Annex of Additional Notes

A: Regression of the Nodes

We are going to derive the expression for the period of regression of the nodes, the slow motion of the plane of an orbit inclined to the equatorial plane. The nodes are the points of intersection of the orbit with the equatorial plane. We make several simplifying approximations: first, the satellite moves in a circular orbit with radius \( b \) and with the normal to the orbital plane \( \hat{n} \). We imagine the satellite moving so rapidly round its orbit that if we average in time, it looks like a circular ring of material spinning. Then we ask for the gravitational couple exerted on this ring by the \( J_2 \) term in the potential. Then we solve the equation of motion of a spinning body acted on by a couple at right angles to its angular momentum, which is the direction that the couple acts. The whole thing is a serious exercise in the vector cross product. A more modern approach would use Lagrangian dynamics. In the final analysis, however, the motion of the satellite is calculated today by solving the equation of motion numerically in a gravitational field given by a full spherical harmonic expansion.

First we need to define the position of the satellite, \( S \), as a function of time. If \( \hat{z} \) is the normal to the equatorial plane (the spin axis of the earth) and \( \hat{x} \) lies in the equatorial plane along the nodal line (we’ll take \( \hat{x} \) to be fixed, though we’ll show soon that it is in fact moving very slowly), then we define \( \hat{p} \) to be the unit vector in the orbital plane at right angles to \( \hat{x} \):

\[
\hat{p} = \hat{n} \times \hat{x}
\]

(A.1)

so that \( \hat{x} \), \( \hat{p} \) and \( \hat{n} \) form a right-handed set of orthogonal coordinate axes. Then \( S \) moves in a circle given by

\[
r(t) = b\hat{x}(t) = b(\hat{x} \cos \omega t + \hat{p} \sin \omega t)
\]

(A.2)
where to a first approximation, we have from Kepler’s 3rd law (aided by Newton) that
\[ \omega^2 = \frac{Gm_E}{b^3}. \]  
(A.3)

At every point in the orbit S experiences a gravitational couple about the center of mass of the earth given by:
\[ \mathbf{T} = \mathbf{r} \times (-m \nabla V) \]  
(A.4)

and we shall use MacCullagh for the potential V:
\[ V(r, \theta) = -Gm_E \left[ \frac{1}{r} - J_2 \frac{a^2}{r^3} P_2(\cos \theta) \right] \]  
(A.5)

where \( \cos \theta = \hat{z} \cdot \hat{r} \) and \( \theta \) is the ordinary colatitude, measured from the spin axis. Since \( V \) doesn’t depend on \( \phi \), the longitude, equation (A.4) can be written:
\[ \mathbf{T} = -m \mathbf{r} \times \left[ \frac{\hat{r}}{r} \frac{\partial V}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial V}{\partial \theta} \right] \]  
(A.6)
\[ = -m \frac{\mathbf{r}}{r} \times \hat{\theta} \frac{\partial V}{\partial \theta} \]  
(A.7)
\[ = -m \hat{\mathbf{r}} \times \hat{\theta} \frac{\partial V}{\partial \theta} = -m \hat{\phi} \frac{\partial V}{\partial \theta}. \]  
(A.8)

Note that \( \hat{\phi} \), the unit vector in the longitudinal direction at S is just
\[ \hat{\phi} = \frac{\hat{z} \times \mathbf{r}}{\sin \theta}. \]  
(A.9)

From (A.5) and then (A.3)
\[ \frac{\partial V}{\partial \theta} = -3Gm_E J_2 a^2 \frac{\sin \theta \cos \theta}{b^3} = -3J_2 a^2 \omega^2 \sin \theta \cos \theta. \]  
(A.10)

So by (A.7) and (A.9), the instantaneous couple is given by
\[ \mathbf{T} = 3mJ_2 a^2 \omega^2 \hat{\mathbf{z}} \times \hat{\mathbf{r}} \cos \theta \]  
(A.11)
\[ = 3mJ_2 a^2 \omega^2 \hat{\mathbf{z}} \times \hat{\mathbf{r}} \hat{\mathbf{z}} \cdot \mathbf{r}. \]  
(A.12)

