as it stands. Now rearrange and take the Fourier transform, recalling that $\mathcal{F}\,[g'] = 2\pi i f \mathcal{F}\,[g]$:

$$\mathcal{F}\,[R_{X'}] = -\mathcal{F}\left[\frac{d^2R_X}{dt^2}\right] = 4\pi^2 f^2 \mathcal{F}\,[R_X]\,. \tag{3.18}$$

Since we know the FT of R_X is just the PSD we see that

$$S_{dX/dt}(f) = 4\pi^2 f^2 S_X(f)\,. \tag{3.19}$$