Annex of Additional Notes

P: Jeans’ Rule

We prove Jeans’ rule, that the wavelength of a spherical harmonic of degree \( l \) is approximately

\[
\Lambda = \frac{2\pi}{l + \frac{1}{2}}
\]

(1)

For this we need a quantitative idea of what “wavelength” actually means, and this gives rise to confusion in some quarters. The so-called *sectoral* harmonic, Re \( Y^m_l \), seems to some people (e.g., Merrill, McElhinny & McFadden, 1996, p 36) to be at variance with Jeans’ rule near \( \theta = 0 \) and \( \pi \). We will discuss this puzzle.

First the proof of Jean’s formula. We will look at a local patch of the unit sphere and assume the variations of the functions are on scales much shorter than the curvature. We set up a Cartesian axis system with \( \hat{z} \) in the local \( r \) direction.

Then the eigenfunction property (Property 2, or (7.2) plus (7.2) can be approximated by

\[
\frac{\partial^2 u_l}{\partial x^2} + \frac{\partial^2 u_l}{\partial y^2} = -l(l+1)u_l .
\]

(2)

To interpret \( Y^m_l \) as a kind of wave we substitute into (2) a *plane wave*; we try

\[
u_l(x, y) = \exp(2\pi i(k_x x + k_y y)).
\]

(3)

Plugging this into (2) gives a condition on the wavenumbers:

\[
-4\pi^2 (k_x^2 + k_y^2) = -l(l+1) .
\]

(4)

The wavelength of a plane wave, \( \Lambda \), is the reciprocal of the magnitude of its wavevector; thus

\[
\Lambda = \frac{1}{|k|} = \frac{1}{\sqrt{k_x^2 + k_y^2}}
\]

(5)

\[
= \frac{2\pi}{\sqrt{l(l+1)}}.
\]

(6)

If \( l > 1 \) we can perform a power series expansion on the radical

\[
\Lambda = \frac{2\pi}{l + \frac{1}{2} - \frac{1}{8l} + \frac{1}{16l^2} + O(l^3)}
\]

(7)

\[
= \frac{2\pi}{l + \frac{1}{2}} + O(l^{-2})
\]

(8)

which is Jean’s formula. Even for \( l = 2 \) the approximation isn’t bad: the exact value 2.44948... versus 2.5.

But notice, to form an arbitrary spherical harmonic of degree \( l \) we will usually have to add together several waves like (3), all with the same \( |k| \) but with differing directions.

Now the apparent paradox. From the notes (equation (7.17)) or the recipe for zeros of \( Y^m_l, p 24 \) the peaks and troughs of the function Re \( Y^m_l \) near the north
pole are distributed as shown in the figure:

$$\text{Re } Y_l^l(\theta, \phi) = c_l \sin^l \theta \cos \phi$$

(9)

The apparent wavelength, as defined by the distance between two peaks, varies with colatitude, and is always less than $2\pi/(l + \frac{1}{2})$.

Interestingly, there are two, equally correct, but seemingly quite different resolutions of this paradox. In the simpler of the two, we go back to (3) and observe that nothing in the analysis restricts the wavenumbers to be real, provided the sum of the squares is $l(l+1)/4\pi^2$. When one is near the pole, the $\theta$ behavior is like $\sin^l \theta$, a very rapidly decaying function as we go northward, which we will designate the local $y$ direction. Suppose instead of cosine behavior in $y$ we approximate the decay by exponential decrease; now (3) can be written

$$u_l(x, y) = \exp(-2\mu y) \exp(2\pi ik_x x)$$

(10)

where we have substituted $k_y = i\mu$, for a real value of $\mu$. Then (4) shows that

$$k_x = \left( \frac{\mu^2 + l(l+1)}{4\pi^2} \right)^{\frac{1}{2}} \frac{l + \frac{1}{2}}{2\pi}, \text{ if } \mu > 1/4\pi.$$  

(11)

This equation shows that $k_x$ can be much larger than $1/\Lambda_l$, which means that the distance in the $x$ direction (the $\phi$ direction on the sphere) between peaks can be much smaller than given by the Jeans' formula, even though the wavenumber magnitude is exactly $\sqrt{l(l+1)/2\pi}$, so that the wavelength (strictly defined as the reciprocal of the wavenumber magnitude) is still approximately $2\pi/(l + \frac{1}{2})$. In an ordinary plane wave, the amplitude normal to the direction of propagation is constant; in a so-called evanescent wave like this one, that amplitude varies rapidly along the lines of constant phase. The amplitude along the constant

Figure d
phase lines near the pole is varying like $\theta^l$, which is obviously not constant, so interpretation of the sectoral harmonic near the poles as a single plane wave is inappropriate.

In the second perspective, the issue is really the (false) perception that a wavefield composed of waves with a constant wavelength $\lambda$ must always have features with scale $\frac{1}{2}\lambda$. At the poles, $Y_l^j$ behaves like $\sin^l \theta e^{il\phi}$. We will look at the question from the plane-wave aspect; if $f(r, \phi)$ behaves like $r^l \cos l\phi$ at the origin, it can be built from the convergence of $l$ plane waves, all of the same wavelength, and distributed evenly in direction.

First, here is a MATLAB script that generates a set of evenly disposed plane waves, summed together. It can be downloaded and you should run it to see the results.

```matlab
% focus
% Generate L plane waves equally distributed in azimuth
% each of wavelength lam, summed and weighted to cancel at
% the origin. In this realization L should be an even integer
L = 6;
N=51; % size of sample array
lam = 20;
X=ones(N,1) *[0:N-1]; Y=[0 : N-1]' *ones(1,N);
W=zeros(size(X));
for j =0 : L-1
    kx = X*cos(j*pi/L) + Y*sin(j*pi/L);
    U=cos((2*pi/lam) * kx);
    W=W+( -1)^j * U;
    contourf(U); axis equal;
    pause(0.5);
end
contourf(W); axis equal;
```

All of the plane waves in this script have the same wavelength, but near the focus they exhibit the same pattern as the $Y_l^j$ spherical harmonic, a set of lines of constant phase radiating from a single point. A quantitative analysis of the sum of $l$ plane standing starts from the following equation, with $l$ even, which describes the wavefield of the program: if $k_j$ are a set of $l$ wavenumbers uniformly spaced in $(0, \pi)$ then

$$w(r) = \sum_j (-1)^j \cos 2\pi k_j \cdot r.$$  \hfill (12)

Writing out the dot product explicitly, we have

$$w(x, y) = \sum_{k=0}^{l-1} (-1)^k \cos \frac{2\pi k \pi}{l} \left( x \cos \frac{k\pi}{l} + y \sin \frac{k\pi}{l} \right).$$  \hfill (13)

To show this really looks like $\sin^l \theta \cos l\phi$ analytically requires Bessel functions, and so I will omit the math.