Our next job is to average over time. In (A.12) only \( \dot{\mathbf{r}} \) varies rapidly in time through (A.2). We plug (A.2) into (A.12) and time average over a complete orbit, denoted by<br>
\[ < \cdot > :=< \mathbf{T} > = 3mJ_2 a^2 \omega^2 <(\hat{\mathbf{z}} \times \hat{\mathbf{x}} \cos \omega t + \hat{\mathbf{z}} \times \hat{\mathbf{p}} \sin \omega t)(\hat{\mathbf{z}} \cdot \mathbf{r} \cos \omega t + \hat{\mathbf{z}} \cdot \hat{\mathbf{p}} \sin \omega t)> \]  
(A.13)
\[ = 3mJ_2 a^2 \omega^2 <(\hat{\mathbf{z}} \times \hat{\mathbf{x}} \cos \omega t + \hat{\mathbf{z}} \times \hat{\mathbf{p}} \sin \omega t) \hat{\mathbf{z}} \cdot \hat{\mathbf{p}} \sin \omega t> \].  
(A.14)

When we average in time the term in \( \sin \omega t \cos \omega t \) goes to zero; the term in \( \sin^2 \omega t \) averages to 1/2. So we have
\[
\langle T \rangle = \frac{3}{2} m J_2 a^2 \omega^2 (\hat{z} \times \hat{p})( \hat{z} \cdot \hat{p}) .
\]
(A.15)

The vectors \( \hat{n} \), \( \hat{z} \) and \( \hat{p} \) all lie in the same plane, and the angle between the first two is called \( \Theta \). So we can rewrite (A.15):

\[
\langle T \rangle = -\frac{3}{2} m J_2 a^2 \omega^2 \cos \Theta \hat{z} \times \hat{n} .
\]
(A.16)

We want to set up a differential equation for the angular momentum of the satellite. First note that the angular momentum of \( S \) about the origin is just

\[
L = m b^2 \omega \hat{n} .
\]
(A.17)

So (A.16) becomes

\[
\langle T \rangle = -\frac{3 a^2}{2b^2} J_2 \omega \cos \Theta \hat{z} \times L .
\]
(A.18)

Finally we recall that the rate of change of angular momentum equals the couple; thus at last we get

\[
\frac{dL}{dt} = -\frac{3 a^2}{2b^2} J_2 \omega \cos \Theta \hat{z} \times L .
\]
(A.19)

The solution to a vector differential equation of the form

\[
\frac{dL}{dt} = \Omega \hat{z} \times L
\]
(A.20)

is a vector \( L \) that moves in a cone with angular velocity \( \Omega \) in the positive sense. The negative sign on \( \Omega \) reverses the sense, and makes the precession retrograde, opposite to the angular momentum, and this is true for either sign of \( L \) in (A.17). This gives the required answer equation (4.10) in the notes.

Just for completeness, we derive the solution of (A.20). First dot with \( \hat{z} \)

\[
\hat{z} \cdot \frac{dL}{dt} = \frac{d\hat{z} \cdot L}{dt} = 0 .
\]
(A.21)

Thus \( \hat{z} \cdot L \) is constant; therefore we decompose \( L \) into a constant part and an orthogonal time varying part: \( L = \hat{z} (\hat{z} \cdot L) + l \) which we substitute into (A.20):

\[
\frac{dl}{dt} = \Omega \hat{z} \times l
\]
(A.22)

obtaining the same equation as before, but now there is no \( \hat{z} \) component in the vector. Dot this equation with \( l \) itself:

\[
l \cdot \frac{dl}{dt} = \frac{1}{2} \frac{dl \cdot l}{dt} = 0
\]
(A.23)

so \( l \) has constant length and must move on a circular path in the plane orthogonal to the \( \hat{z} \) direction. Differentiate (A.22) again:

\[
\frac{d^2l}{dt^2} = \Omega \hat{z} \times \frac{dl}{dt} = \Omega \hat{z} \times (\Omega \hat{z} \times l) .
\]
(A.24)

Applying elementary vector identities to the right side, we find
\[
\frac{d^2 l}{dt^2} = -\Omega^2 l.
\] (A.25)

By this equation in a suitable Cartesian frame in the plane, each component of \( l \) executes simple harmonic motion with period \( 2\pi/\Omega \); thus \( l \) traverses a circular path at uniform speed with the same period which is the result we wanted. But this solution works for either direction of rotation; how can we show that positive \( \Omega \) corresponds to a positive sense of direction? You work it out.